THEORY OF FUNCTIONS

OF A

COMPLEX VARIABLE.

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OF A

COMPLEX VARIABLE

BY

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PREFACE.

AMONG the many advances in the progress of mathematical science during the last forty years, not the least remarkable are those in the theory of functions. The contributions that are still being made to it testify to its vitality: all the evidence points to the continuance of its growth. And, indeed, this need cause no surprise. Few subjects can boast such varied processes, based upon methods so distinct from one another as are those originated by Cauchy, by Weierstrass, and by Riemann. Each of these methods is sufficient in itself to provide a complete development; combined, they exhibit an unusual wealth of ideas and furnish unsurpassed resources in attacking new problems.

It is difficult to keep pace with the rapid growth of the literature which is due to the activity of mathematicians, especially of continental mathematicians: and there is, in consequence, sufficient reason for considering that some marshalling of the main results is at least desirable and is, perhaps, necessary. Not that there is any dearth of treatises in French and in German: but, for the most part, they either expound the processes based upon some single method or they deal with the discussion of some particular branch of the theory.

The present treatise is an attempt to give a consecutive account of what may fairly be deemed the principal branches of the whole subject. It may be that the next few years will see additions as important as those of the last few years: this account would then be insufficient for its purpose, notwithstanding the breadth of range over which it may seem at present to extend. My hope is that the book, so far as it goes, may assist mathematicians, by lessening the labour of acquiring a proper knowledge of the subject, and by indicating the main lines, on which recent progress has been achieved.

No apology is offered for the size of the book. Indeed, if there were to be an apology, it would rather be on the ground of the too brief treatment of some portions and the omissions of others. The detail in the exposition of the elements of several important branches has prevented a completeness of treatment of those branches: but this fulness of initial explanations is deliberate, my opinion being that students will thereby become better qualified to read the great classical memoirs, by the study of which effective progress can best be made. And limitations of space have compelled me to exclude some branches which otherwise would have found a place. Thus the theory of functions of a real variable is left undiscussed: happily, the treatises of Dini, Stolz, Tannery and Chrystal are sufficient to supply the omission. Again, the theory of functions of more than one complex variable receives only a passing mention; but in this case, as in most cases, where the consideration is brief, references are given which will enable the student to follow the development to such extent as he may desire. Limitation in one other direction has been imposed: the treatise aims at dealing with the general theory of functions and it does not profess to deal with special classes of functions. I have not hesitated to use examples of special classes: but they are used merely as illustrations of the general theory, and references are given to other treatises for the detailed exposition of their properties.

The general method which is adopted is not limited so that it may conform to any single one of the three principal independent methods, due to Cauchy, to Weierstrass and to Riemann respectively: where it has been convenient to do so, I have combined ideas and processes derived from different methods.

The book may be considered as composed of five parts.

The first part, consisting of Chapters I—VII, contains the theory of uniform functions: the discussion is based upon power-series, initially connected with Cauchy's theorems in integration, and the properties established are chiefly those which are contained in the memoirs of Weierstrass and Mittag-Leffler.

The second part, consisting of Chapters VIII—XIII, contains the theory of multiform functions, and of uniform periodic functions which are derived through the inversion of integrals of algebraic functions. The method adopted in this part is Cauchy's, as used by Briot and Bouquet in their three memoirs and in their treatise on elliptic functions: it is the method that has been followed by Hermite and others to obtain the properties of various kinds of periodic functions. A chapter has been devoted to the proof of Weierstrass's results relating to functions that possess an addition-theorem.

The third part, consisting of Chapters XIV—XVIII, contains the development of the theory of functions according to the method initiated by Riemann in his memoirs. The proof which is given of the existence-theorem is substantially due to Schwarz; in the rest of this part of the book, I have derived great assistance from Neumann's treatise on Abelian functions, from Fricke's treatise on Klein's theory of modular functions, and from many memoirs by Klein.

The fourth part, consisting of Chapters XIX and XX, treats of conformal representation. The fundamental theorem, as to the possibility of the conformal representation of surfaces upon one another, is derived from the existence-theorem: it is a curious fact that the actual solution, which has been proved to exist in general, has been obtained only for cases in which there is distinct limitation.

The fifth part, consisting of Chapters XXI and XXII, contains an introduction to the theory of Fuchsian or automorphic functions, based upon the researches of Poincaré and Klein: the discussion is restricted to the elements of this newly-developed theory.

The arrangement of the subject-matter, as indicated in this abstract of the contents, has been adopted as being the most convenient for the continuous exposition of the theory. But the arrangement does not provide an order best adapted to one who is reading the subject for the first time. I have therefore ventured to prefix to the Table of Contents a selection of Chapters that will probably form a more suitable introduction to the subject for such a reader; the remaining Chapters can then be taken in an order determined by the branch of the subject which he wishes to follow out.

In the course of the preparation of this book, I have consulted many treatises and memoirs. References to them, both general and particular, are freely made: without making precise reservations as to independent contributions of my own, I wish in this place to make a comprehensive acknowledgement of my obligations to such works. A number of examples occur in the book: most of them are extracted from memoirs, which do not lie close to the direct line of development of the general theory but contain results that provide interesting special illustrations. My intention has been to give the author's name in every case where a result has been extracted from a memoir: any omission to do so is due to inadvertence.

Substantial as has been the aid provided by the treatises and memoirs to which reference has just been made, the completion of the book in the correction of the proof-sheets has been rendered easier to me by the unstinted and untiring help rendered by two friends. To Mr William Burnside, M.A., formerly Fellow of

Pembroke College, Cambridge, and now Professor of Mathematics at the Royal Naval College, Greenwich, I am under a deep debt of gratitude: he has used his great knowledge of the subject in the most generous manner, making suggestions and criticisms that have enabled me to correct errors and to improve the book in many respects. Mr H. M. Taylor, M.A., Fellow of Trinity College, Cambridge, has read the proofs with great care: the kind assistance that he has given me in this way has proved of substantial service and usefulness in correcting the sheets. I desire to recognise most gratefully my sense of the value of the work which these gentlemen have done.

It is but just on my part to state that the willing and active co-operation of the Staff of the University Press during the progress of printing has done much to lighten my labour.

It is, perhaps, too ambitious to hope that, on ground which is relatively new to English mathematics, there will be freedom from error or obscurity and that the mode of presentation in this treatise will command general approbation. In any case, my aim has been to produce a book that will assist mathematicians in acquiring a knowledge of the theory of functions: in proportion as it may prove of real service to them, will be my reward.

A. R. FORSYTH.

TRINITY COLLEGE, CAMBRIDGE.
25 *February*, 1893.

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The following course is recommended, in the order specified, to those who are reading the subject for the first time: *The theory of uniform functions*, Chapters I—V; *Conformal representation*, Chapter XIX; *Multiform functions and uniform periodic functions*, Chapters VIII—XI; *Riemann's surfaces, and Riemann's theory of algebraic functions and their integrals*, Chapters XIV—XVI, XVIII.

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CHAPTER I.

General Introduction.

1. Algebraical operations are either direct or inverse. Without entering into a general discussion of the nature of irrational and of imaginary quantities, it will be sufficient to point out that direct algebraical operations on numbers that are positive and integral lead to numbers of the same character; and that inverse algebraical operations on numbers that are positive and integral lead to numbers, which may be negative or fractional or irrational, or to numbers which may not even fall within the class of real quantities. The simplest case of occurrence of a quantity, which is not real, is that which arises when the square root of a negative quantity is required.

Combinations of the various kinds of quantities that may occur are of the form $x + iy$, where x and y are real and i, the non-real element of the quantity, denotes the square root of -1. It is found that, when quantities of this character are subjected to algebraical operations, they always lead to quantities of the same formal character; and it is therefore inferred that the most general form of algebraical quantity is $x + iy$.

Such a quantity $x + iy$, for brevity denoted by z, is usually called a *complex* variable*; it therefore appears that the complex variable is the most general form of algebraical quantity which obeys the fundamental laws of ordinary algebra.

2. The most general complex variable is that, in which the constituents x and y are independent of one another and (being real quantities) are separately capable of assuming all values from $-\infty$ to $+\infty$; thus a doubly-infinite variation is possible for the variable. In the case of a real variable, it is convenient to use the customary geometrical representation by measurement of distance along a straight line; so also in the case of a complex

* The conjugate complex, viz. $x - iy$, is frequently denoted by z_0.

variable, it is convenient to associate a geometrical representation with the algebraical expression; and this is the well-known representation of the variable $x+iy$ by means of a point with coordinates x and y referred to rectangular axes*. The complete variation of the complex variable z is represented by the aggregate of all possible positions of the associated point, which is often called the point z; the special case of real variables being evidently included in it because, when $y=0$, the aggregate of possible points is the line which is the range of geometrical variation of the real variable.

The variation of z is said to be *continuous* when the variations of x and y are continuous. Continuous variation of z between two given values will thus be represented by continuous variation in the position of the point z, that is, by a continuous curve (not necessarily of continuous curvature) between the points corresponding to the two values. But since an infinite number of curves can be drawn between two points in a plane, continuity of line is not sufficient to specify the variation of the complex variable; and, in order to indicate any special mode of variation, it is necessary to assign, either explicitly or implicitly, some determinate law connecting the variations of x and y or, what is the same thing, some determinate law connecting x and y. The analytical expression of this law is the equation of the curve which represents the aggregate of values assumed by the variable between the two given values.

In such a case the variable is often said to *describe* the part of the curve between the two points. In particular, if the variable resume its initial value, the representative point must return to its initial position; and then the variable is said to describe the whole curve†.

When a given closed curve is continuously described by the variable, there are two directions in which the description can take place. From the analogy of the description of a straight line by a point representing a real variable, one of these directions is considered as positive and the other

* This method of geometrical representation of imaginary quantities, ordinarily assigned to Gauss, was originally developed by Argand who, in 1806, published his "*Essai sur une manière de représenter les quantités imaginaires dans les constructions géométriques.*" This tract was republished in 1874 as a second edition (Gauthier-Villars); an interesting preface is added to it by Hoüel, who gives an account of the earlier history of the publications associated with the theory.

Other references to the historical development are given in Chrystal's *Text-book of Algebra*, vol. i, pp. 248, 249; in Holzmüller's *Einführung in die Theorie der isogonalen Verwandschaften und der conformen Abbildungen, verbunden mit Anwendungen auf mathematische Physik*, pp. 1—10, 21—23; in Schlömilch's *Compendium der höheren Analysis*, vol. ii, p. 38 (note); and in Casorati, *Teorica delle funzioni di variabili complesse*, only one volume of which was published. In this connection, an article by Cayley (*Quart. Journ. of Math.*, vol. xxii, pp. 270—308) may be consulted with advantage.

† In these elementary explanations, it is unnecessary to enter into any discussion of the effects caused by the occurrence of singularities in the curve.

as negative. The usual convention under which one of the directions is selected as the positive direction depends upon the conception that the curve is the boundary, partial or complete, of some area; under it, that direction is taken to be *positive* which is such that the bounded area lies to the left of the direction of description. It is easy to see that the same direction is taken to be positive under an equivalent convention which makes it related to the normal drawn outwards from the bounded area in the same way as the positive direction of the axis of y is to the positive direction of the axis of x in plane coordinate geometry.

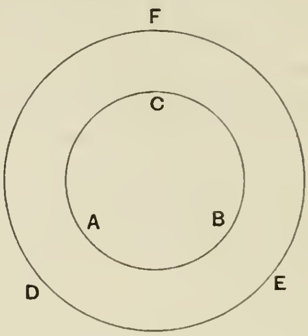

Fig. 1.

Thus in the figure (fig. 1), the positive direction of description of the outer curve for the area included by it is DEF; the positive direction of description of the inner curve for the area without it (say, the area excluded by it) is ACB; and for the area between the curves the positive direction of description of the boundary, which consists of two parts, is DEF, ACB.

3. Since the position of a point in a plane can be determined by means of polar coordinates, it is convenient in the discussion of complex variables to introduce two quantities corresponding to polar coordinates.

In the case of the variable z, one of these quantities is $(x^2+y^2)^{\frac{1}{2}}$, the positive sign being always associated with it; it is called the *modulus** of the variable and it is denoted, sometimes by mod. z, sometimes by $|z|$.

The other is θ, the angular coordinate of the point z; it is called the *argument* (and, less frequently, the *amplitude*) of the variable. It is measured in the trigonometrically positive sense, and is determined by the equations

$$x=|z|\cos\theta, \quad y=|z|\sin\theta,$$

so that $z=|z|e^{\theta i}$. The actual value depends upon the way in which the variable has acquired its value; when variation of the argument is considered, its initial value is usually taken to lie between 0 and 2π or, less frequently, between $-\pi$ and $+\pi$.

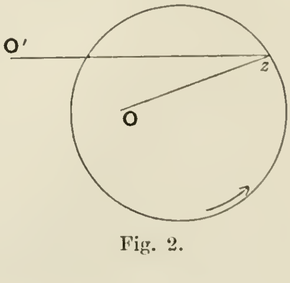

Fig. 2.

As z varies in position, the values of $|z|$ and θ vary. When z has completed a positive description of a closed curve, the modulus of z returns to the initial value whether the origin

* *Der absolute Betrag* is often used by German writers.

be without, within or on the curve. The argument of z resumes its initial value, if the origin O' (fig. 2) be without the curve; but, if the origin O be within the curve, the value of the argument is increased by 2π when z returns to its initial position.

If the origin be on the curve, the argument of z undergoes an abrupt change by π as z passes through the origin; and the change is an increase or a decrease according as the variable approaches its limiting position on the curve from without or from within. No choice need be made between these alternatives; for care is always exercised to choose curves which do not introduce this element of doubt.

4. Representation on a plane is obviously more effective for points at a finite distance from the origin than for points at a very great distance.

One method of meeting the difficulty of representing great values is to introduce a new variable z' given by $z'z = 1$: the part of the new plane for z' which lies quite near the origin corresponds to the part of the old plane for z which is very distant. The two planes combined give a complete representation of variation of the complex variable.

Another method, in many ways more advantageous, is as follows. Draw a sphere of unit diameter, touching the z-plane at the origin O (fig. 3) on the under side: join a point z in the plane to O', the other extremity of the diameter through O, by a straight line cutting the sphere in Z. Then Z is a unique representative of z, that is, a single point on the sphere corresponds to a single point on the plane: and therefore the variable can be represented on the surface of the sphere. With this mode of

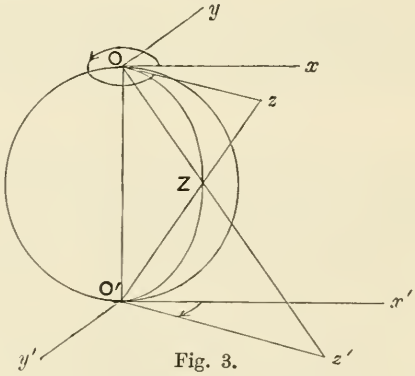

Fig. 3.

representation, O' evidently corresponds to an infinite value of z: and points at a very great distance in the z-plane are represented by points in the immediate vicinity of O' on the sphere. The sphere thus has the advantage of putting in evidence a part of the surface on which the variations of great values of z can be traced*, and of exhibiting the uniqueness of $z=\infty$ as a value of the variable, a fact that is obscured in the representation on a plane.

The former method of representation can be deduced by means of the sphere. At O' draw a plane touching the sphere: and let the straight line OZ cut this plane in z'. Then z' is a point uniquely determined by Z and therefore uniquely determined by z. In this new z'-plane take axes parallel to the axes in the z-plane.

The points z and z' move in the same direction in space round OO' as an axis. If we make the upper side of the z-plane correspond to the lower side of the z'-plane, and take the usual positive directions in the planes, being the positive trigonometrical directions for a spectator looking at the surface of the plane in which the description takes place, we have these directions indicated by the arrows at O and at O' respectively, so that the senses of positive rotations in the two planes are opposite in space. Now it is evident from the geometry that Oz and $O'z'$ are parallel; hence, if θ be the argument of the point z and θ' that of the point z' so that θ is the angle from Ox to Oz and θ' the angle from $O'x'$ to $O'z'$, we have

$$\theta+\theta'=2\pi.$$

Further, by similar triangles, $$\frac{Oz}{OO'}=\frac{OO'}{O'z'},$$

that is, $$Oz\,.\,O'z'=OO'^2=1.$$

Now, if z and z' be the variables, we have

$$z=Oz\,.\,e^{\theta i},\quad z'=O'z'\,.\,e^{\theta' i},$$

so that $$zz'=Oz\,.\,O'z'\,.\,e^{(\theta+\theta')\,i}$$
$$=1,$$

which is the former relation.

The z'-plane can therefore be taken as the lower side of a plane touching the sphere at O' when the z-plane is the upper side of a plane touching it at O. The part of the z-plane at a very great distance is represented on the sphere by the part in the immediate vicinity of O': and this part of the sphere is represented on the z'-plane by its portion in the immediate vicinity of O', which therefore is a space wherein the variations of infinitely great values of z can be traced.

But it need hardly be pointed out that any special method of representation of the variable is not essential to the development of the theory of functions; and, in particular, the foregoing representation of the variable, when it has very great values, merely provides a convenient method of dealing with quantities that tend to become infinite in magnitude.

* This sphere is sometimes called Neumann's sphere; it is used by him for the representation of the complex variable throughout his treatise *Vorlesungen über Riemann's Theorie der Abel'schen Integrale* (Leipzig, Teubner, 2nd edition, 1884).

5. The simplest propositions relating to complex variables will be assumed known. Among these are, the geometrical interpretation of operations such as addition, multiplication, root-extraction; some of the relations of complex variables occurring as roots of algebraical equations with real coefficients; the elementary properties of functions of complex variables which are algebraical and integral, or exponential, or circular functions; and simple tests of convergence of infinite series and of infinite products*.

6. All ordinary operations effected on a complex variable lead, as already remarked, to other complex variables; and any definite quantity, thus obtained by operations on z, is necessarily a function of z.

But if a complex variable w be given as a complex function of x and y without any indication of its source, the question as to whether w is or is not a function of z requires a consideration of the general idea of functionality.

It is convenient to postulate $u + iv$ as a form of the complex variable w, where u and v are real. Since w is initially unrestricted in variation, we may so far regard the quantities u and v as independent and therefore as any functions of x and y, the elements involved in z. But more explicit expressions for these functions are neither assigned nor supposed.

The earliest occurrence of the idea of functionality is in connection with functions of real variables; and then it is coextensive with the idea of dependence. Thus, if the value of X depends on that of x and on no other variable magnitude, it is customary to regard X as a function of x; and there is usually an implication that X is derived from x by some series of operations†.

A detailed knowledge of z determines x and y uniquely; hence the values of u and v may be considered as known and therefore also w. Thus the value of w is dependent on that of z, and is independent of the values of variables unconnected with z; therefore, with the foregoing view of functionality, w is a function of z.

It is, however, equally consistent with that view to regard w as a complex function of the two independent elements from which z is constituted; and we are then led merely to the consideration of functions of two real independent variables with (possibly) imaginary coefficients.

* These and other introductory parts of the subject are discussed in Chrystal's *Text-book of Algebra* and in Hobson's *Treatise on Plane Trigonometry*.

They are also discussed at some length in the recently published translation, by G. L. Cathcart, of Harnack's *Elements of the differential and integral calculus* (Williams and Norgate, 1891), the second and the fourth books of which contain developments that should be consulted in special relation with the first few chapters of the present treatise.

These books, together with Neumann's treatise cited in the note on p. 5, will hereafter be cited by the names of their respective authors.

† It is not important for the present purpose to keep in view such mathematical expressions as have intelligible meanings only when the independent variable is confined within limits.

Both of these aspects of the dependence of w on z require that z be regarded as a composite quantity involving two independent elements which can be considered separately. Our purpose, however, is to regard z as the most general form of algebraical variable and therefore as an irresoluble entity; so that, as this preliminary requirement in regard to z is unsatisfied, neither of the aspects can be adopted.

7. Suppose that w is regarded as a function of z in the sense that it can be constructed by definite operations on z regarded as an irresoluble magnitude, the quantities u and v arising subsequently to these operations by the separation of the real and the imaginary parts when z is replaced by $x+iy$. It is thereby assumed that one series of operations is sufficient for the simultaneous construction of u and v, instead of one series for u and another series for v as in the general case of a complex function in § 6. If this assumption be justified by the same forms resulting from the two different methods of construction, it follows that the two series of operations, which lead in the general case to u and to v, must be equivalent to the single series and must therefore be connected by conditions; that is, u and v as functions of x and y must have their functional forms related.

We thus take

$$u+iv=w=f(z)=f(x+iy)$$

without any specification of the form of f. When this postulated equation is valid, we have

$$\frac{\partial w}{\partial x}=\frac{dw}{dz}\frac{\partial z}{\partial x}=f'(z)=\frac{dw}{dz},$$

$$\frac{\partial w}{\partial y}=\frac{dw}{dz}\frac{\partial z}{\partial y}=if'(z)=i\frac{dw}{dz},$$

and therefore

$$\frac{\partial w}{\partial x}=\frac{1}{i}\frac{\partial w}{\partial y}=\frac{dw}{dz} \qquad \text{............................ (1),}$$

equations from which the functional form has disappeared. Inserting the value of w, we have

$$i\frac{\partial}{\partial x}(u+iv)=\frac{\partial}{\partial y}(u+iv),$$

whence, after equating real and imaginary parts,

$$-\frac{\partial v}{\partial x}=\frac{\partial u}{\partial y}, \quad \frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \qquad \text{............................ (2).}$$

These are necessary relations between the functional forms of u and v.

These relations are easily seen to be sufficient to ensure the required functionality. For, on taking $w=u+iv$, the equations (2) at once lead to

$$\frac{\partial w}{\partial x}=\frac{1}{i}\frac{\partial w}{\partial y},$$

that is, to

$$\frac{\partial w}{\partial x}+i\frac{\partial w}{\partial y}=0,$$

a linear partial differential equation of the first order. To obtain the most general solution, we form a subsidiary system

$$\frac{dx}{1}=\frac{dy}{i}=\frac{dw}{0}.$$

It possesses the integrals w, $x+iy$; and then from the known theory of such equations we infer that every quantity w satisfying the equation can be expressed as a function of $x+iy$, i.e., of z. The conditions (2) are thus proved to be sufficient, as well as necessary.

8. The preceding determination of the necessary and sufficient conditions of functional dependence is based upon the existence of a functional form; and yet that form is not essential, for, as already remarked, it disappears from the equations of condition. Now the postulation of such a form is equivalent to an assumption that the function can be numerically calculated for each particular value of the independent variable, though the immediate expression of the assumption has disappeared in the present case. Experience of functions of real variables shews that it is often more convenient to use their properties than to possess their numerical values. This experience is confirmed by what has preceded. The essential conditions of functional dependence are the equations (1), and they express a property of the function w, viz., that the value of the ratio $\frac{dw}{dz}$ is the same as that of $\frac{\partial w}{\partial x}$, or, in other words, it is independent of the manner in which dz ultimately vanishes by the approach of the point $z+dz$ to coincidence with the point z. We are thus led to an entirely different definition of functionality, viz.:

A complex quantity w is a function of another complex quantity z, when they change together in such a manner that the value of $\frac{dw}{dz}$ is independent of the value of the differential element dz.

This is Riemann's definition*; we proceed to consider its significance.

We have

$$\begin{aligned}\frac{dw}{dz}&=\frac{du+idv}{dx+idy}\\&=\left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right)\frac{dx}{dx+idy}+\left(\frac{\partial u}{\partial y}+i\frac{\partial v}{\partial y}\right)\frac{dy}{dx+idy}.\end{aligned}$$

Let ϕ be the argument of dz; then

$$\frac{dx}{dx+idy}=\frac{\cos\phi}{\cos\phi+i\sin\phi}=\tfrac{1}{2}(1+e^{-2\phi i}),$$

$$\frac{idy}{dx+idy}\qquad\qquad=\tfrac{1}{2}(1-e^{-2\phi i}),$$

* *Ges. Werke*, p. 5; a modified definition is adopted by him, ib., p. 81.

and therefore

$$\frac{dw}{dz} = \tfrac{1}{2}\left\{\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} - i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}\right\} + \tfrac{1}{2}e^{-2\phi i}\left\{\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y}\right\}.$$

Since $\frac{dw}{dz}$ is to be independent of the value of the differential element dz, it must be independent of ϕ the argument of dz; hence the coefficient of $\epsilon^{-2\phi i}$ in the preceding expression must vanish, which can happen only if

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \quad \text{.........................} \quad (2).$$

These are necessary conditions; they are evidently also sufficient to make $\frac{dw}{dz}$ independent of the value of dz and therefore, by the definition, to secure that w is a function of z.

By means of the conditions (2), we have

$$\frac{dw}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial w}{\partial x},$$

and also
$$\frac{dw}{dz} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = \frac{1}{i}\frac{\partial w}{\partial y},$$

agreeing with the former equations (1) and immediately derivable from the present definition by noticing that dx and idy are possible forms of dz.

It should be remarked that equations (2) are the conditions necessary and sufficient to ensure that each of the expressions

$$udx - vdy \quad \text{and} \quad vdx + udy$$

is a perfect differential—a result of great importance in many investigations in the region of mathematical physics.

When the conditions (2) are expressed, as is sometimes convenient, in terms of derivatives with regard to the modulus of z, say r, and the argument of z, say θ, they take the new forms

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta} \quad \text{.......................} \quad (2)'.$$

We have so far assumed that the function has a differential coefficient—an assumption justified in the case of functions which ordinarily occur. But functions do occur which have different values in different regions of the z-plane, and there is then a difficulty in regard to the quantity $\frac{dw}{dz}$ at the boundaries of such regions; and functions do occur which, though themselves definite in value in a given region, do not possess a differential coefficient at all points in that region. The consideration of such functions is not of substantial importance at present: it belongs to another part of our subject.

It must not be inferred that, because $\frac{dw}{dz}$ is independent of the direction in which dz vanishes when w is a function of z, therefore $\frac{dw}{dz}$ has only one value. The number of its values is dependent on the number of values of w: no one of its values is dependent on dz.

A quantity, defined as a function by Riemann on the basis of this property, is sometimes* called an analytical function; but it seems preferable to reserve the term analytical in order that it may be associated hereafter (§ 34) with an additional quality of the functions.

9. The geometrical interpretation of complex variability leads to important results when applied to two variables w and z which are functionally related.

Let P and p be two points in different planes, or in different parts of the same plane, representing w and z respectively; and suppose that P and p are at a finite distance from the points (if any) which cause discontinuity in the relationship. Let q and r be any two other points, $z + dz$ and $z + \delta z$, in the immediate vicinity of p; and let Q and R be the corresponding points, $w + dw$ and $w + \delta w$, in the immediate vicinity of P. Then

$$dw = \frac{dw}{dz} dz, \quad \delta w = \frac{dw}{dz} \delta z,$$

the value of $\frac{dw}{dz}$ being the same for both equations, because, as w is a function of z, that quantity is independent of the differential element of z. Hence

$$\frac{\delta w}{dw} = \frac{\delta z}{dz},$$

on the ground that $\frac{dw}{dz}$ is neither zero nor infinite at z, which is assumed not to be a point of discontinuity in the relationship. Expressing all the differential elements in terms of their moduli and arguments, let

$$dz = \sigma e^{\theta i}, \quad dw = \eta e^{\phi i},$$

$$\delta z = \sigma' e^{\theta' i}, \quad \delta w = \eta' e^{\phi' i},$$

and let these values be substituted in the foregoing relation; then

$$\frac{\eta'}{\eta} = \frac{\sigma'}{\sigma},$$

$$\phi' - \phi = \theta' - \theta.$$

Hence the triangles QPR and qpr are similar to one another, though not necessarily similarly situated. Moreover the directions originally chosen for pq and pr are quite arbitrary. Thus it appears that *a functional relation*

* Harnack, § 84.

between two complex variables establishes the similarity of the corresponding infinitesimal elements of those parts of two planes which are in the immediate vicinity of the points representing the two variables.

The magnification of the w-plane relative to the z-plane at the corresponding points P and p is the ratio of two corresponding infinitesimal lengths, say of QP and qp. This is the modulus of $\frac{dw}{dz}$; if it be denoted by m, we have

$$m^2 = \left|\frac{dw}{dz}\right|^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2$$
$$= \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}.$$

Evidently the quantity m, in general, depends on the variables and therefore it changes from one point to another; hence the functional relation between w and z does not, in general, establish similarity of finite parts of the two planes corresponding to one another through the relation.

It is easy to prove that $w = az + b$, where a and b are constants, is the only relation which establishes similarity of finite parts; and that, with this relation, a must be a real constant in order that the similar parts may be similarly situated.

If $u + iv = w = \phi(z)$, the curves $u = \text{constant}$ and $v = \text{constant}$ cut at right angles; a special case of the proposition that, if $\phi(x + iy) = u + ve^{\lambda i}$, where λ is a real constant and u, v are real, then $u = \text{constant}$ and $v = \text{constant}$ cut at an angle λ.

The process, which establishes the infinitesimal similarity of two planes by means of a functional relation between the variables of the planes, may be called the *conformal representation* of one plane on another*.

The discussion of detailed questions connected with the conformal representation is deferred until the later part of the treatise, principally in order to group all such investigations together; but the first of the two chapters, devoted to it, need not be deferred so late and an immediate reading of some portion of it will tend to simplify many of the explanations relative to functional relations as they occur in the early chapters of this treatise.

10. The analytical conditions of functionality, under either of the adopted definitions, are the equations (2). From them it at once follows that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$
$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0;$$

* By Gauss (*Ges. Werke*, t. iv, p. 262) it was styled *conforme Abbildung*, the name universally adopted by German mathematicians. The French title is *représentation conforme*; and, in England, Cayley has used *orthomorphosis* or *orthomorphic transformation*.

so that neither the real nor the imaginary part of a complex function can be arbitrarily assumed.

If either part be given, the other can be deduced; for example, let u be given; then we have

$$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$$
$$= -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy,$$

and therefore, except as to an additive constant, the value of v is

$$\int \left(-\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy \right).$$

In particular, when u is an integral function, it can be resolved into the sum of homogeneous parts

$$u_1 + u_2 + u_3 + \ldots;$$

and then, again except as to an additive constant, v can similarly be expressed in the form

$$v_1 + v_2 + v_3 + \ldots.$$

It is easy to prove that

$$mv_m = y \frac{\partial u_m}{\partial x} - x \frac{\partial u_m}{\partial y},$$

by means of which the value of v can be obtained.

The case, when u is homogeneous of zero dimensions, presents no difficulty; for we then have

$$u = b + a\theta,$$
$$v = c - a \log r,$$

where a, b, c are constants.

Similarly for other special cases; and, in the most general case, only a quadrature is necessary.

The tests of functional dependence of one complex on another are of effective importance in the case when the supposed dependent complex arises in the form $u + iv$, where u and v are real; the tests are, of course, superfluous when w is explicitly given as a function of z. When w does arise in the form $u + iv$ and satisfies the conditions of functionality, perhaps the simplest method (other than by inspection) of obtaining the explicit expression in terms of z is to substitute $z - iy$ for x in $u + iv$; the simplified result must be a function of z alone.

11. Conversely, when w is explicitly given as a function of z and it is divided into its real and its imaginary parts, these parts individually satisfy the foregoing conditions attaching to u and v. Thus $\log r$, where r is the distance of a point z from a point a, is the real part of $\log (z - a)$ and therefore satisfies the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Again, ϕ, the angular coordinate of z relative to the same point a, is the real part of $-i\log(z-a)$ and satisfies the same equation: the more usual form of ϕ being $\tan^{-1}\{(y-y_0)/(x-x_0)\}$, where $a=x_0+iy_0$. Again, if a point z be distant r from a and r' from b, then $\log(r/r')$, being the real part of $\log\{(z-a)/(z-b)\}$, is a solution of the same equation.

The following example, the result of which will be useful subsequently*, uses the property that the value of the derivative is independent of the differential element.

Consider a function
$$u+iv=w=\log\frac{z-c}{z-c'},$$
where c' is the inverse of c with regard to a circle centre the origin O and radius R. Then
$$u=\log\left|\frac{z-c}{z-c'}\right|,$$
and the curves $u=$ constant are circles. Let (fig. 4) $Oc=r$, $xOc=a$ so that $c=re^{ai}$, $c'=\frac{R^2}{r}e^{ai}$; then if
$$\left|\frac{z-c}{z-c'}\right|=\frac{r}{R}\lambda,$$

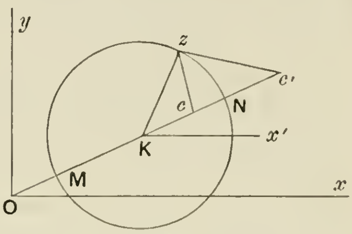

Fig. 4.

the values of λ for points in the interior of the circle of radius R vary from zero, when circle $u=$constant is the point c, to unity, when the circle $u=$constant is the circle of radius R. Let the point $K\,(=\theta e^{ai})$ be the centre of the circle determined by a value of λ, and let its radius be $\rho\,(=\frac{1}{2}MN)$. Then since
$$\frac{cM}{c'M}=\frac{r}{R}\lambda=\frac{cN}{c'N},$$
we have
$$\frac{r+\rho-\theta}{\frac{R^2}{r}+\rho-\theta}=\frac{r}{R}\lambda=\frac{\theta+\rho-r}{\frac{R^2}{r}-\theta-\rho},$$
whence
$$\rho=\frac{\lambda R\,(R^2-r^2)}{R^2-r^2\lambda^2},\quad \theta=\frac{R^2r\,(1-\lambda^2)}{R^2-r^2\lambda^2}.$$
Now if dn be an element of the normal drawn inwards at z to the circle NzM, we have
$$dz=dx+idy=-dn\,.\cos\psi-idn\,.\sin\psi$$
$$=-e^{i\psi}dn,$$
where $\psi\,(=zKx')$ is the argument of z relative to the centre of the circle. Hence, since
$$\frac{dw}{dz}=\frac{1}{z-c}-\frac{1}{z-c'},$$
we have
$$\frac{du}{dn}+i\frac{dv}{dn}=\frac{dw}{dn}=\left(\frac{1}{z-c'}-\frac{1}{z-c}\right)e^{\psi i}.$$
But
$$z=\theta e^{ai}+\rho e^{\psi i},$$
so that
$$z-c=\frac{\lambda\,(R^2-r^2)}{R^2-r^2\lambda^2}\,(Re^{\psi i}-\lambda re^{ai}),$$
and
$$z-c'=\frac{R}{r}\,\frac{R^2-r^2}{R^2-r^2\lambda^2}\,(\lambda re^{\psi i}-Re^{ai});$$
and therefore
$$\frac{du}{dn}+i\frac{dv}{dn}=\frac{R^2-r^2\lambda^2}{R^2-r^2}\,e^{\psi i}\left\{\frac{r}{R}\,\frac{1}{\lambda re^{\psi i}-Re^{ai}}-\frac{1}{\lambda}\,\frac{1}{Re^{\psi i}-\lambda re^{ai}}\right\}.$$

* In § 217, in connection with the investigations of Schwarz, by whom the result is stated, *Ges. Werke*, t. ii, p. 183.

Hence, equating the real parts, it follows that

$$\frac{du}{dn} = -\frac{(R^2 - r^2\lambda^2)^2}{\lambda R (R^2 - r^2) \{R^2 - 2Rr\lambda \cos(\psi - a) + \lambda^2 r^2\}},$$

the differential element dn being drawn inwards from the circumference of the circle.

The application of this method is evidently effective when the curves u=constant, arising from a functional expression of w in terms of z, are a family of non-intersecting algebraical curves.

12. As the tests which are sufficient and necessary to ensure that a complex quantity is a function of z have been given, we shall assume that all complex quantities dealt with are functions of the complex variable (§§ 6, 7). Their characteristic properties, their classification, and some of the simpler applications will be considered in the succeeding chapters.

Some initial definitions and explanations will now be given.

(i). It has been assumed that the function considered has a differential coefficient, that is, that the rate of variation of the function in any direction is independent of that direction by being independent of the mode of change of the variable. We have already decided (§ 8) not to use the term analytical for such a function. It is often called *monogenic*, when it is necessary to assign a specific name; but for the most part we shall omit the name, the property being tacitly assumed*.

We can at once prove from the definition that, when the derivative $w_1 \left(= \frac{dw}{dz}\right)$ exists, it is itself a function. For $w_1 = \frac{\partial w}{\partial x} = \frac{1}{i}\frac{\partial w}{\partial y}$ are equations which, when satisfied, ensure the existence of w_1; hence

$$\begin{aligned} \frac{1}{i}\frac{\partial w_1}{\partial y} &= \frac{1}{i}\frac{\partial}{\partial y}\left(\frac{\partial w}{\partial x}\right) \\ &= \frac{\partial}{\partial x}\left(\frac{1}{i}\frac{\partial w}{\partial y}\right) \\ &= \frac{\partial w_1}{\partial x}, \end{aligned}$$

shewing, as in § 8, that the derivative $\frac{dw_1}{dz}$ is independent of the direction in which dz vanishes. Hence w_1 is a function of z.

Similarly for all the derivatives in succession.

(ii). Since the functional dependence of a complex is ensured only if the value of the derivative of that complex be independent of the manner in which the point $z + dz$ approaches to coincidence with z, a question naturally

* This is in fact done by Riemann, who calls such a dependent complex simply a *function*. Weierstrass, however, has proved (§ 85) that the idea of a monogenic function of a complex variable and the idea of dependence expressible by arithmetical operations are not coextensive. The definition is thus necessary; but the practice indicated in the text will be adopted, as non-monogenic functions will be of relatively rare occurrence.

suggests itself as to the effect on the character of the function that may be caused by the manner in which the variable itself has come to the value of z.

If a function have only one value for each given value of the variable, whatever be the manner in which the variable has come to that value, the function is called *uniform**. Hence two different paths from a point a to a point z give at z the same value for any uniform function; and a closed curve, beginning at any point and completely described by the z-variable, will lead to the initial value of w, the corresponding w-curve being closed, if z have passed through no point which makes w infinite.

The simplest class of uniform functions is constituted by algebraical rational functions.

(iii). If a function have more than one value for any given value of the variable, or if its value can be changed by modifying the path in which the variable reaches that given value, the function is called *multiform*†. Characteristics of curves, which are graphs of multiform functions corresponding to a z-curve, will hereafter be discussed.

One of the simplest classes of multiform functions is constituted by algebraical irrational functions.

(iv). A multiform function has a number of different values for the same value of z, and these values vary with z: the aggregate of the variations of any one of the values is called a *branch* of the function. Although the function is multiform for unrestricted variation of the variable, it often happens that a branch is uniform when the variable is restricted to particular regions in the plane.

(v). A point in the plane, at which two or more branches of a multiform function assume the same value, is called a *branch-point*‡ of the function; the relations of the branches in the immediate vicinity of a branch-point will hereafter be discussed.

(vi). A function which is monogenic, uniform and continuous over any part of the z-plane is called *holomorphic*§ over that part of the plane. When a function is called holomorphic without any limitation, the usual implication is that the character is preserved over the whole of the plane which is not at infinity.

The simplest example of a holomorphic function is a rational integral algebraical polynomial.

* Also *monodromic*, or *monotropic*; with German writers the title is *eindeutig*, occasionally, *einändrig*.

† Also *polytropic*; with German writers the title is *mehrdeutig*.

‡ Also *critical point*, which, however, is sometimes used to include all special points of a function; with German writers the title is *Verzweigungspunkt*, and sometimes *Windungspunkt*. French writers use *point de ramification*, and Italians *punto di giramento* and *punto di diramazione*.

§ Also *synectic*.

(vii). A *root* (or a *zero*) of a function is a value of the variable for which the function vanishes.

The simplest case of occurrence of roots is in a rational integral algebraical function, various theorems relating to which (e.g., the number of roots included within a given contour) will be found in treatises on the theory of equations.

(viii). The *infinities* of a function are the points at which the value of the function is infinite. Among them, the simplest are the *poles** of the function, a pole being an infinity such that in its immediate vicinity the reciprocal of the function is holomorphic.

Infinities other than poles (and also the poles) are called the *singular points* of the function: their classification must be deferred until after the discussion of properties of functions.

(ix). A function which is monogenic, uniform and, except at poles, continuous, is called a *meromorphic* function†. The simplest example is a rational algebraical fraction.

13. The following functions give illustrations of some of the preceding definitions.

(*a*) In the case of a meromorphic function

$$w = \frac{F(z)}{f(z)},$$

where F and f are rational algebraical functions without a common factor, the roots are the roots of $F(z)$ and the poles are the roots of $f(z)$. Moreover, according as the degree of F is greater or is less than that of f, $z=\infty$ is a pole or a zero of w.

(*b*) If w be a polynomial of order n, then each simple root of w is a branch-point and a zero of $w^{\frac{1}{m}}$, where m is a positive integer; $z=\infty$ is a pole of w; and $z=\infty$ is a pole but not a branch-point or is an infinity (though not a pole) and a branch-point of $w^{\frac{1}{2}}$ according as n is even or odd.

(*c*) In the case of the function

$$w = \frac{1}{\operatorname{sn}\frac{1}{z}}$$

(the notation being that of Jacobian elliptic functions), the zeros are given by

$$\frac{1}{z} = iK' + 2mK + 2m'iK',$$

for all positive and negative integral values of m and of m'. If we take

$$\frac{1}{z} = iK' + 2mK + 2m'iK' + \zeta,$$

* Also *polar discontinuities*; also (§ 32) *accidental singularities*.

† Sometimes *regular*, but this term will be reserved for the description of another property of functions.

where ζ may be restricted to values that are not large, then

$$w = (-1)^m k \operatorname{sn} \zeta,$$

so that, in the neighbourhood of a zero, w behaves like a holomorphic function. There is evidently a doubly-infinite system of zeros: they are distinct from one another except at the origin, where an infinite number practically coincide.

The infinities of w are given by

$$\frac{1}{z} = 2nK + 2n'iK',$$

for all positive and negative integral values of n and of n'. If we take

$$\frac{1}{z} = 2nK + 2n'iK' + \zeta,$$

then

$$\frac{1}{w} = (-1)^n \operatorname{sn} \zeta,$$

so that, in the immediate vicinity of $\zeta = 0$, $\frac{1}{w}$ is a holomorphic function. Hence $\zeta = 0$ is a pole of w. There is thus evidently a doubly-infinite system of poles; they are distinct from one another except at the origin, where an infinite number practically coincide. But the origin is not a pole; the function, in fact, is there not determinate, for it has an infinite number of zeros and an infinite number of infinities, and the variations of value are not necessarily exhausted.

For the function $\dfrac{1}{\operatorname{sn}\frac{1}{z}}$, the origin is a point which will hereafter be called an *essential singularity*.

CHAPTER II.

Integration of Uniform Functions.

14. The definition of an integral, that is adopted when the variables are complex, is the natural generalisation of that definition for real variables in which it is regarded as the limit of the sum of an infinite number of infinitesimally small terms. It is as follows:—

Let a and z be any two points in the plane; and let them be connected by a curve of specified form, which is to be the path of variation of the independent variable. Let $f(z)$ denote any function of z; if any infinity of $f(z)$ lie in the vicinity of the curve, the line of the curve will be chosen so as not to pass through that infinity. On the curve, let any number of points $z_1, z_2, \ldots, z_n$ in succession be taken between a and z; then, if the sum

$$(z_1 - a) f(a) + (z_2 - z_1) f(z_1) + \ldots + (z - z_n) f(z_n)$$

have a limit, when n is indefinitely increased so that the infinitely numerous points are in indefinitely close succession along the whole of the curve from a to z, that limit is called the integral of $f(z)$ between a and z. It is denoted, as in the case of real variables, by

$$\int_a^z f(z)\, dz.$$

The limit, as the value of the integral, is associated with a particular curve: in order that the integral may have a definite value, the curve (called the *path of integration*) must, in the first instance, be specified*. The integral of any function whatever may not be assumed to depend in general only upon the limits.

15. Some inferences can be made from the definition.

(I.) *The integral along any path from a to z passing through a point ζ is the sum of the integrals from a to ζ and from ζ to z along the same path.*

* This specification is tacitly supplied when the variables are real: the variable point moves along the axis of x.

Analytically, this is expressed by the equation

$$\int_a^z f(z)\,dz = \int_a^\zeta f(z)\,dz + \int_\zeta^z f(z)\,dz,$$

the paths on the right-hand side combining to form the path on the left.

(II.) *When the path is described in the reverse direction, the sign of the integral is changed*: that is,

$$\int_a^z f(z)\,dz = -\int_z^a f(z)\,dz,$$

the curve of variation between a and z being the same.

(III.) *The integral of the sum of a finite number of terms is equal to the sum of the integrals of the separate terms, the path of integration being the same for all.*

(IV.) *If a function $f(z)$ be finite and continuous along any finite line between two points a and z, the integral $\int_a^z f(z)\,dz$ is finite.*

Let I denote the integral, so that we have I as the limit of

$$\sum_{r=0}^{n} (z_{r+1} - z_r) f(z_r):$$

hence

$$|I| = \text{limit of } \left| \sum_{r=0}^{n} (z_{r+1} - z_r) f(z_r) \right|$$
$$< \ldots\ldots\ldots\ldots\ldots \Sigma |z_{r+1} - z_r|\,|f(z_r)|.$$

Because $f(z)$ is finite and continuous, its modulus is finite and therefore must have a superior limit, say M, for points on the line. Thus

$$|f(z_r)| \leqslant M$$

so that

$$|I| < \text{limit of } M\Sigma |z_{r+1} - z_r|$$
$$< MS,$$

where S is the finite length of the path of integration. Hence the modulus of the integral is finite; the integral itself is therefore finite.

No limitation has been assigned to the path, except finiteness in length; the proposition is still true when the curve is a closed curve of finite length.

Hermite and Darboux have given an expression for the integral which leads to the same result. We have as above

$$I = \int_a^z f(z)\,dz,$$

and

$$|I| < \int_a^z |f(z)|\,|dz|$$
$$= \theta \int_a^z |f(z)|\,|dz|,$$

where θ is a real positive quantity less than unity. The last integral involves only real variables; hence* for some point ξ lying between a and z, we have

$$\int_a^z |f(z)|\,|dz| = |f(\xi)| \int_a^z |dz|$$

$$= S|f(\xi)|,$$

so that
$$|I| = \theta S\,|f(\xi)|.$$

It therefore follows that there is some argument α such that, if $\lambda = \theta e^{i\alpha}$,

$$I = \lambda S f(\xi).$$

This form proves the finiteness of the integral; and the result is the generalisation† to complex variables of the theorem just quoted for real variables.

(V.) *When a function is expressed in the form of a series, which converges uniformly and unconditionally, the integral of the function along any path of finite length is the sum of the integrals of the terms of the series along the same path, provided that path lies within the circle of convergence of the series:* —a result, which is an extension of (III.) above.

Let $u_0 + u_1 + u_2 + \ldots$ be the converging series; take

$$f(z) = u_0 + u_1 + \ldots + u_n + R,$$

where $|R|$ can be made infinitesimally small with indefinite increase of n, because the series converges uniformly and unconditionally. Then by (III.), or immediately from the definition of the integral, we have

$$\int_a^z f(z)\,dz = \int_a^z u_0 dz + \int_a^z u_1 dz + \ldots + \int_a^z u_n dz + \int_a^z R dz,$$

the path of integration being the same for all the integrals. Hence, if

$$\Theta = \int_a^z f(z)\,dz - \sum_{m=0}^{n} \int_a^z u_m dz,$$

we have
$$\Theta = \int_a^z R dz.$$

Let R' be the greatest value of $|R|$ for points in the path of integration from a to z, and let S be the length of this path, so that S is finite; then, by (IV.),

$$|\Theta| < SR'.$$

Now S is finite; and, as n is increased indefinitely, the quantity R' tends towards zero as a limit for all points within the circle of convergence and therefore for all points on the path of integration provided that the path lie within the circle of convergence. When this proviso is satisfied, $|\Theta|$ becomes infinitesimally small and therefore also Θ becomes infinitesimally small with

* Todhunter's *Integral Calculus* (4th ed.), § 40; Williamson's *Integral Calculus*, (6th ed.), § 96.

† Hermite, *Cours à la faculté des sciences de Paris* (4ème éd., 1891), p. 59, where the reference to Darboux is given.

indefinite increase of n. Hence, under the conditions stated in the enunciation, we have

$$\int_a^z f(z)\,dz - \sum_{m=0}^{\infty} \int_a^z u_m dz = 0,$$

which proves the proposition.

16. The following lemma* is of fundamental importance.

Let any region of the plane, on which the z-variable is represented, be bounded by one or more simple† curves which do not meet one another: each curve that lies entirely in the finite part of the plane will be considered to be a closed curve.

If p and q be any two functions of x and y, which, for all points within the region or along its boundary, are uniform, finite and continuous, then the integral

$$\iint \left(\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}\right) dxdy,$$

extended over the whole area of the region, is equal to the integral

$$\int (pdx + qdy),$$

taken in a positive direction round the whole boundary of the region.

(As the proof of the proposition does not depend on any special form of region, we shall take the area to be (fig. 5) that which is included by the curve $Q_1P_1Q_3'P_3'$ and excluded by $P_2'Q_2'P_3Q_3$ and excluded by $P_1'P_2$. The positive directions of description of the curves are indicated by the arrows; and for integration in the area the positive directions are those of increasing x and increasing y.)

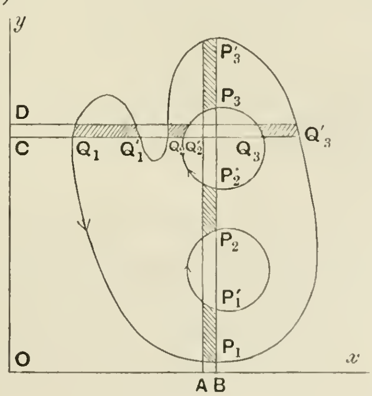

Fig. 5.

* It is proved by Riemann, *Ges. Werke*, p. 12, and is made by him (as also by Cauchy) the basis of certain theorems relating to functions of complex variables.

† A curve is called simple, if it have no multiple points. The aim, in constituting the boundary from such curves is to prevent the superfluous complexity that arises from duplication of area on the plane. If, in any particular case, multiple points existed, the method of meeting the difficulty would be to take each simple loop as a boundary.

First, suppose that both p and q are real. Then, integrating with regard to x, we have*

$$\iint \frac{\partial q}{\partial x}\, dxdy = \int [qdy],$$

where the brackets imply that the limits are to be introduced. When the limits are introduced along a parallel $CQ_1Q_1'\ldots$ to the axis of x, then, since $CQ_1Q_1'\ldots$ gives the direction of integration, we have

$$[qdy] = -q_1dy_1 + q_1'dy_1' - q_2dy_2 + q_2'dy_2' - q_3dy_3 + q_3'dy_3',$$

where the various differential elements are the projections on the axis of y of the various elements of the boundary at points along $CQ_1Q_1'\ldots$.

Now when integration is taken in the positive direction round the whole boundary, the part of $\int qdy$ arising from the elements of the boundary at the points on $CQ_1Q_1'\ldots$ is the foregoing sum. For at Q_3' it is $q_3'dy_3'$ because the positive element dy_3', which is equal to CD, is in the positive direction of boundary integration; at Q_3 it is $-q_3dy_3$ because the positive element dy_3, also equal to CD, is in the negative direction of boundary integration; at Q_2' it is $q_2'dy_2'$, for similar reasons; at Q_2 it is $-q_2dy_2$, for similar reasons; and so on. Hence

$$[qdy],$$

corresponding to parallels through C and D to the axis of x, is equal to the part of $\int qdy$ taken along the boundary in the positive direction for all the elements of the boundary that lie between those parallels. Then when we integrate for all the elements CD by forming $\int [qdy]$, an equivalent is given by the aggregate of all the parts of $\int qdy$ taken in the positive direction round the whole boundary; and therefore

$$\iint \frac{\partial q}{\partial x}\, dxdy = \int qdy,$$

on the suppositions stated in the enunciation.

Again, integrating with regard to y, we have

$$\begin{aligned}\iint \frac{\partial p}{\partial y}\, dxdy &= \int [pdx] \\ &= -p_1dx_1 + p_1'dx_1' - p_2dx_2 + p_2'dx_2' - p_3dx_3 + p_3'dx_3',\end{aligned}$$

when the limits are introduced along a parallel $BP_1P_1'\ldots$ to the axis of y: the various differential elements are the projections on the axis of x of the various elements of the boundary at points along $BP_1P_1'\ldots$.

It is proved, in the same way as before, that the part of $-\int pdx$ arising from the positively-described elements of the boundary at the points on $BP_1P_1'\ldots$ is the foregoing sum. At P_3' the part of $\int pdx$ is $-p_3'dx_3'$, because the positive element dx_3', which is equal to AB, is in the negative direction

* It is in this integration, and in the corresponding integration for p, that the properties of the function q are assumed: any deviation from uniformity, finiteness or continuity within the region of integration would render necessary some equation different from the one given in the text.

of boundary integration; at P_3 it is $p_3 dx_3$, because the positive element dx_3, also equal to AB, is in the positive direction of boundary integration; and so on for the other terms. Hence

$$-[pdx],$$

corresponding to parallels through A and B to the axis of y, is equal to the part of $\int pdx$ taken along the boundary in the positive direction for all the elements of the boundary that lie between those parallels. Hence integrating for all the elements AB, we have as before

$$\iint \frac{\partial p}{\partial y} dxdy = -\int pdx,$$

and therefore
$$\iint \left(\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}\right) dxdy = \int (pdx + qdy).$$

Secondly, suppose that p and q are complex. When they are resolved into real and imaginary parts, in the forms $p' + ip''$ and $q' + iq''$ respectively, then the conditions as to uniformity, finiteness and continuity, which apply to p and q, apply also to p', q', p'', q''. Hence

$$\iint \left(\frac{\partial q'}{\partial x} - \frac{\partial p'}{\partial y}\right) dxdy = \int (p'dx + q'dy),$$

and
$$\iint \left(\frac{\partial q''}{\partial x} - \frac{\partial p''}{\partial y}\right) dxdy = \int (p''dx + q''dy),$$

and therefore
$$\iint \left(\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}\right) dxdy = \int (pdx + qdy)$$

which proves the proposition.

No restriction on the properties of the functions p and q at points that lie without the region is imposed by the proposition. They may have infinities outside, they may cease to be continuous at outside points or they may have branch-points outside; but so long as they are finite and continuous everywhere inside, and in passing from one point to another always acquire at that other the same value whatever be the path of passage in the region, that is, so long as they are uniform in the region, the lemma is valid.

17. The following theorem due to Cauchy* can now be proved:—

If a function $f(z)$ be holomorphic throughout any region of the z-plane, then the integral $\int f(z)\,dz$, taken round the whole boundary of that region, is zero.

We apply the preceding result by assuming

$$p = f(z), \quad q = ip = if(z);$$

owing to the character of $f(z)$, these suppositions are consistent with the

* For an account of the gradual development of the theory and, in particular, for a statement of Cauchy's contributions to the theory (with references), see Casorati, *Teorica delle funzioni di variabili complesse*, pp. 64—90, 102—106. The general theory of functions, as developed by Briot and Bouquet in their treatise *Théorie des fonctions elliptiques*, is based upon Cauchy's method.

conditions under which the lemma is valid. Since p is a function of z, we have, at every point of the region,

$$\frac{\partial p}{\partial x} = \frac{1}{i}\frac{\partial p}{\partial y},$$

and therefore, in the present case,

$$\frac{\partial q}{\partial x} = i\frac{\partial p}{\partial x} = \frac{\partial p}{\partial y}.$$

There is no discontinuity or infinity of p or q within the region; hence

$$\iint \left(\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}\right) dxdy = 0,$$

the integral being extended over the region. Hence also

$$\int (pdx + qdy) = 0,$$

when the integral is taken round the whole boundary of the region. But

$$\begin{aligned} pdx + qdy &= pdx + ipdy \\ &= pdz \\ &= f(z)\, dz, \end{aligned}$$

and therefore $$\int f(z)\, dz = 0,$$

the integral being taken round the whole boundary of the region within which $f(z)$ is holomorphic.

It should be noted that the theorem requires no limitation on the character of $f(z)$ for points z that are not included in the region.

Some important propositions can be derived by means of the theorem, as follows.

18. *When a function $f(z)$ is holomorphic over any continuous region of the plane, the integral $\int_a^z f(z)\, dz$ is a holomorphic function of z provided the points z and a as well as the whole path of integration lie within that region.*

The general definition (§ 14) of an integral is associated with a specified path of integration. In order to prove that the integral is a holomorphic function of z, it will be necessary to prove (i) that the integral acquires the same value in whatever way the point z is attained, that is, that the value is independent of the path of integration, (ii) that it is finite, (iii) that it is continuous, and (iv) that it is monogenic.

Let two paths $a\gamma z$ and $a\beta z$ between a and z be drawn (fig. 6) in the continuous region of the plane within which $f(z)$ is holomorphic. The line $a\gamma z\beta a$ is a contour over the area of which $f(z)$ is holomorphic; and therefore $\int f(z)\, dz$ vanishes when the integral is taken along $a\gamma z\beta a$. Dividing the integral into two parts and implying by z_γ, z_β that the point z has been reached by the paths $a\gamma z$, $a\beta z$ respectively, we have

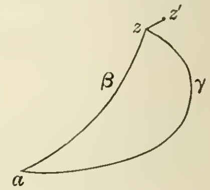

Fig. 6.

$$\int_a^{z_\gamma} f(z)\,dz + \int_{z_\beta}^{a} f(z)\,dz = 0,$$

and therefore
$$\int_a^{z_\gamma} f(z)\,dz = -\int_{z_\beta}^{a} f(z)\,dz$$
$$= \int_a^{z_\beta} f(z)\,dz.$$

Thus the value of the integral is independent of the way in which z has acquired its value; and therefore $\int_a^z f(z)\,dz$ is uniform in the region. Denote it by $F(z)$.

Secondly, $f(z)$ is finite for all points in the region and, after the result of § 17, we naturally consider only such paths between a and z as are finite in length, the distance between a and z being finite; hence (§ 15, IV.) the integral $F(z)$ is finite for all points z in the region.

Thirdly, let z' $(= z + \delta z)$ be a point infinitesimally near to z; and consider $\int_a^{z'} f(z)\,dz$. By what has just been proved, the path from a to z' can be taken $a\beta zz'$; therefore

$$\int_a^{z'} f(z)\,dz = \int_a^{z} f(z)\,dz + \int_z^{z'} f(z)\,dz$$

or
$$\int_a^{z+\delta z} f(z)dz - \int_a^{z} f(z)\,dz = \int_z^{z+\delta z} f(z)\,dz,$$

so that
$$F(z+\delta z) - F(z) = \int_z^{z+\delta z} f(z)\,dz.$$

Now at points in the infinitesimal line from z to z', the value of the continuous function $f(z)$ differs only by an infinitesimal quantity from its value at z; hence the right-hand side is

$$\{f(z) + \epsilon\}\,\delta z,$$

where $|\epsilon|$ is an infinitesimal quantity vanishing with δz. It therefore follows that

$$F(z+\delta z) - F(z)$$

is an infinitesimal quantity with a modulus of the same order of small quantities as $|\delta z|$. Hence $F(z)$ is continuous for points z in the region.

Lastly, we have

$$\frac{F(z+\delta z) - F(z)}{\delta z} = f(z) + \epsilon\,;$$

and therefore
$$\frac{F(z+\delta z) - F(z)}{\delta z}$$

has a limit when δz vanishes; and this limit, $f(z)$, is independent of the way in which δz vanishes. Hence $F(z)$ has a differential coefficient; the integral is monogenic for points z in the region.

Hence $F(z)$, which is equal to

$$\int_a^z f(z)\,dz,$$

is uniform, finite, continuous and monogenic; it is therefore a holomorphic function of z.

As in § 16 for the functions p and q, so here for $f(z)$, no restriction is placed on properties of $f(z)$ at points that do not lie within the region; so that elsewhere it may have infinities, or discontinuities or branch points. The properties, essential to secure the validity of the proposition, are (i) that no infinities or discontinuities lie within the region, and (ii) that the same value of $f(z)$ is acquired by whatever path in the continuous region the variable reaches its position z.

Corollary. *No change is caused in the value of the integral of a holomorphic function between two points when the path of integration between the points is deformed in any manner, provided only that, during the deformation, no part of the path passes outside the boundary of the region within which the function is holomorphic.*

This result is of importance, because it permits special forms of the path of integration without affecting the value of the integral.

19. *When a function $f(z)$ is holomorphic over a part of the plane bounded by two simple curves (one lying within the other), equal values of $\int f(z)\,dz$ are obtained by integrating round each of the curves in a direction, which—relative to the area enclosed by each—is positive.*

The ring-formed portion of the plane (fig. 1, p. 3) which lies between the two curves being a region over which $f(z)$ is holomorphic, the integral $\int f(z)\,dz$ taken in the positive sense round the whole of the boundary of the included portion is zero. The integral consists of two parts: first, that round the outer boundary the positive sense of which is DEF; and second, that round the inner boundary the positive sense of which for the portion of area between ABC and DEF is ACB. Denoting the value of $\int f(z)\,dz$ round DEF by (DEF), and similarly for the other, we have

$$(ACB) + (DEF) = 0.$$

The direction of an integral can be reversed if its sign be changed, so that $(ACB) = -(ABC)$; and therefore

$$(ABC) = (DEF).$$

But (ABC) is the integral $\int f(z)\,dz$ taken round ABC, that is, round the curve in a direction which, relative to the area enclosed by it, is positive.

The proposition is therefore proved.

The remarks made in the preceding case as to the freedom from limitations on the character of the function outside the portion are valid also in this case.

COROLLARY I. *When the integral of a function is taken round the whole of any simple curve in the plane, no change is caused in its value by continuously deforming the curve into any other simple curve provided that the function is holomorphic over the part of the plane in which the deformation is effected.*

COROLLARY II. *When a function $f(z)$ is holomorphic over a continuous portion of a plane bounded by any number of simple non-intersecting curves, all but one of which are external to one another and the remaining one of which encloses them all, the value of the integral $\int f(z)\,dz$ taken positively round the single external curve is equal to the sum of the values taken round each of the other curves in a direction which is positive relative to the area enclosed by it.*

These corollaries are of importance in finding the value of the integral of a meromorphic function round a curve which encloses one or more of the poles. The fundamental theorem for such integrals, also due to Cauchy, is the following.

20. *Let $f(z)$ denote a function which is holomorphic over any region in the z-plane and let a denote any point within that region, which is not a zero of $f(z)$; then*

$$f(a) = \frac{1}{2\pi i}\int \frac{f(z)}{z-a}\,dz,$$

the integral being taken positively round the whole boundary of the region.

With a as centre and a very small radius ρ, describe a circle C, which will be assumed to lie wholly within the region; this assumption is justifiable because the point a lies within the region. Because $f(z)$ is holomorphic over the assigned region, the function $f(z)/(z-a)$ is holomorphic over the whole of the region excluded by the small circle C. Hence, by Corollary II. of § 19, we have

$$\int_B \frac{f(z)}{z-a}\,dz = \int_C \frac{f(z)}{z-a}\,dz,$$

the notation implying that the integrations are taken round the whole boundary B and round the circumference of C respectively.

For points on the circle C, let $z - a = \rho e^{\theta i}$, so that θ is the variable for the circumference and its range is from 0 to 2π; then we have

$$\frac{dz}{z-a} = i\,d\theta.$$

Along the circle $f(z) = f(a + \rho e^{\theta i})$; the quantity ρ is very small and f is finite and continuous over the whole of the region so that $f(a + \rho e^{\theta i})$ differs from $f(a)$ only by a quantity which vanishes with ρ. Let this difference be ϵ, which is a continuous small quantity; then $|\epsilon|$ is a small quantity which, for every point on the circumference of C, vanishes with ρ. Then

$$\int_C \frac{f(z)}{z-a}\,dz = i\int_0^{2\pi}\{f(a)+\epsilon\}\,d\theta$$

$$= 2\pi i f(a) + i\int_0^{2\pi}\epsilon d\theta.$$

If E denote the value of the integral on the right-hand side, and η the greatest value of the modulus of ϵ along the circle, then, as in § 15,

$$|E| < \int_0^{2\pi}|\epsilon|\,d\theta$$

$$< \int_0^{2\pi}\eta d\theta$$

$$< 2\pi\eta.$$

Now let the radius of the circle diminish to zero: then η also diminishes to zero and therefore $|E|$, necessarily positive, becomes less than any finite quantity however small, that is, E is itself zero; and thus we have

$$\int_C \frac{f(z)}{z-a}\,dz = 2\pi i f(a),$$

which proves the theorem.

This result is the simplest case of the integral of a meromorphic function. The subject of integration is $\dfrac{f(z)}{z-a}$, a function which is monogenic and uniform throughout the region and which, everywhere except at $z=a$, is finite and continuous; moreover, $z=a$ is a pole, because in the immediate vicinity of a the reciprocal of the subject of integration, viz. $\dfrac{z-a}{f(z)}$, is holomorphic.

The theorem may therefore be expressed as follows:

If $g(z)$ be a meromorphic function, which in the vicinity of a can be expressed in the form $\dfrac{f(z)}{z-a}$ where $f(a)$ is not zero and which at all other points in a region enclosing a is holomorphic, then

$$\frac{1}{2\pi i}\int g(z)\,dz = \text{limit of } (z-a)\,g(z) \text{ when } z=a,$$

the integral being taken round a curve in the region enclosing the point a.

The pole a of the function $g(z)$ is said to be simple, or of the first order, or of multiplicity unity.

Corollary. The more general case of a meromorphic function with a finite number of poles can easily be deduced. Let these be $a_1, \ldots, a_n$ each assumed to be simple; and let

$$G(z) = (z-a_1)(z-a_2)\ldots(z-a_n).$$

Let $f(z)$ be a holomorphic function within a region of the z-plane bounded by a simple contour enclosing the n points $a_1, a_2, \dots a_n$, no one of which is a zero of $f(z)$. Then since

$$\frac{1}{G(z)} = \sum_{r=1}^{n} \frac{1}{G'(a_r)} \frac{1}{z - a_r},$$

we have
$$\frac{f(z)}{G(z)} = \sum_{r=1}^{n} \frac{1}{G'(a_r)} \frac{f(z)}{z - a_r}.$$

We therefore have
$$\int \frac{f(z)}{G(z)} dz = \sum_{r=1}^{n} \frac{1}{G'(a_r)} \int \frac{f(z)}{z - a_r} dz,$$

each integral being taken round the boundary. But the preceding proposition gives

$$\int \frac{f(z)}{z - a_r} dz = 2\pi i f(a_r),$$

because $f(z)$ is holomorphic over the whole region included in the contour; and therefore

$$\int \frac{f(z)}{G(z)} dz = 2\pi i \sum_{r=1}^{n} \frac{f(a_r)}{G'(a_r)},$$

the integral on the left-hand side being taken in the positive direction*.

The result just obtained expresses the integral of the meromorphic function round a contour which includes a finite number of its simple poles. It can be otherwise obtained by means of Corollary II. of § 19, by adopting a process similar to that adopted above, viz., by making each of the curves in the Corollary quoted small circles round the points $a_1, \dots, a_n$ with ultimately vanishing radii.

21. The preceding theorems have sufficed to evaluate the integral of a function with a number of simple poles: we now proceed to obtain further theorems, which can be used among other purposes to evaluate the integral of a function with poles of order higher than the first.

We still consider a function $f(z)$ which is holomorphic within a given region. Then, if a be a point within the region which is not a zero of $f(z)$, we have

$$f(a) = \frac{1}{2\pi i} \int \frac{f(z)}{z - a} dz,$$

the point a being neither on the boundary nor within an infinitesimal distance of it. Let $a + \delta a$ be any other point within the region; then

$$f(a + \delta a) = \frac{1}{2\pi i} \int \frac{f(z)}{z - a - \delta a} dz,$$

* We shall for the future assume that, if no direction for a complete integral be specified, the positive direction is taken.

and therefore

$$f(a+\delta a)-f(a)=\frac{1}{2\pi i}\int\left(-\frac{1}{z-a}+\frac{1}{z-a-\delta a}\right)f(z)\,dz$$

$$=\frac{1}{2\pi i}\int\left\{\frac{\delta a}{(z-a)^2}+\frac{(\delta a)^2}{(z-a)^2(z-a-\delta a)}\right\}f(z)\,dz,$$

the integral being in every case taken round the boundary.

Since $f(z)$ is monogenic, the definition of $f'(a)$, the first derivative of $f(a)$, gives $f'(a)$ as the limit of

$$\frac{f(a+\delta a)-f(a)}{\delta a},$$

when δa ultimately vanishes; hence we may take

$$\frac{f(a+\delta a)-f(a)}{\delta a}=f'(a)+\sigma,$$

where σ is a quantity which vanishes with δa and is therefore such that $|\sigma|$ also vanishes with δa. Hence

$$\{f'(a)+\sigma\}\,\delta a=\frac{\delta a}{2\pi i}\int\left\{\frac{1}{(z-a)^2}+\frac{\delta a}{(z-a)^2(z-a-\delta a)}\right\}f(z)\,dz;$$

dividing out by δa and transposing, we have

$$f'(a)-\frac{1}{2\pi i}\int\frac{f(z)}{(z-a)^2}dz=-\sigma+\frac{\delta a}{2\pi i}\int\frac{f(z)}{(z-a)^2(z-a-\delta a)}dz.$$

As yet, there is no limitation on the value of δa; we now proceed to a limit by making $a+\delta a$ approach to coincidence with a, viz., by making δa ultimately vanish. Taking moduli of each of the members of the last equation, we have

$$\left|f'(a)-\frac{1}{2\pi i}\int\frac{f(z)}{(z-a)^2}dz\right|=\left|-\sigma+\frac{\delta a}{2\pi i}\int\frac{f(z)}{(z-a)^2(z-a-\delta a)}dz\right|$$

$$<|\sigma|+\frac{|\delta a|}{2\pi}\left|\int\frac{f(z)}{(z-a)^2(z-a-\delta a)}dz\right|.$$

Let the greatest modulus of $\dfrac{f(z)}{(z-a)^2(z-a-\delta a)}$ for points z along the boundary be M, which is a finite quantity on account of the conditions applying to $f(z)$ and the fact that the points a and $a+\delta a$ are not infinitesimally near the boundary. Then, by § 15,

$$\left|\int\frac{f(z)}{(z-a)^2(z-a-\delta a)}dz\right|$$

$$<MS,$$

where S is the whole length of the boundary, a finite quantity. Hence

$$\left|f'(a)-\frac{1}{2\pi i}\int\frac{f(z)}{(z-a)^2}dz\right|<|\sigma|+\frac{|\delta a|}{2\pi}MS.$$

When we proceed to the limit in which δa vanishes, we have $|\delta a| = 0$ and $|\sigma| = 0$, ultimately; hence the modulus on the left-hand side ultimately vanishes and therefore the quantity to which that modulus belongs is itself zero, that is,

$$f'(a) - \frac{1}{2\pi i}\int \frac{f(z)}{(z-a)^2}\,dz = 0,$$

so that
$$f'(a) = \frac{1}{2\pi i}\int \frac{f(z)}{(z-a)^2}\,dz.$$

This theorem evidently corresponds in complex variables to the well-known theorem of differentiation with respect to a constant under the integral sign when all the quantities concerned are real.

Proceeding in the same way, we can prove that

$$\frac{f'(a+\delta a) - f'(a)}{\delta a} = \frac{2\,!}{2\pi i}\int \frac{f(z)}{(z-a)^3}\,dz + \theta,$$

where θ is a small quantity which vanishes with δa. Moreover the integral on the right-hand side is finite, for the subject of integration is everywhere finite along the path of integration which itself is of finite length. Hence, first, a small change in the independent variable leads to a change of the same order of small quantities in the value of the function $f'(a)$, which shews that $f'(a)$ is a continuous function. Secondly, denoting

$$f'(a+\delta a) - f'(a)$$

by $\delta f'(a)$, we have the limiting value of $\dfrac{\delta f'(a)}{\delta a}$ equal to the integral on the right-hand side when δa vanishes, that is, the derivative of $f'(a)$ has a value independent of the form of δa and therefore $f'(a)$ is monogenic. Denoting this derivative by $f''(a)$, we have

$$f''(a) = \frac{2\,!}{2\pi i}\int \frac{f(z)}{(z-a)^3}\,dz.$$

Thirdly, the function $f'(a)$ is uniform; for it is the limit of the value of $\dfrac{f(a+\delta a) - f(a)}{\delta a}$ and both $f(a)$ and $f(a+\delta a)$ are uniform. Lastly, it is finite; for (§ 15) it is the value of the integral $\dfrac{1}{2\pi i}\displaystyle\int \frac{f(z)}{(z-a)^2}\,dz$, in which the length of the path is finite and the subject of integration is finite at every point of the path.

Hence $f'(a)$ is continuous, monogenic, uniform, and finite throughout the whole of the region in which $f(z)$ has these properties: it is a holomorphic function. Hence:—

When a function is holomorphic in any region of the plane bounded *by a simple curve, its derivative is also holomorphic within that region.* And, by repeated application of this theorem:—

When a function is holomorphic in any region of the plane bounded by a simple curve, it has an unlimited number of successive derivatives each of which is holomorphic within the region.

All these properties have been shewn to depend simply upon the holomorphic character of the fundamental function; but the inferences relating to the derivatives have been proved only for points within the region and not for points on the boundary. If the foregoing methods be used to prove them for points on the boundary, they require that a consecutive point shall be taken in any direction; in the absence of knowledge about the fundamental function for points outside (even though just outside) no inferences can be justifiably drawn.

An illustration of this statement is furnished by the hypergeometric series which, together with all its derivatives, is holomorphic within a circle of radius unity and centre the origin; and the series converges unconditionally everywhere on the circumference, provided $\gamma > \alpha + \beta$. But the corresponding condition for convergence on the circumference ceases to be satisfied for some one of the derivatives and for all which succeed it: as such functions do not then converge unconditionally, the circumference of the circle must be excluded from the region within which the derivatives are holomorphic.

22. Expressions for the first and the second derivatives have been obtained.

By a process similar to that which gives the value of $f'(a)$, the derivative of order n is obtainable in the form

$$f^{(n)}(a) = \frac{n!}{2\pi i}\int \frac{f(z)}{(z-a)^{n+1}}\,dz,$$

the integral being taken round the whole boundary of the region or round any curves which arise from deformation of the boundary, provided that no point of the curves in the final or any intermediate form is indefinitely near to a.

In the case when the curve of integration is a circle, no point of which circle may lie outside the boundary of the region, we have a modified form for $f^{(n)}(a)$.

For points along the circumference of the circle with centre a and radius r, let

$$z - a = re^{\theta i},$$

so that as before

$$\frac{dz}{z-a} = i d\theta:$$

then 0 and 2π being taken as the limits of θ, we have

$$f^{(n)}(a) = \frac{n!}{2\pi r^n}\int_0^{2\pi} e^{-n\theta i} f(a + re^{\theta i})\,d\theta.$$

Let M be the greatest value of the modulus of $f(z)$ for points on the circumference (or, as it may be convenient to consider, of points on or within the circumference): then

$$|f^{(n)}(a)| < \frac{n!}{2\pi r^n} \int_0^{2\pi} |e^{-n\theta i}| \, |f(a + re^{\theta i})| \, d\theta$$
$$< \frac{n!}{2\pi r^n} \int_0^{2\pi} M d\theta$$
$$< n! \frac{M}{r^n}.$$

Now, let there be a function $\phi(z)$ defined by the equation

$$\phi(z) = \frac{M}{1 - \dfrac{z-a}{r}},$$

which can evidently be expanded in a series of ascending powers of $z - a$ that converges within the circle. The series is

$$\phi(z) = M\left\{1 + \frac{z-a}{r} + \frac{(z-a)^2}{r^2} + \ldots\right\},$$

so that
$$\frac{d^n\phi(z)}{dz^n} = n!\frac{M}{r^n}\left\{1 + (n+1)\frac{z-a}{r} + \ldots\right\}.$$

Hence
$$\left[\frac{d^n\phi(z)}{dz^n}\right]_{z=a} = n!\frac{M}{r^n},$$

so that, if the value of the nth derivative of $\phi(z)$, when $z = a$, be denoted by $\phi^{(n)}(a)$, we have

$$|f^{(n)}(a)| < \phi^{(n)}(a).$$

These results can be extended to functions of more than one variable: the proof is similar to the foregoing proof. When the variables are two, say z and z', the results may be stated as follows:—

For all points z within a given simple curve C in the z-plane and all points z' within a given simple curve C' in the z'-plane, let $f(z, z')$ be a holomorphic function; then, if a be any point within C and a' any point within C',

$$\frac{n!\,n'!}{(2\pi i)^2} \iint \frac{f(z, z')}{(z-a)^{n+1}(z'-a')^{n'+1}} dz dz' = \frac{\partial^{n+n'} f(a, a')}{\partial a^n \partial a'^{n'}},$$

where n and n' are any integers and the integral is taken positively round the two curves C and C'.

If M be the greatest value of $|f(z, z')|$ for points z and z' within their respective regions when the curves C and C' are circles of radii r, r' and centres a, a', then

$$\left|\frac{\partial^{n+n'} f(a, a')}{\partial a^n \partial a'^{n'}}\right| < n!\,n'!\,\frac{M}{r^n r'^{n'}};$$

and if
$$\phi(z, z') = \frac{M}{\left(1 - \frac{z-a}{r}\right)\left(1 - \frac{z'-a'}{r'}\right)},$$

then
$$\left|\frac{\partial^{n+n'} f(a, a')}{\partial a^n \partial a'^{n'}}\right| < \frac{\partial^{n+n'}\phi(z, z')}{\partial z^n \partial z'^{n}},$$

when $z = a$ and $z' = a'$ in the derivative of $\phi(z, z')$.

23. All the integrals of meromorphic functions that have been considered have been taken along complete curves: it is necessary to refer to integrals along curves which are lines only from one point to another. A single illustration will suffice at present.

Consider the integral $\int_{z_0}^{z} \frac{f(z)}{z-a}\,dz$; the function $f(z)$ is supposed holomorphic in the given region, and z and z_0 are any two points in that region. Let some curves joining z to z_0 be drawn as in the figure (fig. 7).

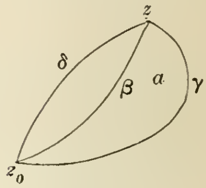

Fig. 7.

Then $\frac{f(z)}{z-a}$ is holomorphic over the whole area enclosed by $z_0 \beta z \delta z_0$: and therefore we have $\int \frac{f(z)}{z-a}\,dz = 0$ when taken round the boundary of that area. Hence as in the earlier case we have
$$\int_{z_0}^{z_\delta} \frac{f(z)}{z-a}\,dz = \int_{z_0}^{z_\beta} \frac{f(z)}{z-a}\,dz.$$

The point a lies within the area enclosed by $z_0 \gamma z \beta z_0$, and the function $\frac{f(z)}{z-a}$ is holomorphic, except in the immediate vicinity of $z = a$; hence
$$\int \frac{f(z)}{z-a}\,dz = 2\pi i f(a),$$
the integral on the left-hand side being taken round $z_0 \gamma z \beta z_0$. Hence
$$\int_{z_0}^{z_\gamma} \frac{f(z)}{z-a}\,dz = \int_{z_0}^{z_\beta} \frac{f(z)}{z-a}\,dz + 2\pi i f(a).$$

Denoting $\frac{f(z)}{z-a}$ by $g(z)$, the function $g(z)$ has one pole a in the region considered.

The preceding results are connected only with the simplest form of meromorphic functions; other simple results can be derived by means of the other theorems proved in §§ 17—21. Those which have been obtained are sufficient however to shew that: *The integral of a meromorphic function $\int g(z)\,dz$ from one point to another of the region of the function is not in general a uniform function.* The value of the integral is not altered by any deformation of the path which does not meet or cross a pole of the function; but the value is altered when the path of integration is so deformed as to pass over one or more poles. Therefore *it is necessary to specify the path of integration when the subject of integration is a meromorphic function;* only partial deformations of the path of integration are possible without modifying the value of the integral.

24. The following additional propositions* are deduced from limiting cases of integration round complete curves. In the first, the curve becomes indefinitely small; in the second, it becomes infinitely large. And in neither, are the properties of the functions to be integrated limited as in the preceding propositions, so that the results are of wider application.

I. *If $f(z)$ be a function which, whatever be its character at a, has no infinities and no branch-points in the immediate vicinity of a, the value of $\int f(z)\,dz$ taken round a small circle with its centre at a tends towards zero when the circle diminishes in magnitude so as ultimately to be merely the point a, provided that, as $|z-a|$ diminishes indefinitely, the limit of $(z-a)f(z)$ tend uniformly to zero.*

Along the small circle, initially taken to be of radius r, let

$$z - a = re^{\theta i},$$

so that

$$\frac{dz}{z-a} = i\,d\theta,$$

and therefore

$$\int f(z)\,dz = i\int_0^{2\pi} (z-a)\,f(z)\,d\theta.$$

Hence

$$\begin{aligned} \left|\int f(z)\,dz\right| &= \left|\int_0^{2\pi} (z-a)\,f(z)\,d\theta\right| \\ &< \int_0^{2\pi} |(z-a)\,f(z)|\,d\theta \\ &< \int_0^{2\pi} M\,d\theta \\ &< 2\pi M', \end{aligned}$$

where M' is the greatest value of M, the modulus of $(z-a)f(z)$, for points on the circumference. Since $(z-a)f(z)$ tends uniformly to the limit zero as $|z-a|$ diminishes indefinitely, $|\int f(z)\,dz|$ is ultimately zero. Hence the integral itself $\int f(z)\,dz$ is zero, under the assigned conditions.

Note. If the integral be extended over only part of the circumference of the circle, it is easy to see that, under the conditions of the proposition, the value of $\int f(z)\,dz$ still tends towards zero.

COROLLARY. *If $(z-a)f(z)$ tend uniformly to a limit k as $|z-a|$ diminishes indefinitely, the value of $\int f(z)\,dz$ taken round a small circle centre a tends towards $2\pi ik$ in the limit.*

* The form of the first two propositions, which is adopted here, is due to Jordan, *Cours d'Analyse*, t. ii, §§ 285, 286.

Thus the value of $\int \frac{dz}{(a^2-z^2)^{\frac{1}{2}}}$, taken round a very small circle centre a, where a is not the origin, is zero: the value of $\int \frac{dz}{(a-z)(a+z)^{\frac{1}{2}}}$, round the same circle is $\frac{\pi}{i}\left(\frac{2}{a}\right)^{\frac{1}{2}}$.

Neither the theorem nor the corollary will apply to a function, such as $\operatorname{sn}\frac{1}{z-a}$, which has the point a for an essential singularity: the value of $(z-a)\operatorname{sn}\frac{1}{z-a}$, as $|z-a|$ diminishes indefinitely, does not tend (§ 13) to a uniform limit. As a matter of fact, the function $\operatorname{sn}\frac{1}{z-a}$ has an infinite number of poles in the immediate vicinity of a as the limit $z=a$ is being reached.

II. *Whatever be the character of a function $f(z)$ for infinitely large values of z, the value of $\int f(z)\,dz$, taken round a circle with the origin for centre, tends towards zero as the circle becomes infinitely large, provided that, as $|z|$ increases indefinitely, the limit of $zf(z)$ tend uniformly to zero.*

Along a circle, centre the origin and radius R, we have $z=Re^{\theta i}$, so that

$$\frac{dz}{z}=id\theta,$$

and therefore

$$\int f(z)\,dz=i\int_0^{2\pi} zf(z)\,d\theta.$$

Hence

$$\begin{aligned}\left|\int f(z)\,dz\right| &= \left|\int_0^{2\pi} zf(z)\,d\theta\right| \\ &< \int_0^{2\pi} |zf(z)|\,d\theta \\ &< \int_0^{2\pi} M d\theta \\ &< 2\pi M',\end{aligned}$$

where M' is the greatest value of M, the modulus of $zf(z)$, for points on the circumference. When R increases indefinitely, the value of M' is zero on the hypothesis in the proposition; hence $\left|\int f(z)\,dz\right|$ is ultimately zero. Therefore the value of $\int f(z)\,dz$ tends towards zero, under the assigned conditions.

Note. If the integral be extended along only a portion of the circumference, the value of $\int f(z)\,dz$ still tends towards zero.

COROLLARY. *If $zf(z)$ tend uniformly to a limit k as $|z|$ increases indefinitely, the value of $\int f(z)\,dz$, taken round a very large circle, centre the origin, tends towards $2\pi ik$.*

Thus the value of $\int(1-z^n)^{-\frac{1}{2}}dz$ round an infinitely large circle, centre the origin, is zero if $n>2$, and is 2π if $n=2$.

III. *If all the infinities and the branch-points of a function lie in a finite region of the z-plane, then the value of $\int f(z)\,dz$ round any simple curve, which* *includes all those points, is zero, provided the value of* $zf(z)$, *as* $|z|$ *increases indefinitely, tends uniformly to zero.*

The simple curve can be deformed continuously into the infinite circle of the preceding proposition, without passing over any infinity or any branch-point; hence, if we assume that the function exists all over the plane, the value of $\int f(z)\,dz$ is, by Cor. I. of § 19, equal to the value of the integral round the infinite circle, that is, by the preceding proposition, to zero.

Another method of stating the proof of the theorem is to consider the corresponding simple curve on Neumann's sphere (§ 4). The surface of the sphere is divided into two portions by the curve*: in one portion lie all the singularities and the branch-points, and in the other portion there is no critical point whatever. Hence in this second portion the function is holomorphic; since the area is bounded by the curve we see that, on passing back to the plane, the excluded area is one over which the function is holomorphic. Hence, by § 19, the integral round the curve is equal to the integral round an infinite circle having its centre at the origin and is therefore zero, as before.

COROLLARY. *If, under the same circumstances, the value of* $zf(z)$, *as* $|z|$ *increases indefinitely, tend uniformly to* k, *then the value of* $\int f(z)\,dz$ *round the simple curve is* $2\pi i k$.

Thus the value of $\int \frac{dz}{(a^2-z^2)^{\frac{1}{2}}}$ along any simple curve which encloses the two points a and $-a$ is 2π; the value of

$$\int \frac{dz}{\{(1-z^2)(1-k^2z^2)\}^{\frac{1}{2}}}$$

round any simple curve enclosing the four points $1, -1, \frac{1}{k}, -\frac{1}{k}$, is zero, k being a non-vanishing constant; and the value of $\int (1-z^{2n})^{-\frac{1}{2}}dz$, taken round a circle, centre the origin and radius greater than unity, is zero when n is an integer greater than 1.

But the value of
$$\int \frac{dz}{\{(z-e_1)(z-e_2)(z-e_3)\}^{\frac{1}{2}}}$$
round any circle, which has the origin for centre and includes the three distinct points e_1, e_2, e_3, is not zero. The subject of integration has $z=\infty$ for a branch-point, so that the condition in the proposition is not satisfied; and the reason that the result is no longer valid is that the deformation into an infinite circle, as described in Cor. I. of § 19, is not possible because the infinite circle would meet the branch-point at infinity.

25. The further consideration of integrals of functions, that do not possess the character of uniformity over the whole area included by the curve of integration, will be deferred until Chap. IX. Some examples of the theorems proved in the present chapter will now be given.

* The fact that a single path of integration is the boundary of two portions of the surface of the sphere, within which the function may have different characteristic properties, will be used hereafter (§ 104) to obtain a relation between the two integrals that arise according as the path is deformed within one portion or within the other.

Ex. 1. It is sufficient merely to mention the indefinite integrals (that is, integrals from an arbitrary point to a point z) of rational, integral, algebraical functions. After the preceding explanations it is evident that they follow the same laws as integrals of similar functions of real variables.

Ex. 2. Consider the integral $\int \frac{dz}{(z-a)^{n+1}}$, taken round a simple curve.

When n is 0, the value of the integral is zero if the curve do not include the point a, and it is $2\pi i$ if the curve include the point a.

When n is a positive integer, the value of the integral is zero if the curve do not include the point a (by § 17), and the value of the integral is still zero if the curve do include the point a (by § 22, for the function $f(z)$ of the text is 1 and all its derivatives are zero). Hence the value of the integral round any curve, which does not pass through a, is zero.

We can now at once deduce, by § 20, the result that, *if a holomorphic function be constant along any simple closed curve within its region, it is constant over the whole area within the curve.* For let t be any point within the curve, z any point on it, and C the constant value of the function for all the points z; then

$$\phi(t) = \frac{1}{2\pi i}\int \frac{\phi(z)}{z-t}\,dz,$$

the integral being taken round the curve, so that

$$\begin{aligned}\phi(t) &= \frac{C}{2\pi i}\int \frac{dz}{z-t}\\ &= C\end{aligned}$$

by the above result, since the point t lies within the curve.

Ex. 3. Consider the integral $\int e^{-z^2}dz$.

In any finite part of the plane, the function e^{-z^2} is holomorphic; therefore (§ 17) the integral round the boundary of a rectangle (fig. 8), bounded by the lines $x = \pm a$, $y=0$, $y=b$, is zero: and this boundary can be extended, provided the deformation remain in the region where the function is holomorphic. Now as a tends towards infinity, the modulus of e^{-z^2}, being $e^{-x^2+y^2}$, tends towards zero when y remains finite; and therefore the preceding rectangle can be extended towards infinity in the direction of the axis of x, the side b of the rectangle remaining unaltered.

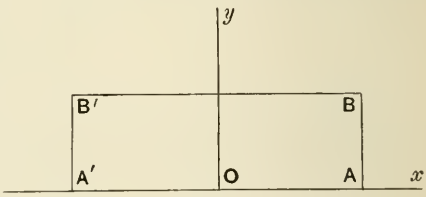

Fig. 8.

Along $A'A$, we have $z=x$: so that the value of the integral along the part $A'A$ of the boundary is $\int_{-a}^{a} e^{-x^2}dx$.

Along AB, we have $z=a+iy$, so that the value of the integral along the part AB is $i\int_0^b e^{-(a+iy)^2}dy$.

Along BB', we have $z=x+ib$, so that the value of the integral along the part BB' is $\int_a^{-a} e^{-(x+ib)^2}dx$.

Along $B'A'$, we have $z=-a+iy$, so that the value of the integral along the part $B'A'$ is $i\int_b^0 e^{-(-a+iy)^2}dy$.

The second of these portions of the integral is $e^{-a^2} . i . \int_0^b e^{y^2-2ayi} dy$, which is easily seen to be zero when the (real) quantity a is infinite.

Similarly the fourth of these portions is zero.

Hence as the complete integral is zero, we have, on passing to the limit,

$$\int_{-\infty}^{\infty} e^{-x^2} dx + \int_{\infty}^{-\infty} e^{-x^2-2ibx+b^2} dx = 0,$$

whence

$$e^{b^2} \int_{-\infty}^{\infty} e^{-x^2-2ibx} dx = \int_{-\infty}^{\infty} e^{-x^2} dx = \pi^{\frac{1}{2}},$$

or

$$\int_{\infty}^{\infty} e^{-x^2} (\cos 2bx - i \sin 2bx)\, dx = \pi^{\frac{1}{2}} e^{-b^2};$$

and therefore, on equating real parts, we obtain the well-known result

$$\int_{-\infty}^{\infty} e^{-x^2} \cos 2bx\, dx = \pi^{\frac{1}{2}} e^{-b^2}.$$

This is only one of numerous examples* in which the theorems in the text can be applied to obtain the values of definite integrals with real limits and real variables.

Ex. 4. Consider the integral $\int \frac{z^{n-1}}{1+z} dz$, where n is a real positive quantity less than unity.

The only infinities of the subject of integration are the origin and the point -1; the branch-points are the origin and $z=\infty$. Everywhere else in the plane the function behaves like a holomorphic function; and, therefore, when we take any simple closed curve enclosing neither the origin nor the point -1, the integral of the function round that curve is zero.

We shall assume that the curve lies on the positive side of the axis of x and that it is made up of :—

(i) a semicircle C_3 (fig. 9), centre the origin and radius R which is made to increase indefinitely :

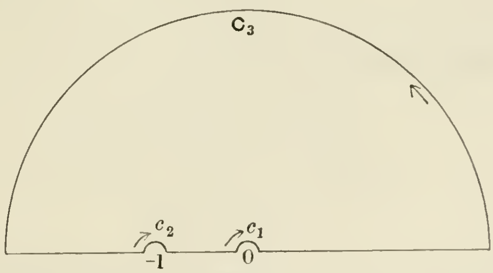

Fig. 9.

(ii) two semicircles, c_1 and c_2, with their centres at 0 and -1 respectively, and with radii r and r', which ultimately are made infinitesimally small :

(iii) the diameter of C_3 along the axis of x excepting those ultimately infinitesimal portions which are the diameters of c_1 and of c_2.

The subject of integration is uniform within the area thus enclosed although it is not uniform over the whole plane. We shall take that value of z^{n-1} which has its argument equal to $(n-1)\,\theta$, where θ is the argument of z.

* See Briot and Bouquet, *Théorie des fonctions elliptiques*, (2nd ed.), pp. 141 et sqq., from which examples 3 and 4 are taken.

The integral round the boundary is made up of four parts.

(a) The integral round C_3. The value of $z \cdot \frac{z^{n-1}}{1+z}$, as $|z|$ increases indefinitely, tends uniformly to the limit zero; hence, as the radius of the semicircle is increased indefinitely, the integral round C_3 vanishes (§ 24, II., *Note*).

(b) The integral round c_1. The value of $z \cdot \frac{z^{n-1}}{1+z}$, as $|z|$ diminishes indefinitely, tends uniformly to the limit zero; hence as the radius of the semicircle is diminished indefinitely, the integral round c_1 vanishes (§ 24, I., *Note*).

(c) The integral round c_2. The value of $(1+z)\frac{z^{n-1}}{1+z}$, as $|1+z|$ diminishes indefinitely for points in the area, tends uniformly to the limit $(-1)^{n-1}$, i.e., to the limit $e^{(n-1)\pi i}$. Hence this part of the integral is

$$e^{(n-1)\pi i}\int\frac{dz}{1+z},$$

being taken in the direction indicated by the arrow round c_2, the infinitesimal semicircle. Evidently $\frac{dz}{1+z}=id\theta$ and the limits are π to 0, so that this part of the whole integral is

$$e^{(n-1)\pi i}\int_\pi^0 id\theta$$

$$=-i\pi e^{(n-1)\pi i}$$

$$=i\pi e^{n\pi i}.$$

(d) The integral along the axis of x. The parts at -1 and at 0 which form the diameters of the small semicircles are to be omitted; so that the value is

$$\left\{\int_{-\infty}^{-1-r'}+\int_{-1+r'}^{-r}+\int_r^\infty\right\}\frac{x^{n-1}}{1+x}dx.$$

This is what Cauchy calls the principal value* of the integral

$$\int_{-\infty}^\infty\frac{x^{n-1}}{1+x}dx.$$

Since the whole integral is zero, we have

$$i\pi e^{n\pi i}+\int_{-\infty}^\infty\frac{x^{n-1}}{1+x}dx=0.$$

Let $$P=\int_0^\infty\frac{x^{n-1}}{1+x}dx,\quad P'=\int_{-\infty}^0\frac{x^{n-1}}{1+x}dx,$$

and $$Q=\int_0^\infty\frac{x^{n-1}}{1-x}dx,$$

principal values being taken in each case. Then, taking account of the arguments, we have

$$P'=\int_0^\infty\frac{(-x)^{n-1}}{1-x}dx$$

$$=(-1)^{n-1}\int_0^\infty\frac{x^{n-1}dx}{1-x}$$

$$=e^{(n-1)\pi i}Q.$$

Since $$i\pi e^{n\pi i}+P+P'=0,$$

we have $$P-e^{n\pi i}Q=-i\pi e^{n\pi i},$$

* Williamson's *Integral Calculus*, § 104.

so that $$P - Q\cos n\pi = \pi \sin n\pi, \quad Q \sin n\pi = \pi \cos n\pi.$$

Hence $$\int_0^\infty \frac{x^{n-1}}{1+x}\,dx = P = \pi \operatorname{cosec} n\pi,$$

$$\int_0^\infty \frac{x^{n-1}}{1-x}\,dx = Q = \pi \cot n\pi.$$

Ex. 5. In the same way it may be proved that

$$\int_{-\infty}^\infty \frac{\cos ax}{1+x^{2n}}\,dx = -i\frac{\pi}{n}\sum_{r=1}^{n} \omega^{2r-1} e^{ai\omega^{2r-1}},$$

where n is an integer, a is positive and ω is $e^{i\frac{\pi}{2n}}$.

Ex. 6. By considering the integral $\int e^{-z} z^{n-1} dz$ round the contour of the sector of a circle of radius r, bounded by the radii $\theta=0$, $\theta=a$, where a is less than $\frac{1}{2}\pi$ and n is positive, it may be proved that

$$\int_0^\infty \{r^{n-1} e^{-r\cos a} \cos(\beta + r\sin a)\}\, dr = \Gamma(n)\cos(\beta + na),$$

on proceeding to the limit when r is made infinite. (Briot and Bouquet.)

Ex. 7. Consider the integral $\int \frac{dz}{z^n - 1}$, where n is an integer. The subject of integration is meromorphic; it has for its poles (each of which is simple) the n points ω^r for $r=0$, 1, ..., $n-1$, where ω is a primitive nth root of unity; and it has no other infinities and no branch-points. Moreover the value of $\frac{z}{z^n-1}$, as $|z|$ increases indefinitely, tends uniformly to the limit zero; hence (§ 24, III.) the value of the integral, taken round a circle centre the origin and radius > 1, is zero.

This result can be derived by means of Corollary II. in § 19. Surround each of the poles with an infinitesimal circle having the pole for centre; then the integral round the circle of radius > 1 is equal to the sum of the values of the integral round the infinitesimal circles. The value round the circle having ω^r for its centre is, by § 20,

$$2\pi i \left(\text{limit of } \frac{z-\omega^r}{z^n-1}, \text{ when } z = \omega^r\right)$$

$$= \frac{2\pi i}{n}\omega^{n-r}.$$

Hence the integral round the large circle

$$= \frac{2\pi i}{n}\sum_{r=0}^{n-1} \omega^{n-r}$$

$$= 0.$$

Ex. 8. Hitherto, in all the examples considered, the poles that have occurred have been simple: but the results proved in § 21 enable us to obtain the integrals of functions which have multiple poles within an area. As an example, consider the integral $\int \frac{dz}{(1+z^2)^{n+1}}$ round any curve which includes the point i but not the point $-i$, these points being the two poles of the subject of integration, each of multiplicity $n+1$.

We have seen that $$f^n(a)=\frac{n!}{2\pi i}\int\frac{f(z)}{(z-a)^{n+1}}dz,$$
where $f(z)$ is holomorphic throughout the region bounded by the curve round which the integral is taken.

In the present case a is i, and $f(z)=\frac{1}{(z+i)^{n+1}}$; so that
$$f^n(z)=\frac{2n!}{n!}\frac{(-1)^n}{(z+i)^{2n+1}},$$
and therefore $$f^n(i)=\frac{2n!}{n!}\frac{(-1)^n}{(2i)^{2n+1}}=-\frac{2n!}{n!}2^{-2n-1}i.$$
Hence we have $$\int\frac{dz}{(1+z^2)^{n+1}}=\frac{2\pi i}{n!}f^n(i)$$
$$=\frac{2n!}{n!\,n!}\frac{\pi}{2^{2n}}.$$

In the case of the integral of a function round a simple curve which contains several of its poles, we first (§ 20) resolve the integral into the sum of the integrals round simple curves each containing only one of the points, and then determine each of the latter integrals as above.

Another method that is sometimes possible makes use of the expression of the uniform function in partial fractions. After Ex. 2, we need retain only those fractions which are of the form $\frac{A}{z-a}$: the integral of such a fraction is $2\pi iA$, and the value of the whole integral is therefore $2\pi i\Sigma A$. It is thus sufficient to obtain the coefficients of the inverse first powers which arise when the function is expressed in partial fractions corresponding to each pole. Such a coefficient A, the coefficient of $\frac{1}{z-a}$ in the expansion of the function, is called by Cauchy the *residue* of the function relative to the point.

For example,
$$\frac{1}{(z^3+1)^2}=\tfrac{2}{9}\left\{\frac{1}{z+1}+\frac{\omega}{z+\omega}+\frac{\omega^2}{z+\omega^2}\right\}+\tfrac{1}{9}\left\{\frac{1}{(z+1)^2}+\frac{\omega^2}{(z+\omega)^2}+\frac{\omega}{(z+\omega^2)^2}\right\},$$
so that the residues relative to the points -1, $-\omega$, $-\omega^2$ are $\frac{2}{9}$, $\frac{2}{9}\omega$, $\frac{2}{9}\omega^2$ respectively. Hence if we take a semicircle, of radius >1 and centre the origin with its diameter along the axis of y, so as to lie on the positive side of the axis of y, the area between the semi-circumference and the diameter includes the two points $-\omega$ and $-\omega^2$; and therefore the value of
$$\int\frac{dz}{(z^3+1)^2},$$
taken along the semi-circumference and the diameter, is
$$2\pi i\left(\tfrac{2}{9}\omega+\tfrac{2}{9}\omega^2\right);$$
i.e., the value is $-\frac{4}{9}\pi i$.

CHAPTER III.

Expansion of Functions in Series of Powers.

26. We are now in a position to obtain the two fundamental theorems relating to the expansion of functions in series of powers of the variable: they are due to Cauchy and Laurent respectively.

Cauchy's theorem is as follows*:—

When a function is holomorphic over the area of a circle of centre a, it can be expanded as a series of positive integral powers of $z-a$ converging for all points within the circle.

Let z be any point within the circle; describe a concentric circle of radius r such that

$$|z-a|=\rho<r<R,$$

where R is the radius of the given circle. If t denote a current point on the circumference of the new circle, we have

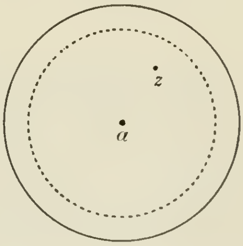

Fig. 10.

$$f(z)=\frac{1}{2\pi i}\int\frac{f(t)}{t-z}\,dz$$

$$=\frac{1}{2\pi i}\int\frac{f(t)}{t-a}\frac{dt}{1-\dfrac{z-a}{t-a}},$$

the integral extending along the whole circumference of radius r. Now

$$\frac{1}{1-\dfrac{z-a}{t-a}}=1+\frac{z-a}{t-a}+\left(\frac{z-a}{t-a}\right)^2+\ldots\ldots+\left(\frac{z-a}{t-a}\right)^n+\frac{\left(\dfrac{z-a}{t-a}\right)^{n+1}}{1-\dfrac{z-a}{t-a}},$$

so that, by § 14 (III.), we have

$$f(z)=\frac{1}{2\pi i}\int\frac{f(t)}{t-a}\,dt+\frac{z-a}{2\pi i}\int\frac{f(t)}{(t-a)^2}\,dt+\ldots\ldots+\frac{(z-a)^n}{2\pi i}\int\frac{f(t)}{(t-a)^{n+1}}\,dt$$
$$+\frac{1}{2\pi i}\int\frac{f(t)}{t-z}\left(\frac{z-a}{t-a}\right)^{n+1}dt.$$

* *Exercices d'Analyse et de Physique Mathématique*, t. ii, pp. 50 et seq.; the memoir was first made public at Turin in 1832.

Now $f(t)$ is holomorphic over the whole area of the circle; hence, if t be not actually on the boundary of the region (§§ 21, 22), a condition secured by the hypothesis $r < R$, we have

$$f^{(s)}(a) = \frac{s!}{2\pi i}\int \frac{f(t)}{(t-a)^{s+1}}\,dt,$$

and therefore

$$f(z) = f(a) + (z-a)f'(a) + \ldots\ldots + \frac{(z-a)^n}{n!}f^{(n)}(a) + \frac{(z-a)^{n+1}}{2\pi i}\int \frac{f(t)}{t-z}\frac{dt}{(t-a)^{n+1}}.$$

Let the last term be denoted by L. Since $|z-a| = \rho$ and $|t-a| = r$, it is at once evident that $|t-z| \geqslant r-\rho$. Let M be the greatest value of $|f(t)|$ for points along the circle of radius r; then M must be finite, owing to the initial hypothesis relating to $f(z)$. Taking

$$t - a = re^{\theta i}$$

so that

$$dt = i(t-a)\,d\theta,$$

we have

$$\begin{aligned}|L| &= \frac{\rho^{n+1}}{2\pi}\left|\int_0^{2\pi}\frac{f(t)}{t-z}\frac{d\theta}{(t-a)^n}\right| \\ &< \frac{\rho^{n+1}}{2\pi}\frac{1}{r^n(r-\rho)}\int_0^{2\pi}|f(t)|\,d\theta \\ &< \frac{\rho^{n+1}}{r^n(r-\rho)}M \\ &< \left(\frac{\rho}{r}\right)^{n+1} M\left(1-\frac{\rho}{r}\right)^{-1}.\end{aligned}$$

Now r was chosen to be greater than ρ; hence as n becomes infinitely large, we have $\left(\frac{\rho}{r}\right)^{n+1}$ infinitesimally small. Also $M\left(1-\frac{\rho}{r}\right)^{-1}$ is finite. Hence as n increases indefinitely, the limit of $|L|$, necessarily not negative, is infinitesimally small and therefore, in the same case, L tends towards zero.

It thus appears, exactly as in § 15 (V.), that, when n is made to increase without limit, the difference between the quantity $f(z)$ and the first $n+1$ terms of the series is ultimately zero; hence the series is a converging series having $f(z)$ as the limit of the sum, so that

$$f(z) = f(a) + (z-a)f'(a) + \frac{(z-a)^2}{2!}f''(a) + \ldots\ldots + \frac{(z-a)^n}{n!}f^{(n)}(a) + \ldots\ldots,$$

which proves the proposition under the assigned conditions. It is the form of Taylor's expansion for complex variables.

Note. The series on the right-hand side is frequently denoted by $P(z-a)$, where P is a general symbol for a converging series of positive integral powers of $z-a$: it is also sometimes* denoted by $P(z|a)$. Con-

* Weierstrass, *Abh. aus der Functionenlehre*, p. 1.

formably with this notation, a series of negative integral powers of $z-a$ would be denoted by $P\left(\frac{1}{z-a}\right)$: a series of negative integral powers of z either by $P\left(\frac{1}{z}\right)$ or by $P(z\,|\,\infty)$, the latter implying a series proceeding in positive integral powers of a quantity which vanishes when z is infinite, i.e., in positive integral powers of $\frac{1}{z}$.

If, however, the circle can be made of infinitely great radius so that the function $f(z)$ is holomorphic over the finite part of the plane, the equivalent series is denoted by $G(z-a)$ and it converges over the whole plane. Conformably with this notation, a series of negative integral powers of $z-a$ which converges over the whole plane is denoted by $G\left(\frac{1}{z-a}\right)$.

27. The following remarks on the proof and on inferences from it should be noticed.

(i) In order that $\frac{1}{t-z}$ may be expanded in the required form, the point z must be taken actually within the area of the circle of radius R; and therefore the convergence of the series $P(z-a)$ is not established for points on the circumference.

(ii) The coefficients of the powers of $z-a$ in the series are the values of the function and its derivatives at the centre of the circle; and the character of the derivatives is sufficiently ensured (§ 21) by the holomorphic character of the function for all points within the region. It therefore follows that, if a function be holomorphic within a region bounded by a circle of centre a, its expansion in a series of ascending powers of $z-a$ converging for all points within the circle depends only upon the values of the function and its derivatives at the centre.

But instead of having the values of the function and of all its derivatives at the centre of the circle, it will suffice to have the values of the holomorphic function itself over any small region at a or along any small line through a, the region or the line not being infinitesimal. The values of the derivatives at a can be found in either case; for $f'(b)$ is the limit of $\{f(b+\delta b)-f(b)\}/\delta b$, so that the value of the first derivative can be found for any point in the region or on the line, as the case may be; and so for all the derivatives in succession.

(iii) The form of Maclaurin's series for complex variables is at once derivable by supposing the centre of the circle at the origin. We then infer that, *if a function be holomorphic over a circle, centre the origin, it can be* *represented in the form of a series of ascending, positive, integral powers of the variable given by*

$$f(z) = f(0) + zf'(0) + \frac{z^2}{2\,!} f''(0) + \dots ,$$

where the coefficients of the various powers of z are the values of the derivatives of $f(z)$ at the origin, and the series converges for all points within the circle.

Thus, the function e^z is holomorphic over the finite part of the plane; therefore its expansion is of the form $G(z)$. The function $\log(1+z)$ has a singularity at -1; hence within a circle, centre the origin and radius unity, it can be expanded in the form of an ascending series of positive integral powers of z, it being convenient to choose that one of the values of the function which is zero at the origin. Again, $\tan^{-1} z^2$ has singularities at the four points $z^4 = -1$, which all lie on the circumference; choosing the value at the origin which is zero there, we have a similar expansion in a series, converging for points within the circle.

Similarly for the function $(1+z)^n$, which has -1 for a singularity.

(iv) Darboux's method* of derivation of the expansion of $f(z)$ in positive powers of $z-a$ depends upon the expression, obtained in § 15 (IV.), for the value of an integral. When applied to the general term

$$\frac{1}{2\pi i} \int \left(\frac{z-a}{t-a}\right)^{n+1} f(t)\, dt,$$

$= L$ say, it gives
$$L = \lambda r \left(\frac{z-a}{\zeta - a}\right)^{n+1} f(\zeta),$$

where ζ is some point on the circumference of the circle of radius r, and λ is a complex quantity of modulus not greater than unity. The modulus of $\dfrac{z-a}{\zeta-a}$ is less than a quantity which is less than unity; the terms of the series of moduli are therefore less than the terms of a converging geometric progression, so that they form a converging series; the limit of $|L|$, and therefore of L, can, with indefinite increase of n, be made zero and Taylor's expansion can be derived as before.

Ex. 1. Prove that the arithmetic mean of all values of $z^{-n} \sum\limits_{\nu=0}^{\infty} a_\nu z^\nu$, for points lying along a circle $|z| = r$ entirely contained in the region of continuity, is a_n. (Rouché, Gutzmer.)

Prove also that the arithmetic mean of the squares of the moduli of all values of $\sum\limits_{\nu=0}^{\infty} a_\nu z^\nu$, for points lying along a circle $|z| = r$ entirely contained in the region of continuity, is equal to the sum of the squares of the moduli of the terms of the series for a point on the circle. (Gutzmer.)

Ex. 2. Prove that the function
$$\sum_{n=0}^{\infty} a^n z^{n^2},$$
is finite and continuous, as well as all its derivatives, within and on the boundary of the circle $|z| = 1$, provided $|a| < 1$. (Fredholm.)

* Liouville, 3ème Sér., t. ii, (1876), pp. 291—312.

28. Laurent's theorem is as follows*:—

A function, which is holomorphic in a part of the plane bounded by two concentric circles with centre a and finite radii, can be expanded in the form of a double series of integral powers, positive and negative, of $z-a$, the series converging uniformly and unconditionally in the part of the plane between the circles.

Let z be any point within the region bounded by the two circles of radii R and R'; describe two concentric circles of radii r and r' such that

$$R > r > |z-a| > r' > R'.$$

Denoting by t and by s current points on the circumference of the outer and of the inner circles respectively, and considering the space which lies between them and includes the point z, we have, by § 20,

$$f(z) = \frac{1}{2\pi i}\int \frac{f(t)}{t-z}\,dt - \frac{1}{2\pi i}\int \frac{f(s)}{s-z}\,ds \;\ldots.(a),$$

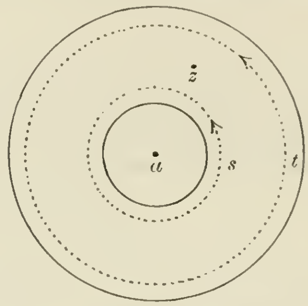

Fig. 11.

a negative sign being prefixed to the second integral because the direction indicated in the figure is the negative direction for the description of the inner circle regarded as a portion of the boundary.

Now we have

$$\frac{t-a}{t-z} = 1 + \frac{z-a}{t-a} + \left(\frac{z-a}{t-a}\right)^2 + \ldots\ldots + \left(\frac{z-a}{t-a}\right)^n + \frac{\left(\dfrac{z-a}{t-a}\right)^{n+1}}{1-\dfrac{z-a}{t-a}},$$

this expansion being adopted with a view to an infinite converging series, because $\left|\dfrac{z-a}{t-a}\right|$ is less than unity for all points t; and hence, by § 15,

$$\int \frac{f(t)}{t-z}\,dt = \int \frac{f(t)}{t-a}\,dt + (z-a)\int \frac{f(t)}{(t-a)^2}\,dt + \ldots\ldots + (z-a)^n \int \frac{f(t)}{(t-a)^{n+1}}\,dt$$
$$+ \int \frac{f(t)}{t-z}\left(\frac{z-a}{t-a}\right)^{n+1} dt.$$

Now each of the integrals, which are the respective coefficients of powers of $z-a$, is finite, because the subject of integration is everywhere finite along the circle of finite radius, by § 15 (IV.). Let the value of

$$\int \frac{f(t)}{(t-a)^{r+1}}\,dt$$

be $2\pi i u_r$: the quantity u_r is not necessarily equal to $f^r(a) \div r!$, because no

* *Comptes Rendus*, t. xvii, (1843), p. 939.

knowledge of the function or of its derivatives is given for a point within the innermost circle of radius R'. Thus

$$\frac{1}{2\pi i}\int\frac{f(t)}{t-z}dt = u_0 + (z-a)u_1 + (z-a)^2 u_2 + \ldots\ldots + (z-a)^n u_n$$
$$+\frac{1}{2\pi i}\int\frac{f(t)}{t-z}\left(\frac{z-a}{t-a}\right)^{n+1}dt.$$

The modulus of the last term is less than

$$\frac{M}{1-\frac{\rho}{r}}\left(\frac{\rho}{r}\right)^{n+1},$$

where ρ is $|z-a|$ and M is the greatest value of $|f(t)|$ for points along the circle. Because $\rho < r$, this quantity diminishes to zero with indefinite increase of n; and therefore the modulus of the expression

$$\frac{1}{2\pi i}\int\frac{f(t)}{t-z}dt - u_0 - (z-a)u_1 - \ldots\ldots - (z-a)^n u_n$$

becomes indefinitely small with increase of n. The quantity itself therefore vanishes in the same limiting circumstance; and hence

$$\frac{1}{2\pi i}\int\frac{f(t)}{t-z}dt = u_0 + (z-a)u_1 + \ldots\ldots + (z-a)^m u_m + \ldots\ldots,$$

so that the first of the integrals is equal to a series of positive powers. This series converges uniformly and unconditionally within the outer circle, for the modulus of the $(m+1)^{\text{th}}$ term is less than

$$M\left(\frac{\rho}{r}\right)^m,$$

which is the $(m+1)^{\text{th}}$ term of a converging series*.

As in § 27, the equivalence of the integral and the series can be affirmed only for points which lie within the outermost circle of radius R.

Again, we have

$$-\frac{z-a}{s-z} = 1 + \frac{s-a}{z-a} + \ldots\ldots + \left(\frac{s-a}{z-a}\right)^n + \frac{\left(\frac{s-a}{z-a}\right)^{n+1}}{1-\frac{s-a}{z-a}},$$

this expansion being adopted with a view to an infinite converging series, because $\left|\frac{s-a}{z-a}\right|$ is less than unity. Hence

$$-\frac{1}{2\pi i}\int\frac{f(s)}{s-z}ds = \frac{1}{z-a}\frac{1}{2\pi i}\int f(s)\,ds + \ldots\ldots + \frac{1}{(z-a)^{n+1}}\frac{1}{2\pi i}\int(s-a)^n f(s)\,ds$$
$$+\frac{1}{2\pi i}\int\left(\frac{s-a}{z-a}\right)^{n+1}\frac{f(s)}{z-s}ds.$$

* Chrystal, ii, 124.

The modulus of the last term is less than

$$\frac{M'}{1-\frac{r'}{\rho}}\left(\frac{r'}{\rho}\right)^{n+2},$$

where M' is the greatest value of $|f(s)|$ for points along the circle of radius r'. With indefinite increase of n, this modulus is ultimately zero; and thus, by an argument similar to the one which was applied to the former integral, we have

$$-\frac{1}{2\pi i}\int\frac{f(s)}{s-z}ds=\frac{v_1}{z-a}+\frac{v_2}{(z-a)^2}+\ldots\ldots+\frac{v_m}{(z-a)^m}+\ldots\ldots,$$

where v_m denotes the integral $\int(s-a)^{m-1}f(s)\,ds$ taken round the circle.

As in the former case, the series is one which converges uniformly and unconditionally; and the equivalence of the integral and the series is valid for points z that lie without the innermost circle of radius R'.

The coefficients of the various negative powers of $z-a$ are of the form

$$\frac{1}{2\pi i}\int\frac{f(s)}{\frac{1}{(s-a)^m}}\,d\left(\frac{1}{s-a}\right),$$

a form that suggests values of the derivatives of $f(s)$ at the point given by $\frac{1}{s-a}=0$, that is, at infinity. But the outermost circle is of finite radius; and no knowledge of the function at infinity, lying without the circle, is given, so that the coefficients of the negative powers may not be assumed to be the values of the derivatives at infinity, just as, in the former case, the coefficients u_r could not be assumed to be the values of the derivatives at the common centres of the circles.

Combining the expressions obtained for the two integrals, we have

$$f(z)=u_0+(z-a)\,u_1+(z-a)^2\,u_2+\ldots$$
$$+(z-a)^{-1}\,v_1+(z-a)^{-2}\,v_2+\ldots.$$

Both parts of the double series converge uniformly and unconditionally for all points in the region between the two circles, though not necessarily for points on the boundary of the region. The whole series therefore converges for all those points: and we infer the theorem as enunciated.

Conformably with the notation (§ 26, note) adopted to represent Taylor's expansion, a function $f(z)$ of the character required by Laurent's Theorem can be represented in the form

$$P_1(z-a)+P_2\left(\frac{1}{z-a}\right),$$

the series P_1 converging within the outer circle and the series P_2 converging without the inner circle; their sum converges for the ring-space between the circles.

29. The coefficient u_0 in the foregoing expansion is

$$\frac{1}{2\pi i}\int \frac{f(t)}{t-a}\,dt,$$

the integral being taken round the circle of radius r. We have

$$\frac{dt}{t-a} = id\theta$$

for points on the circle; and therefore

$$u_0 = \int \frac{d\theta}{2\pi} f(t),$$

so that
$$|u_0| < \int \frac{d\theta}{2\pi} M_t < M',$$

M' being the greatest value of M_t, the modulus of $f(t)$, for points along the circle. If M be the greatest value of $|f(z)|$ for any point in the whole region in which $f(z)$ is defined, so that $M' \leqslant M$, then we have

$$|u_0| < M,$$

that is, the modulus of the term independent of $z-a$ in the expansion of $f(z)$ by Laurent's Theorem is less than the greatest value of $|f(z)|$ at points in the region in which it is defined.

Again, $(z-a)^{-m} f(z)$ is a double series in positive and negative powers of $z-a$, the term independent of $z-a$ being u_m; hence, by what has just been proved, $|u_m|$ is less than $\rho^{-m} M$, where ρ is $|z-a|$. But the coefficient u_m does not involve z, and we can therefore choose a limit for any point z. The lowest limit will evidently be given by taking z on the outer circle of radius R, so that $|u_m| < MR^{-m}$. Similarly for the coefficients v_m; and therefore we have the result:—

If $f(z)$ be expanded as by Laurent's Theorem in the form

$$u_0 + \sum_{m=1}^{\infty} (z-a)^m u_m + \sum_{m=1}^{\infty} (z-a)^{-m} v_m,$$

then
$$|u_m| < MR^{-m}, \quad |v_m| < MR'^m,$$

where M is the greatest value of $|f(z)|$ at points within the region in which $f(z)$ is defined, and R and R' are the radii of the outer and the inner circles respectively.

30. The following proposition is practically a corollary from Laurent's Theorem:—

When a function is holomorphic over all the plane which lies outside a circle of centre a, it can be expanded in the form of a series of negative integral powers of $z-a$, the series converging uniformly and unconditionally everywhere in that part of the plane.

It can be deduced as the limiting case of Laurent's Theorem when the radius of the outer circle is made infinite. We then take r infinitely large, and substitute for t by the relation

$$t - a = re^{\theta i},$$

so that the first integral in the expression (a), p. 47, for $f(z)$ is

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{d\theta}{\dfrac{t-z}{t-a}} f(t).$$

Since the function is holomorphic over the whole of the plane which lies outside the assigned circle, $f(t)$ cannot be infinite at the circle of radius r when that radius increases indefinitely. If it tend towards a (finite) limit k, which must be uniform owing to the hypothesis as to the functional character of $f(z)$, then, since the limit of $(t-z)/(t-a)$ is unity, the preceding integral is equal to k.

The second integral in the same expression (a), p. 47, for $f(z)$ is unaltered by the conditions of the present proposition; hence we have

$$f(z) = k + (z-a)^{-1} v_1 + (z-a)^{-2} v_2 + \dots,$$

the series converging uniformly and unconditionally without the circle, though it does not necessarily converge on the circumference.

The series can be represented in the form

$$P\left(\frac{1}{z-a}\right),$$

conformably with the notation of § 26.

Of the three theorems in expansion which have been obtained, Cauchy's is the most definite, because the coefficients of the powers are explicitly obtained as values of the function and of its derivatives at an assigned point. In Laurent's theorem, the coefficients are not evaluated into simple expressions; and in the corollary from Laurent's theorem the coefficients are, as is easily proved, the values of the function and of its derivatives for infinite values of the variable. The essentially important feature of all the theorems is the expansibility of the function in series under assigned conditions.

31. It was proved (§ 21) that, when a function is holomorphic in any region of the plane bounded by a simple curve, it has an unlimited number of successive derivatives each of which is holomorphic in the region. Hence, by the preceding propositions, each such derivative can be expanded in converging series of integral powers, the series themselves being deducible by differentiation from the series which represents the function in the region.

In particular, when the region is a finite circle of centre a, within which $f(z)$ and consequently all the derivatives of $f(z)$ are expansible in converging series of positive integral powers of $z-a$, the coefficients of the various powers of $z-a$ are—save as to numerical factors—the values of the derivatives at the centre of the circle. Hence it appears that, *when a function is holomorphic over the area of a given circle, the values of the function and all its derivatives at any point z within the circle depend only upon the variable of the point and upon the values of the function and its derivatives at the centre.*

32. Some of the classes of points in a plane that usually arise in connection with uniform functions may now be considered.

(i) A point a in the plane may be such that a function of the variable has a determinate finite value there, always independent of the path by which the variable reaches a; the point a is called an *ordinary* point* of the function. The function, supposed continuous in the vicinity of a, is continuous at a: and it is said to behave *regularly* in the vicinity of an ordinary point.

Let such an ordinary point a be at a distance d, not infinitesimal, from the nearest of the singular points (if any) of the function; and let a circle of centre a and radius just less than d be drawn. The part of the z-plane lying within this circle is called† the *domain* of a; and the function, holomorphic within this circle, is said to behave regularly (or to be regular) in the domain of a. From the preceding section, we infer that a function and its derivatives can be expanded in a converging series of positive integral powers of $z-a$ for all points z in the domain of a, an ordinary point of the function: and the coefficients in the series are the values of the function and its derivatives at a.

The property possessed by the series—that it contains only positive integral powers of $z-a$—at once gives a test that is both necessary and sufficient to determine whether a point is an ordinary point. *If the point a be ordinary, the limit of $(z-a)f(z)$ necessarily is zero when z becomes equal to a.* This necessary condition is also sufficient to ensure that the point is an ordinary point of the function $f(z)$, supposed to be uniform; for, since $f(z)$ is holomorphic, the function $(z-a)f(z)$ is also holomorphic and can be expanded in a series

$$u_0 + u_1(z-a) + u_2(z-a)^2 + \dots,$$

converging in the domain of a. The quantity u_0 is zero, being the value of $(z-a)f(z)$ at a and this vanishes by hypothesis; hence

$$(z-a)f(z) = (z-a)\{u_1 + u_2(z-a) + \dots\},$$

shewing that $f(z)$ is expressible as a series of positive integral powers of $z-a$ converging within the domain of a, or, in other words, that $f(z)$ certainly has a for an ordinary point in consequence of the condition being satisfied.

* Sometimes a *regular* point.

† The German title is *Umgebung*, the French is *domaine*.

(ii) A point a in the plane may be such that a function $f(z)$ of the variable has a determinate infinite value there, always independent of the path by which the variable reaches a, the function behaving regularly for points in the vicinity of a; then $\frac{1}{f(z)}$ has a determinate zero value there, so that a is an ordinary point of $\frac{1}{f(z)}$. The point a is called a *pole* (§ 12) or an *accidental singularity** of the function.

A test, necessary and sufficient to settle whether a point is an accidental singularity of a function will subsequently (§ 42) be given.

(iii) A point a in the plane may be such that $f(z)$ has not a determinate value there, either finite or infinite, though the function is regular for all points in the vicinity of a that are not at merely infinitesimal distances. Thus the origin is of this nature for the functions $e^{\frac{1}{z}}$, $\text{sn}\,\frac{1}{z}$.

Such a point is called† an *essential singularity* of the function. No hypothesis is postulated as to the character of the function for points at infinitesimal distances from the essential singularity, while the relation of the singularity to the function naturally depends upon this character at points near it. There may thus be various kinds of essential singularities all included under the foregoing definition; their classification is effected through the consideration of the character of the function at points in their immediate vicinity. (See § 88.)

One sufficient test of discrimination between an accidental singularity and an essential singularity is furnished by the determinateness of the value at the point. If the reciprocal of the function have the point for an ordinary point, the point is an accidental singularity—it is, indeed, a zero for the reciprocal. But when the point is an essential singularity, the value of the reciprocal of the function is not determinate there; and then the reciprocal, as well as the function, has the point for an essential singularity.

33. It may be remarked at once that there must be at least one infinite value among the values which a function can assume at an essential singularity. For if $f(z)$ cannot be infinite at a, then the limit of $(z-a)f(z)$ is zero when $z=a$, no matter what the non-infinite values of $f(z)$ may be, that is, the limit is a determinate zero. The function $(z-a)f(z)$ is regular in the vicinity of a: hence by the foregoing test for an ordinary point, the point a is ordinary and the value of the uniform function $f(z)$ is

* Weierstrass, *Abh. aus der Functionenlehre*, p. 2, to whom the name is due, calls it *ausserwesentliche singuläre Stelle;* the term *non-essential* is suggested by Mr Cathcart, Harnack, p. 148.

† Weierstrass, l.c., calls it *wesentliche singuläre Stelle.*

determinate, contrary to hypothesis. Hence the function must have at least one infinite value at an essential singularity.

Further, *a uniform function must be capable of assuming any value C at an essential singularity.* For an essential singularity of $f(z)$ is also an essential singularity of $f(z) - C$ and therefore also of $\frac{1}{f(z) - C}$. The last function must have at least one infinite value among the values that it can assume at the point; and, for this infinite value, we have $f(z) = C$ at the point, so that $f(z)$ assumes the assigned value C at the essential singularity*.

34. Let $f(z)$ denote the function represented by a series of powers $P_1(z-a)$, the circle of convergence of which is the domain of the ordinary point a of the function. The region over which the function $f(z)$ is holomorphic may extend beyond the domain of a, although the circumference bounding that domain is the greatest of centre a that can be drawn within the region. The region evidently cannot extend beyond the domain of a in all directions.

Take an ordinary point b in the domain of a. The value at b of the function $f(z)$ is given by the series $P_1(b-a)$, and the values at b of all its derivatives are given by the derived series. All these series converge within the domain of a and they are therefore finite at b; and their expressions involve the values at a of the function and its derivatives.

Let the domain of b be formed. The domain of b may be included in that of a, and then its bounding circle will touch the bounding circle of the domain of a internally. If the domain of b be not entirely included in that of a, part of it will lie outside the domain of a; but it cannot include the whole of the domain of a unless its bounding circumference touch that of the domain of a externally, for otherwise it would extend beyond a in all directions, a result inconsistent with the construction of the domain of a. Hence there must be points excluded from the domain of a which are also excluded from the domain of b.

For all points z in the domain of b, the function can be represented by a series, say $P_2(z-b)$, the coefficients of which are the values at b of the function and its derivatives. Since these values are partially dependent upon the corresponding values at a, the series representing the function may be denoted by $P_2(z-b, a)$.

At a point z in the domain of b lying also in the domain of a, the two series $P_1(z-a)$ and $P_2(z-b, a)$ must furnish the same value for the function $f(z)$; and therefore no new value is derived from the new series P_2

* Weierstrass, l.c., pp. 50—52; Durège, *Elemente der Theorie der Funktionen*, p. 119; Hölder, *Math. Ann.*, t. xx, (1882), pp. 138—143; Picard, "Mémoire sur les fonctions entières," *Annales de l'École Norm. Sup.*, 2me Sér., t. ix, (1880), pp. 145—166, which, in this regard, should be consulted in connection with the developments in Chapter V. See also § 62.

which cannot be derived from the old series P_1. For all such points the new series is of no advantage; and hence, if the domain of b be included in that of a, the construction of the series $P_2(z-b, a)$ is superfluous. Hence in choosing the ordinary point b in the domain of a we choose a point, if possible, that will not have its domain included in that of a.

At a point z in the domain of b, which does not lie in the domain of a, the series $P_2(z-b, a)$ gives a value for $f(z)$ which cannot be given by $P_1(z-a)$. The new series P_2 then gives an additional representation of the function; it is called* a *continuation* of the series which represents the function in the domain of a. The derivatives of P_2 give the values of $f(z)$ for points in the domain of b.

It thus appears that, if the whole of the domain of b be not included in that of a, the function can, by the series which is valid over the whole of the new domain, be continued into that part of the new domain excluded from the domain of a.

Now take a point c within the region occupied by the combined domains of a and b; and construct the domain of c. In the new domain, the function can be represented by a new series, say $P_3(z-c)$, or, since the coefficients (being the values at c of the function and of its derivatives) involve the values at a and possibly also the values at b of the function and of its derivatives, the series representing the function may be denoted by $P_3(z-c, a, b)$. Unless the domain of c include points, which are not included in the combined domains of a and b, the series P_3 does not give a value of the function which cannot be given by P_1 or P_2: we therefore choose c, if possible, so that its domain will include points not included in the earlier domains. At such points z in the domain of c as are excluded from the combined domains of a and b, the series $P_3(z-c, a, b)$ gives a value for $f(z)$ which cannot be derived from P_1 or P_2; and thus the new series is a continuation of the earlier series.

Proceeding in this manner by taking successive points and constructing their domains, we can reach all parts of the plane connected with one another where the function preserves its holomorphic character; their combined aggregate is called† the *region of continuity* of the function. With each domain, constructed so as to include some portion of the region of continuity not included in the earlier domains, a series is associated, which is a continuation of the earlier series and, as such, gives a value of the function not deducible from those earlier series; and all the associated series are ultimately derived from the first.

* Biermann, *Theorie der analytischen Functionen*, p. 170, which may be consulted in connection with the whole of § 34; the German word is *Fortsetzung*.

† Weierstrass, l.c., p. 1.

Each of the continuations is called an *Element* of the function. The aggregate of all the distinct elements is called a *monogenic analytic function*: it is evidently the complete analytical expression of the function in its region of continuity.

Let z be any point in the region of continuity, not necessarily in the circle of convergence of the initial element of the function; a value of the function at z can be obtained through the continuations of that initial element. In the formation of each new domain (and therefore of each new element) a certain amount of arbitrary choice is possible; and there may, moreover, be different sets of domains which, taken together in a set, each lead to z from the initial point. When the analytic function is uniform, as before defined (§ 12), the same value at z for the function is obtained, whatever be the set of domains. If there be two sets of elements, differently obtained, which give at z different values for the function, then the analytic function is multiform, as before defined (§ 12); but not every change in a set of elements leads to a change in the value at z of a multiform function, and the analytic function is uniform within such a region of the plane as admits only equivalent changes of elements.

The whole process is reversible when the function is uniform. We can pass back from any point to any earlier point by the use, if necessary, of intermediate points. Thus, if the point a in the foregoing explanation be not included in the domain of b (there supposed to contribute a continuation of the first series), an intermediate point on a line, drawn in the region of continuity so as to join a and b, would be taken; and so on, until a domain is formed which does include a. The continuation, associated with this domain, must give at a the proper value for the function and its derivatives, and therefore for the domain of a the original series $P_1(z-a)$ will be obtained, that is, $P_1(z-a)$ can be deduced from $P_2(z-b, a)$ the series in the domain of b. This result is general, so that *any one of the continuations of a uniform function, represented by a power-series, can be derived from any other;* and therefore the expression of such a function in its region of continuity is potentially given by one element, for all the distinct elements can be derived from any one element.

35. It has been assumed that the property, characteristic of some of the functions adduced as examples, of possessing either accidental or essential singularities, is characteristic of all functions; it will be proved (§ 40) to hold for every uniform function which is not a mere constant.

The singularities limit the region of continuity; for each of the separate domains is, from its construction, limited by the nearest singularity, and the combined aggregate of the domains constitutes the region of continuity when they form a continuous space*. Hence the complete boundary of the region of continuity is the aggregate of the singularities of the function†.

It may happen that a function has no singularity except at infinity; the region of continuity then extends over the whole finite part of the plane but it does not include the point at infinity.

It follows from the foregoing explanations that, in order to know a uniform analytic function, it is necessary to know some element of the function, which has been shewn to be potentially sufficient for the derivation of the full expression of the function and for the construction of its region of continuity.

36. The method of continuation of a function, which has just been described, is quite general; there is one particular continuation, which is important in investigations on conformal representations. It is contained in the following proposition, due to Schwarz‡:—

If an analytic function w of z be defined only for a region S' in the positive half of the z-plane and if continuous real values of w correspond to continuous real values of z, then w can be continued across the axis of real quantities.

Consider a region S'', symmetrical with S' relative to the axis of real quantities (fig. 12). Then a function is defined for the region S'' by associating a value w_0, the conjugate of w, with z_0, the conjugate of z.

Let the two regions be combined along the portion of the axis of x which is their common boundary; they then form a single region $S' + S''$.

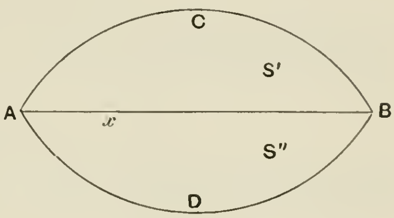

Fig. 12.

Consider the integrals

$$\frac{1}{2\pi i}\int_{S'} \frac{w}{z-\zeta}\,dz \quad\text{and}\quad \frac{1}{2\pi i}\int_{S''} \frac{w_0}{z_0-\zeta}\,dz_0,$$

taken round the boundaries of S' and of S'' respectively. Since w is

* Cases occur in which the region of continuity of a function is composed of isolated spaces, each continuous in itself, but not continuous into one another. The consideration of such cases will be dealt with briefly hereafter, and they are assumed excluded for the present: meanwhile, it is sufficient to note that each continuous space could be derived from an element belonging to some domain of that space and that a new element would be needed for a new space.

† See Weierstrass, l.c., pp. 1—3; Mittag-Leffler, "Sur la représentation analytique des fonctions monogènes uniformes d'une variable indépendante," *Acta Math.*, t. iv, (1884), pp. 1 et seq., especially pp. 1—8.

‡ *Crelle*, t. lxx, (1869), pp. 106, 107, and *Ges. Math. Abh.*, t. ii, pp. 66—68. See also Darboux, *Théorie générale des surfaces*, t. i, § 130.

continuous over the whole area of S' as well as along its boundary and likewise w_0 relative to S'', it follows that, if the point ζ be in S', the value of the first integral is $w(\zeta)$ and that of the second is zero; while, if ζ lie in S'', the value of the first integral is zero and that of the second is $w_0(\zeta)$. Hence the sum of the two integrals represents a unique function of a point in either S' or S''. But the value of the first integral is

$$\frac{1}{2\pi i}\int_B^{A} {}_{(C)}\frac{wdz}{z-\zeta}+\frac{1}{2\pi i}\int_A^B\frac{w(x)\,dx}{x-\zeta},$$

the first being taken along the curve BCA and the second along the axis AxB; and the value of the second integral is

$$\frac{1}{2\pi i}\int_B^A\frac{w_0(x)\,dx}{x-\zeta}+\frac{1}{2\pi i}\int_A^B {}_{(D)}\frac{w_0dz_0}{z_0-\zeta},$$

the first being taken along the axis BxA and the second along the curve ADB. But

$$w_0(x)=w(x),$$

because conjugate values w and w_0 correspond to conjugate values of the argument by definition of w_0 and because w (and therefore also w_0) is real and continuous when the argument is real and continuous. Hence when the sum of the four integrals is taken, the two integrals corresponding to the two descriptions of the axis of x cancel and we have as the sum

$$\frac{1}{2\pi i}\int_B^A {}_{(C)}\frac{wdz}{z-\zeta}+\frac{1}{2\pi i}\int_A^B {}_{(D)}\frac{w_0dz_0}{z_0-\zeta},$$

and this sum represents a unique function of a point in $S'+S''$. These two integrals, taken together, are

$$\frac{1}{2\pi i}\int\frac{w'dz}{z-\zeta},$$

taken round the whole contour of $S'+S''$, where w' is equal to $w(\zeta)$ in the positive half of the plane and to $w_0(\zeta)$ in the negative half.

For all points ζ in the whole region $S'+S''$, this integral represents a single uniform, finite, continuous function of ζ; its value is $w(\zeta)$ in the positive half of the plane and is $w_0(\zeta)$ in the negative half; and therefore $w_0(\zeta)$ is the continuation into the negative half of the plane of the function, which is defined by $w(\zeta)$ for the positive half.

For a point c on the axis of x, we have

$$w(z)-w(c)=A(z-c)+B(z-c)^2+C(z-c)^3+\ldots;$$

and all the coefficients $A, B, C,\ldots$ are real. If, in addition, w be such a function of z that the inverse functional relation makes z a uniform analytic function of w, it is easy to see that A must not vanish, so that the functional relation may be expressed in the form

$$w(z)-w(c)=(z-c)\,P(z-c),$$

where $P(z-c)$ does not vanish when $z=c$.

CHAPTER IV.

General properties of Uniform Functions, particularly of those without Essential Singularities.

37. In the derivation of the general properties of functions, which will be deduced in the present and the next three chapters from the results already obtained, it is to be supposed, in the absence of any express statement to other effect, that the functions are uniform, monogenic and, except at either accidental or essential singularities, continuous*.

Theorem I. *A function, which is constant throughout any region of the plane not infinitesimal in area, or which is constant along any line not infinitesimal in length, is constant throughout its region of continuity.*

For the first part of the theorem, we take any point a in the region of the plane where the function is constant, and we draw a circle of centre a and of any radius, provided only that the circle remains within the region of continuity of the function. At any point z within this circle we have

$$f(z) = f(a) + (z-a) f'(a) + \frac{(z-a)^2}{2!} f''(a) + \dots,$$

a converging series the coefficients of which are the values of the function and its derivatives at a. But

$$f'(a) = \text{Limit of } \frac{f(a+\delta a) - f(a)}{\delta a},$$

which is zero because $f(a+\delta a)$ is the same constant as $f(a)$: so that the first derivative is zero at a. Similarly, all the derivatives can be shewn to be zero at a; hence the above series after its first term is evanescent, and we have

$$f(z) = f(a),$$

that is, the function preserves its constant value throughout its region of continuity.

The second result follows in the same way, when once the derivatives are proved zero. Since the function is monogenic, the value of the first and

* It will be assumed, as in § 35 (note, p. 57), that the region of continuity consists of a single space; functions, with regions of continuity consisting of a number of separated spaces, will be discussed in Chap. VII.

of each of the successive derivatives will be obtained, if we make the differential element of the independent variable vanish along the line.

Now, if a be a point on the line and $a+\delta a$ a consecutive point, we have $f(a+\delta a)=f(a)$; hence $f'(a)$ is zero. Similarly the first derivative at any other point on the line is zero. Therefore we have $f'(a+\delta a)=f'(a)$, for each has just been proved to be zero: hence $f''(a)$ is zero; and similarly the value of the second derivative at any other point on the line is zero. So on for all the derivatives: the value of each of them at a is zero.

Using the same expansion as before and inserting again the zero values of all the derivatives at a, we find that

$$f(z)=f(a),$$

so that under the assigned condition the function preserves its constant value throughout its region of continuity.

It should be noted that, if in the first case the area be so infinitesimally small and in the second the line be so infinitesimally short that consecutive points cannot be taken, then the values at a of the derivatives cannot be proved to be zero and the theorem cannot then be inferred.

COROLLARY I. *If two functions have the same value over any area of their common region of continuity which is not infinitesimally small or along any line in that region which is not infinitesimally short, then they have the same values at all points in their common region of continuity.*

This is at once evident: for their difference is zero over that area or along that line and therefore, by the preceding theorem, their difference has a constant zero value, that is, the functions have the same values, everywhere in their common region of continuity.

But two functions can have the same values at a succession of isolated points, without having the same values everywhere in their common region of continuity; in such a case the theorem does not apply, the reason being that the fundamental condition of equality over a continuous area or along a continuous line is not satisfied.

COROLLARY II. *A function cannot be zero over any continuous area of its region of continuity which is not infinitesimal or along any line in that region which is not infinitesimally short without being zero everywhere in its region of continuity.*

This corollary is deduced in the same manner as that which precedes.

If, then, there be a function which is evidently not zero everywhere, we conclude that *its zeros are isolated points though such points may be multiple zeros.*

Further, *in any finite area of the region of continuity of a function that is subject to variation, there can be at most only a finite number of its zeros, when* *no point of the boundary of the area is infinitesimally near an essential singularity.* For if there were an infinite number of such points in any such region, there must be a cluster in at least one area or a succession along at least one line, infinite in number and so close as to constitute a continuous area or a continuous line where the function is everywhere zero. This would require that the function should be zero everywhere in its region of continuity, a condition excluded by the hypothesis.

And it immediately follows that the points (other than those infinitesimally near an essential singularity) in a region of continuity, at which a function assumes any the same value, are isolated points; and that only a finite number of such points occur in any finite area.

38. THEOREM II. *The multiplicity m of any zero a of a function is finite provided the zero be an ordinary point of the function, which is not zero throughout its region of continuity; and the function can be expressed in the form*

$$(z-a)^m \phi(z),$$

where $\phi(z)$ is holomorphic in the vicinity of a, and a is not a zero of $\phi(z)$.

Let $f(z)$ denote the function; since a is a zero, we have $f(a)=0$. Suppose that $f'(a)$, $f''(a)$,...... vanish: in the succession of the derivatives of f, one of finite order must be reached which does not have a zero value. Otherwise, if all vanish, then the function and all its derivatives vanish at a; the expansion of $f(z)$ in powers of $z-a$ leads to zero as the value of $f(z)$, that is, the function is everywhere zero in the region of continuity, if all the derivatives vanish at a.

Let, then, the mth derivative be the first in the natural succession which does not vanish at a, so that m is finite. Using Cauchy's expansion, we have

$$\begin{aligned} f(z) &= \frac{(z-a)^{(m)}}{m!} f^m(a) + \frac{(z-a)^{(m+1)}}{(m+1)!} f^{m+1}(a) + \dots \\ &= (z-a)^m \phi(z), \end{aligned}$$

where $\phi(z)$ is a function that does not vanish with a and, being the quotient of a converging series by a monomial factor, is holomorphic in the immediate vicinity of a.

COROLLARY I. *If infinity be a zero of a function of multiplicity m and at the same time be an ordinary point of the function, then the function can be expressed in the form*

$$z^{-m} \phi\left(\frac{1}{z}\right),$$

where $\phi\left(\frac{1}{z}\right)$ is a function that is continuous and non-evanescent for infinitely large values of z.

The result can be derived from the expansion in § 30 in the same way as the foregoing theorem from Cauchy's expansion.

COROLLARY II. *The number of zeros of a function, account being taken of their multiplicity, which occur within a finite area of the region of continuity of the function, is finite, when no point of the boundary of the area is infinitesimally near an essential singularity.*

By Corollary II. of § 37, the number of distinct zeros in the limited area is finite, and, by the foregoing theorem, the multiplicity of each is finite; hence, when account is taken of their respective multiplicities, the total number of zeros is still finite.

The result is, of course, a known result for an algebraical polynomial; but the functions in the enunciation are not restricted to be of the type of algebraical polynomials.

Note. It is important to notice, both for the Theorem and for Corollary I., that the zero is an ordinary point of the function under consideration; the implication therefore is that the zero is a definite zero and that in the immediate vicinity of the point the function can be represented in the form $P(z-a)$ or $P\left(\frac{1}{z}\right)$, the function $P(a-a)$ or $P\left(\frac{1}{\infty}\right)$ being always a definite zero.

Instances do occur for which this condition is not satisfied. The point may not be an ordinary point, and the zero value may be an indeterminate zero; or zero may be only one of a set of distinct values though everywhere in the vicinity the function is regular. Thus the analysis of § 13 shews that $z=a$ is a point where the function $\operatorname{sn} \frac{1}{z-a}$ has any number of zero values and any number of infinite values, and there is no indication that there are not also other values at the point. In such a case the preceding proposition does not apply; there may be no limit to the order of multiplicity of the zero, and we certainly cannot infer that any finite integer m can be obtained such that

$$(z-a)^{-m}\phi(z)$$

is finite at the point. Such a point is (§ 32) an essential singularity of the function.

39. THEOREM III. *A multiple zero of a function is a zero of its derivative; and the multiplicity for the derivative is less or is greater by unity according as the zero is not or is at infinity.*

If a be a point in the finite part of the plane which is a zero of $f(z)$ of multiplicity n, we have

$$f(z) = (z-a)^n \phi(z),$$

and therefore $$f'(z) = (z-a)^{n-1}\{n\phi(z) + (z-a)\phi'(z)\}.$$

The coefficient of $(z-a)^{n-1}$ is holomorphic in the immediate vicinity of a and does not vanish for a; hence a is a zero for $f'(z)$ of decreased multiplicity $n-1$.

If $z=\infty$ be a zero of $f(z)$ of multiplicity r, then

$$f(z)=z^{-r}\phi\left(\frac{1}{z}\right),$$

where $\phi\left(\frac{1}{z}\right)$ is holomorphic for very large values of z and does not vanish at infinity. Therefore

$$\begin{aligned} f'(z) &= -rz^{-r-1}\phi\left(\frac{1}{z}\right)-z^{-r-2}\phi'\left(\frac{1}{z}\right) \\ &= z^{-r-1}\left\{-r\phi\left(\frac{1}{z}\right)-\frac{1}{z}\phi'\left(\frac{1}{z}\right)\right\}. \end{aligned}$$

The coefficient of z^{-r-1} is holomorphic for very large values of z, and does not vanish at infinity; hence $z=\infty$ is a zero of $f'(z)$ of increased multiplicity $r+1$.

Corollary I. If a function be finite at infinity, then $z=\infty$ is a zero of the first derivative of multiplicity at least two.

Corollary II. If a be a finite zero of $f(z)$ of multiplicity n, we have

$$\frac{f'(z)}{f(z)}=\frac{n}{z-a}+\frac{\phi'(z)}{\phi(z)}.$$

Now a is not a zero of $\phi(z)$; and therefore $\frac{\phi'(z)}{\phi(z)}$ is finite, continuous, uniform and monogenic in the immediate vicinity of a. Hence, taking the integral of both members of the equation round a circle of centre a and of radius so small as to include no infinity and no zero, other than a, of $f(z)$—and therefore no zero of $\phi(z)$—we have, by § 17 and Ex. 2, § 25,

$$\frac{1}{2\pi i}\int\frac{f'(z)}{f(z)}dz=n.$$

40. THEOREM IV. *A function must have an infinite value for some finite or infinite value of the variable.*

If M be a finite maximum value of the modulus for points in the plane, then (§ 22) we have

$$|f^{(n)}(a)|<\frac{n!\,M}{r^n},$$

where r is the radius of an arbitrary circle of centre a, provided the whole of the circle is in the region of continuity of the function. But as the function is uniform, monogenic, finite and continuous everywhere, this radius can be increased indefinitely; when this increase takes place, the limit of

$$|f^{(n)}(a)|$$

is zero and therefore $f^{(n)}(a)$ vanishes. This is true for all the indices 1, 2,... of the derivatives.

Now the function can be represented at any point z in the vicinity of a by the series

$$f(a) + (z-a) f'(a) + \frac{(z-a)^2}{2!} f''(a) + \ldots,$$

which degenerates, under the present hypothesis, to $f(a)$, so that the function is everywhere constant. Hence, if a function has not an infinity somewhere in the plane, it must be a constant.

The given function is not a constant; and therefore there is no finite limit to the maximum value of its modulus, that is, the function acquires an infinite value somewhere in the plane.

COROLLARY I. *A function must have a zero value for some finite or infinite value of the variable.*

For the reciprocal of a uniform monogenic analytic function is itself a uniform monogenic analytic function; and the foregoing proposition shews that this reciprocal must have an infinite value for some value of the variable, which therefore is a zero of the function.

COROLLARY II. *A function must assume any assigned value at least once.*

COROLLARY III. *Every function which is not a mere constant must have at least one singularity, either accidental or essential.* For it must have an infinite value: if this be a determinate infinity, the point is an accidental singularity (§ 32); if it be an infinity among a set of values at the point, the point is an essential singularity (§§ 32, 33).

41. Among the infinities of a function, the simplest class is that constituted by its accidental singularities, already defined (§ 32) by the property that, in the immediate vicinity of such a point, the reciprocal of the function is regular, the point being an ordinary (zero) point for that reciprocal.

THEOREM V. *A function, which has a point c for an accidental singularity, can be expressed in the form*

$$(z-c)^{-n} \phi(z),$$

where n is a finite positive integer and $\phi(z)$ is a continuous function in the vicinity of c.

Since c is an accidental singularity of the function $f(z)$, the function $\dfrac{1}{f(z)}$ is regular in the vicinity of c and is zero there (§ 32). Hence, by § 38, there is a finite limit to the multiplicity of the zero, say n (which is a positive integer), and we have

$$\frac{1}{f(z)} = (z-c)^n \chi(z),$$

where $\chi(z)$ is uniform, monogenic and continuous in the vicinity of c and is not zero there. The reciprocal of $\chi(z)$, say $\phi(z)$, is also uniform, monogenic and continuous in the vicinity of c, which is an ordinary point for $\phi(z)$; hence we have

$$f(z) = (z-c)^{-n}\phi(z),$$

which proves the theorem.

The finite positive integer n measures the *multiplicity* of the accidental singularity at c, which is sometimes said to be of multiplicity n or of order n.

Another analytical expression for $f(z)$ can be derived from that which has just been obtained. Since c is an ordinary point for $\phi(z)$ and not a zero, this function can be expanded in a series of ascending, positive, integral powers of $z-c$, converging in the vicinity of c, in the form

$$\begin{aligned}\phi(z) &= P(z-c)\\ &= u_0 + u_1(z-c) + \ldots + u_{n-1}(z-c)^{n-1} + u_n(z-c)^n + \ldots\\ &= u_0 + u_1(z-c) + \ldots + u_{n-1}(z-c)^{n-1} + (z-c)^n Q(z-c),\end{aligned}$$

where $Q(z-c)$, a series of positive, integral, powers of $z-c$ converging in the vicinity of c, is a monogenic analytic function of z. Hence we have

$$f(z) = \frac{u_0}{(z-c)^n} + \frac{u_1}{(z-c)^{n-1}} + \ldots + \frac{u_{n-1}}{z-c} + Q(z-c),$$

the indicated expression for $f(z)$, valid in the immediate vicinity of c, where $Q(z-c)$ is uniform, finite, continuous and monogenic.

COROLLARY. *A function, which has $z=\infty$ for an accidental singularity of multiplicity n, can be expressed in the form*

$$z^n\phi\left(\frac{1}{z}\right),$$

where $\phi\left(\frac{1}{z}\right)$ is a continuous function for very large values of $|z|$, and is not zero when $z=\infty$. It can also be expressed in the form

$$a_0z^n + a_1z^{n-1} + \ldots + a_{n-1}z + Q\left(\frac{1}{z}\right),$$

where $Q\left(\frac{1}{z}\right)$ is uniform, finite, continuous and monogenic for very large values of $|z|$.

The derivation of the form of the function in the vicinity of an accidental singularity has been made to depend upon the form of the reciprocal of the function. Whatever be the (finite) order of that point as a zero of the reciprocal, it is assumed that other zeros of the reciprocal are not at merely infinitesimal distances from the point, that is, that other infinities of the function are not at merely infinitesimal distances from the point.

Hence the accidental singularities of a function are isolated points; and there is only a finite number of them in any limited portion of the plane.

42. We can deduce a criterion which determines whether a given singularity of a function $f(z)$ is accidental or essential.

When the point is in the finite part of the plane, say at c, and a finite positive integer n can be found such that

$$(z-c)^n f(z)$$

is not infinite at c, then c is an accidental singularity.

When the point is at infinity and a finite positive integer n can be found such that

$$z^{-n} f(z)$$

is not infinite when $z=\infty$, then $z=\infty$ is an accidental singularity.

If one of these conditions be not satisfied, the singularity at the point is essential. But it must not be assumed that the failure of the limitation to finiteness in the multiplicity of the accidental singularity is the only source or the complete cause of essential singularity.

Since the association of a single factor with the function is effective in preventing an infinite value at the point when one of the conditions is satisfied, it is justifiable to regard the discontinuity of the function at the point as not essential and to call the singularity either non-essential or accidental (§ 32).

43. THEOREM VI. *The poles of a function, that lie in the finite part of the plane, are all the poles (of increased multiplicity) of the derivatives of the function that lie in the finite part of the plane.*

Let c be a pole of the function $f(z)$ of multiplicity p: then, for any point z in the vicinity of c,

$$f(z)=(z-c)^{-p}\phi(z),$$

where $\phi(z)$ is holomorphic in the vicinity of c, and does not vanish for $z=c$. Then we have

$$\begin{aligned} f'(z) &= (z-c)^{-p}\phi'(z) - p(z-c)^{-p-1}\phi(z) \\ &= (z-c)^{-p-1}\{(z-c)\phi'(z) - p\phi(z)\} \\ &= (z-c)^{-p-1}\chi(z), \end{aligned}$$

where $\chi(z)$ is holomorphic in the vicinity of c, and does not vanish for $z=c$.

Hence c is a pole of $f'(z)$ of multiplicity $p+1$. Similarly it can be shewn to be a pole of $f^{(r)}(z)$ of multiplicity $p+r$.

This proves that all the poles of $f(z)$ in the finite part of the plane are poles of its derivatives. It remains to prove that a derivative cannot have a pole which the original function does not also possess.

Let α be a pole of $f'(z)$ of multiplicity m: then, in the vicinity of α, $f'(z)$ can be expressed in the form

$$(z-\alpha)^{-m}\psi(z),$$

where $\psi(z)$ is holomorphic in the vicinity of α and does not vanish for $z=\alpha$. Thus

$$\psi(z)=\psi(\alpha)+(z-\alpha)\psi'(\alpha)+\ldots,$$

and therefore
$$f'(z)=\frac{\psi(\alpha)}{(z-\alpha)^m}+\frac{\psi'(\alpha)}{(z-\alpha)^{m-1}}+\ldots,$$

so that, integrating, we have

$$f(z)=-\frac{\psi(\alpha)}{m(z-\alpha)^{m-1}}-\frac{\psi'(\alpha)}{(m-1)(z-\alpha)^{m-2}}-\ldots,$$

that is, α is a pole of $f(z)$.

An apparent exception occurs in the case when m is unity: for then we have

$$f'(z)=\frac{\psi(\alpha)}{z-\alpha}+\psi'(\alpha)+\frac{z-\alpha}{2!}\psi''(\alpha)+\ldots,$$

the integral of which leads to

$$f(z)=\psi(\alpha)\log(z-\alpha)+\ldots,$$

so that $f(z)$ is no longer uniform, contrary to hypothesis. Hence *a derivative cannot have a simple pole in the finite part of the plane;* and so the exception is excluded.

The theorem is thus proved.

Corollary I. *The r^{th} derivative of a function cannot have a pole in the finite part of the plane of multiplicity less than $r+1$.*

Corollary II. *If c be a pole of $f(z)$ of any order of multiplicity μ and if $f^{(r)}(z)$ be expressed in the form*

$$\frac{a_0}{(z-c)^{\mu+r}}+\frac{a_1}{(z-c)^{\mu+r-1}}+\ldots\ldots,$$

there are no terms in this expression with the indices $-1, -2, \ldots\ldots, -r$.

Corollary III. If c be a pole of $f(z)$ of multiplicity p, we have

$$\frac{f'(z)}{f(z)}=\frac{-p}{z-c}+\frac{\phi'(z)}{\phi(z)},$$

where $\phi(z)$ is a holomorphic function that does not vanish for $z=c$, so that $\frac{\phi'(z)}{\phi(z)}$ is a holomorphic function in the vicinity of c. Taking the integral of $\frac{f'(z)}{f(z)}$ round a circle, with c for centre, with radius so small as to exclude all other poles or zeros of the function $f(z)$, we have

$$\frac{1}{2\pi i}\int\frac{f'(z)}{f(z)}dz=-p.$$

Corollary IV. If a simple closed curve include a number N of zeros of a uniform function $f(z)$ and a number P of its poles, in both of which numbers account is taken of possible multiplicity, and if the curve contain no essential singularity of the function, then

$$\frac{1}{2\pi i}\int \frac{f'(z)}{f(z)}\, dz = N - P,$$

the integral being taken round the curve.

The only infinities of the function $\frac{f'(z)}{f(z)}$ within the curve are the zeros and the poles of $f(z)$. Round each of these draw a circle of radius so small as to include it but no other infinity; then, by Cor. II. § 18, the integral round the closed curve is the sum of the values when taken round these circles. By the Corollary II. § 39 and by the preceding Corollary III., the sum of these values is

$$\begin{aligned} &= \Sigma n - \Sigma p \\ &= N - P. \end{aligned}$$

It is easy to infer the known theorem that the number of roots of an algebraical polynomial of order n is n, as well as the further result that $2\pi(N-P)$ is the variation of the argument of $f(z)$ as z describes the closed curve in a positive sense.

Ex. Prove that, if $F(z)$ be holomorphic over an area, of simple contour, which contains roots $a_1, a_2, \ldots$ of multiplicity $m_1, m_2, \ldots$ and poles $c_1, c_2, \ldots$ of multiplicity $p_1, p_2, \ldots$ respectively of a function $f(z)$ which has no other singularities within the contour, then

$$\frac{1}{2\pi i}\int F(z)\frac{f'(z)}{f(z)}\, dz = \sum_{r=1} m_r F(a_r) - \sum_{r=1} p_r F(c_r),$$

the integral being taken round the contour.

In particular, if the contour contains a single simple root a and no singularity, then that root is given by

$$a = \frac{1}{2\pi i}\int z\frac{f'(z)}{f(z)}\, dz,$$

the integral being taken as before. (Laurent.)

44. Theorem VII. *If infinity be a pole of $f(z)$, it is also a pole of $f'(z)$ only when it is a multiple pole of $f(z)$.*

Let the multiplicity of the pole for $f(z)$ be n; then for very large values of z we have

$$f(z) = z^n \phi\left(\frac{1}{z}\right),$$

where ϕ is holomorphic for very large values of z and does not vanish at infinity; hence

$$f'(z) = z^{n-1}\left\{n\phi\left(\frac{1}{z}\right) - \frac{1}{z}\phi'\left(\frac{1}{z}\right)\right\}.$$

The coefficient of z^{n-1} is holomorphic for very large values of z and does not vanish at infinity; hence infinity is a pole of $f'(z)$ of multiplicity $n-1$.

If n be unity, so that infinity is a simple pole of $f(z)$, then it is not a pole of $f'(z)$; the derivative is then finite at infinity.

45. THEOREM VIII. *A function, which has no singularity in a finite part of the plane, and has $z=\infty$ for a pole, is an algebraical polynomial.*

Let n, necessarily a finite integer, be the order of multiplicity of the pole at infinity: then the function $f(z)$ can be expressed in the form

$$a_0 z^n + a_1 z^{n-1} + \ldots\ldots + a_{n-1} z + Q\left(\frac{1}{z}\right),$$

where $Q\left(\frac{1}{z}\right)$ is a holomorphic function for very large values of z, and is finite (or zero) when z is infinite.

Now the first n terms of the series constitute a function which has no singularities in the finite part of the plane: and $f(z)$ has no singularities in that part of the plane. Hence $Q\left(\frac{1}{z}\right)$ has no singularities in the finite part of the plane: it is finite for infinite values of z. It thus can never have an infinite value: and it is therefore merely a constant, say a_n. Then

$$f(z) = a_0 z^n + a_1 z^{n-1} + \ldots\ldots + a_{n-1} z + a_n,$$

a polynomial of degree equal to the multiplicity of the pole at infinity, supposed to be the only pole of the function.

46. The above result may be obtained in the following manner.

Since $z=\infty$ is a pole of multiplicity n, the limit of $z^{-n} f(z)$ is not infinite when $z=\infty$.

Now in any finite part of the plane the function is everywhere finite, so that we can use the expansion

$$f(z) = f(0) + z f'(0) + \ldots\ldots + \frac{z^n}{n!} f^{(n)}(0) + R,$$

where

$$R = \frac{z^{n+1}}{2\pi i} \int \frac{f(t)}{t^{n+1}} \frac{dt}{t-z},$$

the integral being taken round a circle of any radius r enclosing the point z and having its centre at the origin. As the subject of integration is finite everywhere along the circumference, we have, by Darboux's expression in (IV.) § 14,

$$R = \lambda r \frac{z^{n+1}}{\tau^{n+1}} \frac{f(\tau)}{\tau - z},$$

where τ is some point on the circumference and λ is a quantity of modulus not greater than unity.

Let $\tau = re^{ia}$; then

$$R = \frac{\lambda}{r} z^{n+1} e^{-2ai} \frac{f(\tau)}{\tau^n} \frac{1}{1 - \frac{z}{r} e^{-ai}}.$$

By definition, the limit of $\frac{f(\tau)}{\tau^n}$ as τ (and therefore r) becomes infinitely large is not infinite; in the same case, the limit of $\left(1 - \frac{z}{r} e^{-ai}\right)^{-1}$ is unity. Since $|\lambda|$ is not greater than unity, the limit of λ/r in the same case is zero; hence with indefinite increase of r, the limit of R is zero and so

$$f(z) = f(0) + zf'(0) + \ldots\ldots + \frac{z^n}{n!} f^{(n)}(0),$$

shewing as before that $f(z)$ is an algebraical polynomial.

47. As the quantity n is necessarily a positive integer*, there are two distinct classes of functions discriminated by the magnitude of n.

The first (and the simpler) is that for which n has a finite value. The polynomial then contains only a finite number of terms, each with a positive integral index; and the function is then a *rational, integral, algebraical polynomial* of degree n.

The second (and the more extensive, as significant functions) is that for which n has an infinite value. The point $z = \infty$ is not a pole, for then the function does not satisfy the test of § 42: it is an essential singularity of the function, which is expansible in an infinite converging series of positive integral powers. To functions of this class the general term *transcendental* is applied.

The number of zeros of a function of the former class is known: it is equal to the degree of the function. It has been proved that the zeros of a transcendental function are isolated points, occurring necessarily in finite number in any finite part of the region of continuity of the function, no point on the boundary of the part being infinitesimally near an essential singularity; but no test has been assigned for the determination of the total number of zeros of a function in an infinite part of the region of continuity.

Again, when the zeros of a polynomial are given, a product-expression can at once be obtained that will represent its analytical value. Also we know that, if a be a zero of any uniform analytic function of multiplicity n, the function can be represented in the vicinity of a by the expression

$$(x - a)^n \phi(z),$$

where $\phi(z)$ is holomorphic in the vicinity of a. The other zeros of the function are zeros of $\phi(z)$; this process of modification in the expression

* It is unnecessary to consider the zero value of n, for the function is then a polynomial of order zero, that is, it is a constant.

can be continued for successive zeros so long as the number of zeros taken account of is limited. But when the number of zeros is unlimited, then the inferred product-expression for the original function is not necessarily a converging product; and thus the question of the formal factorisation of a transcendental function arises.

48. THEOREM IX. *A function, all the singularities of which are accidental, is a rational, algebraical, meromorphic function.*

Since all the singularities are accidental, each must be of finite multiplicity; and therefore infinity, if an accidental singularity, is of finite multiplicity. All the other poles are in the finite part of the plane; they are isolated points and therefore only finite in number, so that the total number of distinct poles is finite and each is of finite order. Let them be $a_1, a_2, \ldots\ldots, a_\mu$ of orders $m_1, m_2, \ldots\ldots, m_\mu$ respectively: let m be the order of the pole at infinity: and let the poles be arranged in the sequence of decreasing moduli such that $|a_\mu| > |a_{\mu-1}| > \ldots\ldots > |a_1|$.

Then, since infinity is a pole of order m, we have

$$f(z) = a_m z^m + a_{m-1} z^{m-1} + \ldots\ldots + a_1 z + f_0(z),$$

where $f_0(z)$ is not infinite for infinite values of z. Now the polynomial $\sum_{i=1}^{m} a_1 z^i$ is not infinite for any finite value of z; hence $f_0(z)$ is infinite for all the finite infinities of $f(z)$ and in the same way, that is, the function $f_0(z)$ has $a_1, \ldots\ldots, a_\mu$ for its poles and it has no other singularities.

Again, since a_μ is a finite pole of multiplicity m_μ, we have

$$f_0(z) = \frac{b_{m_\mu}}{(z-a_\mu)^{m_\mu}} + \ldots\ldots + \frac{b_1}{z-a_\mu} + f_1(z),$$

where $f_1(z)$ is not infinite for $z = a_\mu$ and, as $f_0(z)$ is not infinite for $z = \infty$, evidently $f_1(z)$ is not infinite for $z = \infty$. Hence the singularities of $f_1(z)$ are merely the poles $a_1, \ldots\ldots, a_{\mu-1}$; and these are all its singularities.

Proceeding in this manner for the singularities in succession, we ultimately reach a function $f_\mu(z)$ which has only one pole a_1 and no other singularity, so that

$$f_\mu(z) = \frac{k_{m_1}}{(z-a_1)^{m_1}} + \ldots\ldots + \frac{k_1}{z-a_1} + g(z),$$

where $g(z)$ is not infinite for $z = a_1$. But the function $f_\mu(z)$ is infinite only for $z = a_1$, and therefore $g(z)$ has no infinity. Hence $g(z)$ is only a constant, say k_0: thus

$$g(z) = k_0.$$

Combining all these results we have a *finite* number of series to add together: and the result is that

$$f(z) = g_1(z) + \frac{g_2(z)}{g_3(z)},$$

where $g_1(z)$ is the series $k_0 + a_1 z + \ldots\ldots + a_m z^m$, and $\dfrac{g_2(z)}{g_3(z)}$ is the sum of the finite number of fractions. Evidently $g_3(z)$ is the product

$$(z-a_1)^{m_1}(z-a_2)^{m_2}\ldots\ldots(z-a_\mu)^{m_\mu};$$

and $g_2(z)$ is at most of degree

$$m_1 + m_2 + \ldots\ldots + m_\mu - 1.$$

If $F(z)$ denote $g_1(z)\,g_3(z) + g_2(z)$, the form of $f(z)$ is

$$\frac{F(z)}{g_3(z)},$$

that is, $f(z)$ is a rational, algebraical, meromorphic function.

It is evident that, when the function is thus expressed as an algebraical fraction, the degree of $F(z)$ is the sum of the multiplicities of all the poles when infinity is a pole.

Corollary I. *A function, all the singularities of which are accidental, has as many zeros as it has accidental singularities in the plane.*

If $z=\infty$ be a pole, then it follows that, because $f(z)$ can be expressed in the form

$$\frac{F(z)}{g_3(z)},$$

it has as many zeros as $F(z)$, unless one such should be also a zero of $g_3(z)$. But the zeros of $g_3(z)$ are known, and no one of them is a zero of $F(z)$, on account of the form of $f(z)$ when it is expressed in partial fractions. Hence the number of zeros of $f(z)$ is equal to the degree of $F(z)$, that is, it is equal to the number of poles of $f(z)$.

If $z=\infty$ be not a pole, two cases are possible; (i) the function $f(z)$ may be finite for $z=\infty$, or (ii) it may be zero for $z=\infty$. In the former case, the number of zeros is, as before, equal to the degree of $F(z)$, that is, it is equal to the number of infinities.

In the latter case, if the degree of the numerator $F(z)$ be κ less than that of the denominator $g_3(z)$, then $z=\infty$ is a zero of multiplicity κ; and it follows that the number of zeros is equal to the degree of the numerator together with κ, so that their number is the same as the number of accidental singularities.

Corollary II. At the beginning of the proof of the theorem of the present section, it is proved that a function, all the singularities of which are accidental, has only a finite number of such singularities.

Hence, by the preceding Corollary, *such a function can have only a finite number of zeros.*

If, therefore, the number of zeros of a function be infinite, the function must have at least one essential singularity.

COROLLARY III. When a uniform analytic function has no essential singularity, if the (finite) number of its poles, say $c_1, \ldots, c_m$, be m, no one of them being at $z = \infty$, and if the number of its zeros, say $a_1, \ldots, a_m$, be also m, no one of them being at $z = \infty$, then the function is

$$\prod_{r=1}^{m} \left(\frac{z - a_r}{z - c_r} \right),$$

except possibly as to a constant factor.

When $z = \infty$ is a zero of order n, so that the function has $m - n$ zeros, say $a_1, a_2, \ldots$, in the finite part of the plane, the form of the function is

$$\frac{\prod_{r=1}^{m-n} (z - a_r)}{\prod_{r=1}^{m} (z - c_r)};$$

and, when $z = \infty$ is a pole of order p, so that the function has $m - p$ poles, say $c_1, c_2, \ldots$, in the finite part of the plane, the form of the function is

$$\frac{\prod_{r=1}^{m} (z - a_r)}{\prod_{r=1}^{m-p} (z - c_r)}.$$

COROLLARY IV. *All the singularities of rational algebraical meromorphic functions are accidental.*

CHAPTER V.

Transcendental Integral Functions.

49. We now proceed to consider the properties of uniform functions which have essential singularities.

The simplest instance of the occurrence of such a function has already been referred to in § 42; the function has no singularity except at $z = \infty$, and that value is an essential singularity solely through the failure of the limitation to finiteness that would render the singularity accidental. The function is then an integral function of transcendental character; and it is analytically represented (§ 26) by $G(z)$ an infinite series in positive powers of z, which converges everywhere in the finite part of the plane and acquires an infinite value at infinity alone.

The preceding investigations shew that uniform functions, all the singularities of which are accidental, are rational algebraical functions—their character being completely determined by their uniformity and the accidental nature of their singularities, and that among such functions having the same accidental singularities the discrimination is made, save as to a constant factor, by means of their zeros.

Hence the zeros and the accidental singularities of a rational algebraical function determine, save as to a constant factor, an expression of the function which is valid for the whole plane. A question therefore arises how far the zeros and the singularities of a transcendental function determine the analytical expression of the function for the whole plane.

50. We shall consider first how far the discrimination of transcendental integral functions, which have no infinite value except for $z = \infty$, is effected by means of their zeros*.

* The following investigations are based upon the famous memoir by Weierstrass, "Zur Theorie der eindeutigen analytischen Functionen," published in 1876: it is included, pp. 1—52, in the *Abhandlungen aus der Functionenlehre* (Berlin, 1886).

In connection with the product-expression of a transcendental function, Cayley, "Mémoire sur les fonctions doublement périodiques," *Liouville*, t. x, (1845), pp. 385—420, or *Collected Works*, vol. i, pp. 156—182, should be consulted.

Let the zeros $a_1, a_2, a_3, \ldots$ be arranged in order of increasing moduli; a finite number of terms in the series may have the same value so as to allow for the existence of a multiple zero at any point. After the results stated in § 47, it will be assumed that the number of zeros is infinite; that, subject to limited repetition, they are isolated points; and, in the present chapter, that, as n increases indefinitely, the limit of $|a_n|$ is infinity. And it will be assumed that $|a_1| > 0$, so that the origin is temporarily excluded from the series of zeros.

Let z be any point in the finite part of the plane. Then only a limited number of the zeros can lie within and on a circle centre the origin and radius equal to $|z|$; let these be $a_1, a_2, \ldots, a_{k-1}$, and let a_r denote any one of the other zeros. We proceed to form the infinite product of quantities u_r, where u_r denotes

$$\left(1 - \frac{z}{a_r}\right) e^{g_r},$$

and g_r is a rational integral function of z which, being subject to choice, will be chosen so as to make the infinite product converge everywhere in the plane. We have

$$\log u_r = g_r - \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{z}{a_r}\right)^n,$$

a series which converges because $|z| < |a_r|$. Now let

$$g_r = \sum_{n=1}^{s-1} \frac{1}{n} \left(\frac{z}{a_r}\right)^n,$$

then
$$\log u_r = - \sum_{n=s}^{\infty} \frac{1}{n} \left(\frac{z}{a_r}\right)^n,$$

and therefore
$$u_r = e^{-\sum_{n=s}^{\infty} \frac{1}{n} \left(\frac{z}{a_r}\right)^n}.$$

Hence
$$\prod_{r=k}^{\infty} u_r = e^{-\sum_{r=k}^{\infty} \sum_{n=s}^{\infty} \frac{1}{n} \left(\frac{z}{a_r}\right)^n},$$

if the expression on the right-hand side be finite, that is, if the series

$$\sum_{r=k}^{\infty} \sum_{n=s}^{\infty} \frac{1}{n} \left(\frac{z}{a_r}\right)^n$$

converge unconditionally. Denoting the modulus of this series by M, we have

$$M < \sum_{r=k}^{\infty} \sum_{n=s}^{\infty} \frac{1}{n} \left|\frac{z}{a_r}\right|^n,$$

so that
$$sM < \sum_{r=k}^{\infty} \sum_{n=s}^{\infty} \left|\frac{z}{a_r}\right|^n$$

$$< \sum_{r=k}^{\infty} \frac{\left|\frac{z}{a_r}\right|^s}{1 - \left|\frac{z}{a_r}\right|},$$

whence, since $1-\left|\frac{z}{a_k}\right|$ is the smallest of the denominators in terms of the last sum, we have

$$sM\left\{1-\left|\frac{z}{a_k}\right|\right\} < \sum_{r=k}^{\infty}\left|\frac{z}{a_r}\right|^s$$
$$< |z|^s \sum_{r=k}^{\infty}\frac{1}{|a_r|^s}.$$

If, as is not infrequently the case, there be any finite integer s for which (and therefore for all greater indices) the series

$$\sum_{r=1}^{\infty}\frac{1}{|a_r|^s},$$

and therefore the series $\sum_{r=k}^{\infty}|a_r|^{-s}$, converges, we choose s to be that least integer. The value of M then is finite for all finite values of z; the series

$$\sum_{r=k}^{\infty}\sum_{n=s}^{\infty}\frac{1}{n}\left(\frac{z}{a_r}\right)^n$$

converges unconditionally and therefore

$$\prod_{r=k}^{\infty} u_r$$

is a converging product when

$$u_r = \left(1-\frac{z}{a_r}\right)e^{\sum_{n=1}^{s-1}\frac{1}{n}\left(\frac{z}{a_r}\right)^n}.$$

Let the finite product

$$\prod_{m=1}^{k-1}\left\{\left(1-\frac{z}{a_m}\right)e^{\sum_{n=1}^{s-1}\frac{1}{n}\left(\frac{z}{a_m}\right)^n}\right\}$$

be associated as a factor with the foregoing infinite converging product. Then *the expression*

$$f(z) = \prod_{r=1}^{\infty}\left\{\left(1-\frac{z}{a_r}\right)e^{\sum_{n=1}^{s-1}\frac{1}{n}\left(\frac{z}{a_r}\right)^n}\right\}$$

is an infinite product, converging uniformly and unconditionally for all finite values of z, provided the finite integer s be such as to make the series $\sum_{r=1}^{\infty}|a_r|^{-s}$ converge uniformly and unconditionally.

Since the product converges uniformly and unconditionally, no product constructed from its factors u_r, say from all but one of them, can be infinite. Now the factor

$$\left(1-\frac{z}{a_m}\right)e^{\sum_{n=1}^{s-1}\frac{1}{n}\left(\frac{z}{a_m}\right)^n}$$

vanishes for $z=a_m$; hence $f(z)$ vanishes for $z=a_m$. Thus the function, evidently uniform after what has been proved, has the assigned points $a_1, a_2, \ldots$ and no others for its zeros.

Further, $z=\infty$ is an essential singularity of the function; for it is an essential singularity of each of the factors on account of the exponential element in the factor.

51. But it may happen that no finite integer s can be found which will make the series

$$\sum_{r=1}^{\infty} |a_r|^{-s}$$

converge*. We then proceed as follows.

Instead of having the same index s throughout the series, we associate with every zero a_r an integer m_r chosen so as to make the series

$$\sum_{n=1}^{\infty} \left| \frac{1}{a_n} \left(\frac{z}{a_n}\right)^{m_n} \right|$$

a converging series. To obtain these integers, we take any series of decreasing real positive quantities ϵ, ϵ_1, $\epsilon_2, \ldots$, such that (i) ϵ is less than unity and (ii) they form an unconditionally converging series; and we choose integers m_r such that

$$\epsilon^{m_r+1} \leqslant \epsilon_r.$$

These integers make the foregoing series of moduli converge. For, neglecting the limited number of terms for which $|z| \geqslant |a|$, and taking ϵ such that

$$\left|\frac{z}{a_k}\right| \leqslant \epsilon,$$

we have for all succeeding terms

$$\left|\frac{z}{a_r}\right| \leqslant \epsilon,$$

and therefore
$$\left|\frac{z}{a_r}\right|^{m_r+1} \leqslant \epsilon^{m_r+1} \leqslant \epsilon_r.$$

Hence, except for the first $k-1$ terms, the sum of which is finite, we have

$$\sum_{n=k}^{\infty} \left| \frac{1}{a_n} \left(\frac{z}{a_n}\right)^{m_n} \right| \leqslant \frac{1}{|z|}(\epsilon_k + \epsilon_{k+1} + \ldots)$$
$$\leqslant \frac{1}{|z|}(\epsilon + \epsilon_1 + \epsilon_2 + \ldots),$$

which is finite because the series $\epsilon + \epsilon_1 + \epsilon_2 + \ldots$ converges. Hence the series

$$\sum_{n=1}^{\infty} \left| \frac{1}{a_n} \left(\frac{z}{a_n}\right)^{m_n} \right|$$

is a converging series.

* For instance, there is no finite integer s that can make the infinite series

$$(\log 2)^{-s} + (\log 3)^{-s} + (\log 4)^{-s} + \ldots$$

converge. This series is given in illustration by Hermite, *Cours à la faculté des Sciences* (4[me] éd. 1891), p. 86.

Just as in the preceding case a special expression was formed to serve as a typical factor in the infinite product, we now form a similar expression for the same purpose. Evidently

$$1 - x = e^{\log (1-x)} = e^{-\sum\limits_{r=0}^{\infty} \frac{x^{r+1}}{r+1}},$$

if $|x| < 1$. Forming a function $E(x, m)$ defined by the equation

$$E(x, m) = (1 - x)\, e^{\sum\limits_{r=1}^{m} \frac{x^r}{r}},$$

we have

$$E(x, m) = e^{-\sum\limits_{r=1}^{\infty} \frac{x^{m+r}}{m+r}}.$$

In the preceding case it was possible to choose the integer m so that it should be the same for all the factors of the infinite product, which was ultimately proved to converge. Now, we take $x = \dfrac{z}{a_n}$ and associate m_n as the corresponding value of m. Hence, if

$$f(z) = \prod_{n=k}^{\infty} E\left(\frac{z}{a_n}, m_n\right),$$

where $|a_{k-1}| < |z| < |a_k|$, we have

$$f(z) = e^{-\sum\limits_{n=k}^{\infty} \sum\limits_{r=1}^{\infty} \frac{1}{r + m_k} \left(\frac{z}{a_n}\right)^{r+m_n}}.$$

The infinite product represented by $f(z)$ will converge if the double series in the exponential be a converging series.

Denoting the double series by S, we have

$$|S| < \sum_{n=k}^{\infty} \sum_{r=1}^{\infty} \frac{1}{r + m_n} \left|\frac{z}{a_n}\right|^{r+m_n}$$

$$< \sum_{n=k}^{\infty} \sum_{r=1}^{\infty} \left|\frac{z}{a_n}\right|^{r+m_n}$$

$$< \sum_{n=k}^{\infty} \left|\frac{z}{a_n}\right|^{1+m_n} \frac{1}{1 - \left|\dfrac{z}{a_n}\right|},$$

on effecting the summation for r. Let A be the value of $1 - \left|\dfrac{z}{a_k}\right|$; then for all the remaining values of n we have

$$1 - \left|\frac{z}{a_n}\right| > A,$$

and so

$$|S| < \frac{1}{A} \sum_{n=k}^{\infty} \left|\frac{z}{a_n}\right|^{1+m_n}$$

$$< \frac{|z|}{A} \sum_{n=k}^{\infty} \left|\frac{1}{a_n}\left(\frac{z}{a_n}\right)^{m_n}\right|.$$

This series converges; hence for finite values of $|z|$ the value of $|S|$ is finite, so that S is a converging series. Hence it follows that $f(z)$ is an unconditionally converging product. We now associate with $f(z)$ as factors the $k-1$ functions

$$E\left(\frac{z}{a_i}, m_i\right),$$

for $i=1, 2, \ldots, k-1$; their number being finite, their product is finite and therefore the modified infinite product still converges. We thus have

$$G(z) = \prod_{n=1}^{\infty} E\left(\frac{z}{a_n}, m_n\right)$$

an unconditionally converging product.

Since the product $G(z)$ converges unconditionally, no product constructed from its factors E, say from all but one of them, can be infinite. The factor

$$E\left(\frac{z}{a_n}, m_n\right) = \left(1 - \frac{z}{a_n}\right) e^{\sum_{r=1}^{m_n} \frac{1}{r}\left(\frac{z}{a_n}\right)^r}$$

vanishes for the value $z=a_n$ and only for this value; hence $G(z)$ vanishes for $z=a_n$. It therefore appears that $G(z)$ has the assigned points $a_1, a_2, a_3, \ldots$ and no others for its zeros; and from the existence of the exponential in each of the factors it follows that $z=\infty$ is an essential singularity of the factor and therefore it is an essential singularity of the function.

Denoting the series in the exponential by $g_n(z)$, so that

$$g_n(z) = \sum_{r=1}^{m_n} \frac{1}{r}\left(\frac{z}{a_n}\right)^r,$$

we have

$$E\left(\frac{z}{a_n}, m_n\right) = \left(1 - \frac{z}{a_n}\right) e^{g_n(z)};$$

and therefore the function obtained is

$$G(z) = \prod_{n=1}^{\infty} \left\{\left(1 - \frac{z}{a_n}\right) e^{g_n(z)}\right\}.$$

The series g_n usually contains only a limited number of terms; when the number of terms increases without limit, it is only with indefinite increase of $|a_n|$ and the series is then a converging series.

It should be noted that the factors of the infinite product $G(z)$ are the expressions E no one of which, for the purposes of the product, is resoluble into factors that can be distributed and recombined with similarly obtained factors from other expressions E; there is no guarantee that the product of the factors, if so resolved, would converge uniformly and unconditionally, and it is to secure such convergence that the expressions E have been constructed.

It was assumed, merely for temporary convenience, that the origin was not a zero of the required function; there obviously could not be a factor of exactly the same form as the factors E if a were the origin.

If, however, the origin were a zero of order λ, we should have merely to associate a factor z^λ with the function already constructed.

We thus obtain Weierstrass's theorem:—

It is possible to construct a transcendental integral function such that it shall have infinity as its only essential singularity and have the origin (of multiplicity λ), $a_1, a_2, a_3, \ldots$ as zeros; and such a function is

$$z^\lambda \prod_{n=1}^{\infty} \left\{\left(1 - \frac{z}{a_n}\right) e^{g_n(z)}\right\},$$

where $g_n(z)$ is a rational, integral, algebraical function of z, the form of which is dependent upon the law of succession of the zeros.

52. But, unlike uniform functions with only accidental singularities, the function is not unique: there *are an unlimited number of transcendental integral functions with the same series of zeros and infinity as the sole essential singularity*, a theorem also due to Weierstrass.

For, if $G_1(z)$ and $G(z)$ be two transcendental, integral functions with the same series of zeros in the same multiplicity, and $z = \infty$ as their only essential singularity, then

$$\frac{G_1(z)}{G(z)}$$

is a function with no zeros and no infinities in the finite part of the plane. Denoting it by G_2, then

$$\frac{1}{G_2}\frac{dG_2}{dz}$$

is a function which, in the finite part of the plane, has no infinities; and therefore it can be expanded in the form

$$C_1 + 2C_2 z + 3C_3 z^2 + \ldots,$$

a series converging everywhere in the finite part of the plane. Choosing a constant C_0 so that $G_2(0) = e^{C_0}$, we have on integration

$$G_2(z) = e^{\bar{g}(z)},$$

where $$\bar{g}(z) = C_0 + C_1 z + C_2 z^2 + \ldots,$$

and $\bar{g}(z)$ is finite everywhere in the finite part of the plane. Hence it follows that, *if $\bar{g}(z)$ denote any integral function of z which is finite everywhere in the finite part of the plane, and if $G(z)$ be some transcendental integral function with a given series of zeros and $z = \infty$ as its sole essential singularity, all transcendental integral functions with that series of zeros and $z = \infty$ as the sole essential singularity are included in the form*

$$G(z)\, e^{\bar{g}(z)}.$$

Corollary I. *A function which has no zeros in the finite part of the plane, no accidental singularities and $z = \infty$ for its sole essential singularity is necessarily of the form*

$$e^{\bar{g}(z)},$$

where $\bar{g}(z)$ is an integral function of z finite everywhere in the finite part of the plane.

COROLLARY II. *Every transcendental function, which has the same zeros in the same multiplicity as an algebraical polynomial $A(z)$—the number, therefore, being necessarily finite—, which has no accidental singularities and has $z=\infty$ for its sole essential singularity, can be expressed in the form*

$$A(z)\, e^{g(z)}.$$

COROLLARY III. *Every function, which has an assigned series of zeros and an assigned series of poles and has $z=\infty$ for its sole essential singularity, is of the form*

$$\frac{G_0(z)}{G_p(z)} e^{\bar{g}(z)},$$

where the zeros of $G_0(z)$ are the assigned zeros and the zeros of $G_p(z)$ are the assigned poles.

For if $G_p(z)$ be any transcendental integral function, constructed as in the proposition, which has as its zeros the poles of the required function in the assigned multiplicity, the most general form of that function is

$$G_p(z)\, e^{h(z)},$$

where $h(z)$ is integral. Hence, if the most general form of function which has those zeros for its poles be denoted by $f(z)$, we have

$$f(z)\, G_p(z)\, e^{h(z)}$$

as a function with no poles, with infinity as its sole essential singularity, and with the assigned series of zeros. But if $G_0(z)$ be any transcendental integral function with the assigned zeros as its zeros, the most general form of function with those zeros is

$$G_0(z)\, e^{g(z)};$$

and so
$$f(z)\, G_p(z)\, e^{h(z)} = G_0(z)\, e^{g(z)},$$

whence
$$f(z) = \frac{G_0(z)}{G_p(z)} e^{\bar{g}(z)},$$

in which $\bar{g}(z)$ denotes $g(z)-h(z)$.

If the number of zeros be finite, we evidently may take $G_0(z)$ as the algebraical polynomial with those zeros as its only zeros.

If the number of poles be finite, we evidently may take $G_p(z)$ as the algebraical polynomial with those poles as its only zeros.

And, lastly, if a function have a finite number of zeros, a finite number of accidental singularities and $z=\infty$ as its sole essential singularity, it can be expressed in the form

$$\frac{P(z)}{Q(z)} e^{g(z)},$$

where P and Q are rational integral polynomials. This is valid even though the number of assigned zeros be not the same as the number of assigned poles; the sole effect of the inequality of these numbers is to complicate the character of the essential singularity at infinity.

53. It follows from what has been proved that any uniform function, having $z = \infty$ for its sole essential singularity and any number of assigned zeros, can be expressed as a product of expressions of the form

$$\left(1 - \frac{z}{a_n}\right) e^{g_n(z)}.$$

Such a quantity is called* a *primary factor* of the function.

It has also been proved that:—

(i) If there be no zero a_n, the primary factor has the form

$$e^{g_n(z)}.$$

(ii) The exponential index $g_n(z)$ may be zero for individual primary factors, though the number of such factors must, at the utmost, be finite †.

(iii) The factor takes the form z when the origin is a zero.

Hence we have the theorem, due to Weierstrass:—

Every uniform integral function of z can be expressed as a product of primary factors, each of the form

$$(kz + l)\, e^{g(z)},$$

where $g(z)$ is an appropriate integral function of z vanishing with z and where k, l are constants. In particular factors, $g(z)$ may vanish; and either k or l, but not both k and l, may vanish with or without a non-vanishing exponential index $g(z)$.

54. It thus appears that an essential distinction between transcendental integral functions is constituted by the aggregate of their zeros: and we may conveniently consider that *all such functions are substantially the same when they have the same zeros.*

There are a few very simple sets of functions, thus discriminated by their zeros: of each set only one member will be given, and the factor $e^{g(z)}$, which makes the variation among the members of the same set, will be neglected for the present. Moreover, it will be assumed that the zeros are isolated points.

I. There may be a finite number of zeros; the simplest function is then an algebraical polynomial.

* Weierstrass's term is *Primfunction*, l.c., p. 15.

† Unless the *class* (§ 59) be zero, when the index is zero for all the factors.

II. There may be a singly-infinite system of zeros. Various functions will be obtained, according to the law of distribution of the zeros.

Thus let them be distributed according to a law of simple arithmetic progression along a given line. If a be a zero, ω a quantity such that $|\omega|$ is the distance between two zeros and arg. ω is the inclination of the line, we have

$$a + m\omega,$$

for integer values of m from $-\infty$ to $+\infty$, as the expression of the series of the zeros. Without loss of generality we may take a at the origin—this is merely a change of origin of coordinates—and the origin is then a simple zero: the zeros are given by $m\omega$, for integer values of m from $-\infty$ to $+\infty$.

Now $\Sigma \dfrac{1}{m\omega} = \dfrac{1}{\omega} \Sigma \dfrac{1}{m}$ is a diverging series; but an integer s—the lowest value is $s=2$—can be found for which the series $\Sigma \left(\dfrac{1}{m\omega}\right)^s$ converges uniformly and unconditionally. Taking $s = 2$, we have

$$g_m(z) = \sum_{n=1}^{s-1} \frac{1}{n} \left(\frac{z}{a_m}\right)^n = \frac{z}{m\omega},$$

so that the primary factor of the present function is

$$\left(1 - \frac{z}{m\omega}\right) e^{\frac{z}{m\omega}};$$

and therefore, by § 52, the product

$$f(z) = z \prod_{-\infty}^{\infty} \left\{\left(1 - \frac{z}{m\omega}\right) e^{\frac{z}{m\omega}}\right\}$$

converges uniformly and unconditionally for all finite values of z.

The term corresponding to $m = 0$ is to be omitted from the product; and it is unnecessary to assume that the numerical value of the positive infinity for m is the same as that of the negative infinity for m. If, however, the latter assumption be adopted, the expression can be changed into the ordinary product-expression for a sine, by combining the primary factors due to values of m that are equal and opposite: in fact, then

$$f(z) = \frac{\omega}{\pi} \sin \frac{\pi z}{\omega}.$$

This example is sufficient to shew the importance of the exponential term in the primary factor. If the product be formed exactly as for an algebraical polynomial, then the function is

$$z \prod_{m=-q}^{m=p} \left(1 - \frac{z}{m\omega}\right)$$

in the limit when both p and q are infinite. But this is known* to be

$$\left(\frac{q}{p}\right)^{\frac{z}{\omega}} \frac{\omega}{\pi} \sin \frac{\pi z}{\omega}.$$

* Hobson's *Trigonometry*, § 287.

Another illustration is afforded by Gauss's Π-function, which is the limit when k is infinite of

$$\frac{1 \cdot 2 \cdot 3 \ldots\ldots k}{(z+1)(z+2)\ldots\ldots(z+k)} k^z.$$

This is transformed by Gauss* into the reciprocal of the expression

$$(1+z) \prod_{m=2}^{\infty} \left\{\left(1+\frac{z}{m}\right)\left(\frac{m}{m-1}\right)^{-z}\right\},$$

that is, of

$$(1+z) \prod_{m=2}^{\infty} \left\{\left(1+\frac{z}{m}\right) e^{-z \log\left(\frac{m}{m-1}\right)}\right\},$$

the primary factors of which have the same characteristic form as in the preceding investigation, though not the same literal form.

It is chiefly for convenience that the index of the exponential part of the primary factor is taken, in § 50, in the form $\sum_{n=1}^{s-1} \frac{1}{n}\left(\frac{z}{a_r}\right)^n$. With equal effectiveness it may be taken in the form $\sum_{n=1}^{s-1} \frac{1}{n} b_{r,n} z^n$, provided the series

$$\sum_{r=k}^{\infty} \sum_{n=1}^{\infty} \left\{\frac{1}{n}\left(b_{r,n} - a_r^{-n}\right) z^n\right\}$$

converge uniformly and unconditionally.

Ex. 1. Prove that each of the products

$$\Pi \left\{\left(1-\frac{2z}{m\pi}\right) e^{\frac{2z}{m\pi}}\right\}$$

for $m = \pm 1, \pm 3, \pm 5, \ldots\ldots$ to infinity, and

$$\left(1+\frac{2z}{\pi}\right) \prod_{n=-\infty}^{n=\infty} \left[\left\{1-\frac{2z}{(2n-1)\pi}\right\} e^{\frac{z}{n\pi}}\right],$$

the term for $n=0$ being excluded from the latter product, converges uniformly and unconditionally and that each of them is equal to $\cos z$. (Hermite and Weyr.)

Ex. 2. Prove that, if the zeros of a transcendental integral function be given by the series

$$0, \pm\omega, \pm 4\omega, \pm 9\omega, \ldots\ldots \text{ to infinity,}$$

the simplest of the set of functions thereby determined can be expressed in the form

$$\sin\left\{\pi\left(\frac{z}{\omega}\right)^{\frac{1}{2}}\right\} \sin\left\{i\pi\left(\frac{z}{\omega}\right)^{\frac{1}{2}}\right\}.$$

Ex. 3. Construct the set of transcendental integral functions which have in common the series of zeros determined by the law $m^2\omega_1 + 2m\omega_2 + \omega_3$ for all integral values of m between $-\infty$ and $+\infty$; and express the simplest of the set in terms of circular functions.

55. The law of distribution of the zeros, next in importance and substantially next in point of simplicity, is that in which the zeros form a doubly-infinite double arithmetic progression, the points being the ∞^2 intersections of one infinite system of equidistant parallel straight lines with another infinite system of equidistant parallel straight lines.

The origin may, without loss of generality, be taken as one of the zeros. If ω be the coordinate of the nearest zero along the line of one system passing through the origin, and ω' be the coordinate of the nearest zero along

* *Ges. Werke*, t. iii, p. 145; the example is quoted in this connection by Weierstrass, l.c., p. 15.

the line of the other system passing through the origin, then the complete series of zeros is given by

$$\Omega = m\omega + m'\omega',$$

for all integral values of m and all integral values of m' between $-\infty$ and $+\infty$. The system of points may be regarded as doubly-periodic, having ω and ω' for periods.

It must be assumed that the two systems of lines intersect. Otherwise, ω and ω' would have the same argument and their ratio would be a real quantity, say α; and then

$$\frac{\Omega}{\omega} = m + m'\alpha.$$

Whether α be commensurable or incommensurable, the number of pairs of integers, for which $m + m'\alpha$ is zero or may be made less than any small quantity δ, is infinite; and in either case we should have the origin a zero for each such pair, that is, altogether the origin would be a zero of infinite multiplicity. This property of a function is to be considered as excluded, for it would make the origin an essential singularity instead of, as required, an ordinary point of the transcendental integral function. Hence *the ratio of the quantities ω and ω' is not real.*

56. For the construction of the primary factor, it is necessary to render the series

$$\Sigma \Omega^{-s_{m, m'}}$$

converging, by appropriate choice of integers $s_{m,m'}$. It is found to be possible to choose an integer s to be the same for every term of the series, corresponding to the simpler case of the general investigation, given in § 50.

As a matter of fact, the series

$$\Sigma \Omega^{-s}$$

diverges for $s = 1$ (we have not made any assumption that the positive and the negative infinities for m are numerically equal, nor similarly as to m'); the series converges for $s = 2$, but its value depends upon the relative values of the infinities for m and m'; and $s = 3$ is the lowest integral value for which, as for all greater values, the series converges uniformly and unconditionally.

There are various ways of proving the uniform and unconditional convergence of the series $\Sigma \Omega^{-\mu}$ when $\mu > 2$: the following proof is based upon a general method due to Eisenstein*.

First, the series $\sum\limits_{m=-\infty}^{m=\infty} \sum\limits_{n=-\infty}^{n=\infty} (m^2 + n^2)^{-\mu}$ converges uniformly and unconditionally, if $\mu > 1$. Let the series be arranged in partial series: for this purpose,

* *Crelle*, t. xxxv, (1847), p. 161; a geometrical exposition is given by Halphen, *Traité des fonctions elliptiques*, t. i, pp. 358—362.

we choose integers k and l, and include in each such partial series all the terms which satisfy the inequalities

$$2^k < m \leqslant 2^{k+1},$$
$$2^l < n \leqslant 2^{l+1},$$

so that the number of values of m is 2^k and the number of values of n is 2^l.

Then, if $k + l = 2\kappa$, we have

$$2^{2\kappa} < 2^{2\kappa+1} < 2^{2k} + 2^{2l} < m^2 + n^2,$$

so that each term in the partial series $\leqslant \dfrac{1}{2^{2\kappa\mu}}$. The number of terms in the partial series is $2^k \cdot 2^l$, that is, $2^{2\kappa}$: so that the sum of the terms in the partial series is

$$\leqslant \frac{1}{2^{2\kappa(\mu-1)}}.$$

Take the upper limit of k and l to be p, ultimately to be made infinite. Then the sum of all the partial series is

$$\leqslant \sum_{\kappa=0}^{p} \frac{1}{2^{2\kappa(\mu-1)}}$$
$$\leqslant \frac{1 - 2^{-2(p+1)(\mu-1)}}{1 - 2^{-2(\mu-1)}},$$

which, when $p = \infty$, is a finite quantity if $\mu > 1$.

Next, let $\omega = \alpha + \beta i$, $\omega' = \gamma + \delta i$, so that

$$\Omega = m\omega + n\omega' = m\alpha + n\gamma + i(m\beta + n\delta);$$

hence, if $$\theta = m\alpha + n\gamma, \quad \phi = m\beta + n\delta,$$

we have $$|\Omega|^2 = \theta^2 + \phi^2.$$

Now take integers r and s such that

$$r < \theta < r + 1, \quad s < \phi < s + 1.$$

The number of terms Ω satisfying these conditions is definitely finite and is independent of m and n. For since

$$m(\alpha\delta - \beta\gamma) = \theta\delta - \phi\gamma,$$
$$n(\alpha\delta - \beta\gamma) = -\theta\beta + \phi\alpha,$$

and $\alpha\delta - \beta\gamma$ does not vanish because ω'/ω is not purely real, the number of values of m is the integral part of

$$\frac{(r+1)\delta - s\gamma}{\alpha\delta - \beta\gamma}$$

less the integral part of

$$\frac{r\delta - (s+1)\gamma}{\alpha\delta - \beta\gamma},$$

that is, it is the integral part of $(\gamma + \delta)/(\alpha\delta - \beta\gamma)$. Similarly, the number of values of n is the integral part of $(\alpha + \beta)/(\alpha\delta - \beta\gamma)$. Let the product of the last two integers be q; then the number of terms Ω satisfying the inequalities is q.

Then
$$\Sigma\Sigma\,|\,\Omega\,|^{-2\mu} = \Sigma\Sigma\,(\theta^2+\phi^2)^{-\mu}$$
$$\leqslant q\,\Sigma\Sigma\,(r^2+s^2)^{-\mu},$$
which, by the preceding result, is finite when $\mu > 1$. Hence
$$\Sigma\Sigma\,(m\omega + m'\omega')^{-2\mu}$$
converges uniformly and unconditionally when $\mu > 1$; and therefore the least value of s, an integer for which
$$\Sigma\Sigma\,(m\omega + m'\omega')^{-s}$$
converges uniformly and unconditionally, is 3.

The series $\Sigma\Sigma\,(m\omega+m'\omega')^{-2}$ has a finite sum, the value of which depends* upon the infinite limits for the summation with regard to m and m'. This dependence is inconvenient and it is therefore excluded in view of our present purpose.

Ex. Prove in the same manner that the series
$$\ldots\ldots\Sigma\Sigma\Sigma\,(m_1{}^2+m_2{}^2+\ldots\ldots+m_n{}^2)^{-\mu},$$
the multiple summation extending over all integers $m_1, m_2, \ldots\ldots, m_n$ between $-\infty$ and $+\infty$, converges uniformly and unconditionally if $2\mu > n$. (Eisenstein.)

57. Returning now to the construction of the transcendental integral function the zeros of which are the various points Ω, we use the preceding result in connection with § 50 to form the general primary factor. Since $s=3$, we have
$$g(z) = \sum_{n=1}^{s-1}\frac{1}{n}\left(\frac{z}{\Omega}\right)^n$$
$$= \frac{z}{\Omega} + \frac{1}{2}\frac{z^2}{\Omega^2},$$
and therefore the primary factor is
$$\left(1-\frac{z}{\Omega}\right)e^{\frac{z}{\Omega}+\frac{1}{2}\frac{z^2}{\Omega^2}}.$$
Moreover, the origin is a simple zero. Hence, denoting the required function by $\sigma(z)$, we have
$$\sigma(z) = z\prod_{-\infty}^{\infty}\prod_{-\infty}^{\infty}\left\{\left(1-\frac{z}{\Omega}\right)e^{\frac{z}{\Omega}+\frac{1}{2}\frac{z^2}{\Omega^2}}\right\}$$
as a transcendental integral function which, since the product converges uniformly and unconditionally for all finite values of z, exists and has a finite value everywhere in the finite part of the plane; the quantity Ω denotes $m\omega + m'\omega'$, and the double product is taken for all values of m and of m' between $-\infty$ and $+\infty$, simultaneous zero values alone being excluded.

This function will be called Weierstrass's σ-function; it is of importance in the theory of doubly-periodic functions which will be discussed in Chapter XI.

* See a paper by the author, *Quart. Journ. of Math.*, vol. xxi, (1886), pp. 261—280.

Ex. If the doubly-infinite series of zeros be the points given by

$$\Omega = m^2\omega_1 + 2mn\omega_2 + n^2\omega_3,$$

ω_1, ω_2, ω_3 being such complex constants that Ω does not vanish for real values of m and n, then the series

$$\sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} \Omega^{-s}$$

converges for $s=2$. The primary factor is thus

$$\left(1 - \frac{z}{\Omega}\right) e^{\frac{z}{\Omega}},$$

and the simplest transcendental integral function having the assigned zeros is

$$z \prod_{-\infty}^{\infty} \prod_{-\infty}^{\infty} \left\{\left(1 - \frac{z}{\Omega}\right) e^{\frac{z}{\Omega}}\right\}.$$

The actual points that are the zeros are the intersections of two infinite systems of parabolas.

58. One more result—of a negative character—will be adduced in this connection. We have dealt with the case in which the system of zeros is a singly-infinite arithmetical progression of points along one straight line and with the case in which the system of zeros is a doubly-infinite arithmetical progression of points along two different straight lines: it is easy to see that *a uniform transcendental integral function cannot exist with a triply-infinite arithmetical progression of points for zeros.*

A triply-infinite arithmetical progression of points would be represented by all the possible values of

$$p_1\Omega_1 + p_2\Omega_2 + p_3\Omega_3$$

for all possible integer values for p_1, p_2, p_3 between $-\infty$ and $+\infty$, where no two of the arguments of the complex constants Ω_1, Ω_2, Ω_3 are equal. Let

$$\Omega_r = \omega_r + i\omega_r', \quad (r = 1, 2, 3);$$

then, as will be proved (§ 107) in connection with a later proposition, it is possible*—and possible in an unlimited number of ways—to determine integers p_1, p_2, p_3 so that, save as to infinitesimal quantities,

$$\frac{p_1}{\omega_2\omega_3' - \omega_3\omega_2'} = \frac{p_2}{\omega_3\omega_1' - \omega_1\omega_3'} = \frac{p_3}{\omega_1\omega_2' - \omega_2\omega_1'},$$

all the denominators in which equations differ from zero on account of the fact that no two arguments of the three quantities Ω_1, Ω_2, Ω_3 are equal. For each such set of determined integers we have

$$p_1\Omega_1 + p_2\Omega_2 + p_3\Omega_3$$

zero or infinitesimal, so that the origin is a zero of unlimited multiplicity or, in other words, there is a space at the origin containing an unlimited number of zeros. In either case the origin is an essential singularity, contrary to

* Jacobi, *Ges. Werke*, t. ii, p. 27.

the hypothesis that the only essential singularity is for $z = \infty$; and hence a uniform transcendental function cannot exist having a triply-infinite arithmetical succession of zeros.

59. In effecting the formation of a transcendental integral function by means of its primary factors, it was seen that the expression of the primary factor depends upon the values of the integers which make

$$\sum_{n=1}^{\infty} |a_n|^{-m_n-1} |z|^{m_n}$$

a converging series. Moreover, the primary factors are not unique in form, because any finite number of terms of the proper form can be added to the exponential index in

$$\left(1 - \frac{z}{a_n}\right) e^{\sum\limits_{r=1}^{m_n-1} \frac{1}{r} \frac{z^r}{a_n{}^r}},$$

and such terms will only the more effectively secure the convergence of the infinite product. But there is a lower limit to the removal of terms with the highest exponents from the index of the exponential; for there are, in general, minimum values for the integers $m_1, m_2, \ldots$, below which these integers cannot be reduced, if the convergence of the product is to be secured.

The simplest case, in which the exponential must be retained in the primary factor in order to secure the convergence of the infinite product, is that discussed in § 50, viz., when the integers $m_1, m_2, \ldots$ are equal to one another. Let m denote this common value for a given function, and let m be the least integer effective for the purpose: the function is then said* to be of *class* m, and the condition that it should be of class m is that the integer m be the least integer to make the series

$$\sum_{n=1}^{\infty} |a_n|^{-m-1}$$

converge uniformly and unconditionally, the constants a being the zeros of the function.

Thus algebraical polynomials are of class 0; the circular functions $\sin z$ and $\cos z$ are of class 1; Weierstrass's σ-function, and the Jacobian elliptic function $\operatorname{sn} z$ are of class 2, and so on: but in no one of these classes do the functions mentioned constitute the whole of the functions of that class.

60. One or two of the simpler properties of an aggregate of transcendental integral functions of the same class can easily be obtained.

Let a function $f(z)$, of class n, have a zero of order r at the origin and

* The French word is *genre*; the Italian is *genere*. Laguerre (see references on p. 92) appears to have been the first to discuss the *class* of transcendental integral functions.

have $a_1, a_2, \ldots$ for its other zeros, arranged in order of increasing moduli. Then, by § 50, the function $f(z)$ can be expressed in the form

$$f(z) = e^{G(z)} z^r \prod_{i=1}^{\infty} \left\{ \left(1 - \frac{z}{a_i}\right) e^{g_i(z)} \right\},$$

where $g_i(z)$ denotes the series $\sum_{s=1}^{n} \frac{1}{s}\left(\frac{z}{a_i}\right)^s$ and $G(z)$ must be properly determined to secure the equality.

Now the series

$$\sum_{i=1}^{\infty} \frac{1}{a_i^n (a_i - z)}$$

is one which converges uniformly for all values of z that do not coincide with one of the points a, that is, with one of the zeros of the original function. For the sum of the series of the moduli of its terms is

$$\sum_{i=1}^{\infty} \frac{1}{|a_i|^{n+1}} \frac{1}{\left|1 - \frac{z}{a_i}\right|}.$$

Let d be the least of the quantities $\left|1 - \frac{z}{a_i}\right|$, necessarily non-evanescent because z does not coincide with any of the points a; then the sum of the series

$$< \frac{1}{d} \sum_{i=1}^{\infty} \frac{1}{|a_i|^{n+1}},$$

which is a converging series since the function is of class n. Hence the series of moduli converges and therefore the original series converges; let it be denoted by $S(z)$, so that

$$S(z) = \sum_{i=1}^{\infty} \frac{1}{a_i^n (a_i - z)}.$$

We have

$$\begin{aligned} \frac{f'(z)}{f(z)} &= G'(z) + \frac{r}{z} + \sum_{i=1}^{\infty} \frac{1}{a_i} \left\{ 1 + \frac{z}{a_i} + \ldots + \frac{z^{n-1}}{a_i^{n-1}} - \frac{1}{1 - \frac{z}{a_i}} \right\} \\ &= G'(z) + \frac{r}{z} - z^n \sum_{i=1}^{\infty} \frac{1}{a_i^n (a_i - z)} \\ &= G'(z) + \frac{r}{z} - z^n S(z). \end{aligned}$$

Each step of this process is reversible in all cases in which the original product converges; if, therefore, it can be shewn of a function $f(z)$ that $\frac{f'(z)}{f(z)}$ takes this form, the function is thereby proved to be of class n.

If there be no zero at the origin, the term $\frac{r}{z}$ is absent.

If the exponential factor $G(z)$ be a constant so that $G'(z)$ is zero, the function $f(z)$ is said to be a *simple* function of class n.

61. There are one or two criteria to determine the class of a function: the simplest of them is contained in the following proposition, due to Laguerre*.

If, as z tends to the value ∞, a very great value of $|z|$ can be found for which the limit of $z^{-n}\frac{f'(z)}{f(z)}$, where $f(z)$ is a transcendental, integral function, tends uniformly to the value zero, then $f(z)$ is of class n.

Take a circle centre the origin and radius R, equal to this value of $|z|$; then, by § 24, II., the integral

$$\frac{1}{2\pi i}\int \frac{1}{t^n}\frac{f'(t)}{f(t)}\frac{dt}{t-z},$$

taken round the circle, is zero when R becomes indefinitely great. But the value of the integral is, by the Corollary in § 20,

$$\frac{1}{2\pi i}\int^{(0)} \frac{1}{t^n}\frac{f'(t)}{f(t)}\frac{dt}{t-z} + \frac{1}{2\pi i}\int^{(z)} \frac{1}{t^n}\frac{f'(t)}{f(t)}\frac{dt}{t-z} + \frac{1}{2\pi i}\sum_{i=1}^{\infty}\int^{(a_i)} \frac{1}{t^n}\frac{f'(t)}{f(t)}\frac{dt}{t-z},$$

taken round small circles enclosing the origin, the point z, and the points a_i, which are the infinities of the subject of integration; the origin being supposed a zero of $f(t)$ of multiplicity r.

Now
$$\frac{1}{2\pi i}\int^{(z)} \frac{1}{t^n}\frac{f'(t)}{f(t)}\frac{dt}{t-z} = \frac{1}{z^n}\frac{f'(z)}{f(z)},$$

$$\frac{1}{2\pi i}\int^{(a_i)} \frac{1}{t^n}\frac{f'(t)}{f(t)}\frac{dt}{t-z} = \frac{1}{a_i^n}\frac{1}{a_i-z},$$

and
$$\frac{1}{2\pi i}\int^{(0)} \frac{1}{t^n}\frac{f'(t)}{f(t)}\frac{dt}{t-z} = -\frac{\phi(z)}{z^n} - \frac{r}{z^{n+1}},$$

where $\phi(z)$ denotes the integral, algebraical, polynomial

$$\left\{\frac{f'(t)}{f(t)} - \frac{r}{t}\right\} + z\frac{d}{dt}\left\{\frac{f'(t)}{f(t)} - \frac{r}{t}\right\} + \ldots + \frac{z^{n-1}}{n-1!}\frac{d^{n-1}}{dt^{n-1}}\left\{\frac{f'(t)}{f(t)} - \frac{r}{t}\right\}$$

when t is made zero. Hence

$$\frac{1}{z^n}\frac{f'(z)}{f(z)} + \sum_{i=1}^{\infty}\frac{1}{a_i^n(a_i-z)} - \frac{\phi(z)}{z^n} - \frac{r}{z^{n+1}} = 0,$$

and therefore

$$\frac{f'(z)}{f(z)} = \phi(z) + \frac{r}{z} - z^n S(z),$$

which, by § 60, shews that $f(z)$ is of class n.

* *Comptes Rendus*, t. xciv, (1882), p. 636.

COROLLARY. *The product of any finite number of functions of the same class n is a function of class not higher than n; and the class of the product of any finite number of functions of different classes is not greater than the highest class of the component functions.*

The following are the chief references to memoirs discussing the class of functions:

Laguerre, *Comptes Rendus*, t. xciv, (1882), pp. 160—163, pp. 635—638, ib. t. xcv, (1882), pp. 828—831, ib. t. xcviii, (1884), pp. 79—81;

Poincaré, *Bull. des Sciences Math.*, t. xi, (1883), pp. 136—144;

Cesàro, *Comptes Rendus*, t. xcix, (1884), pp. 26—27, followed (p. 27) by a note by Hermite; *Giornale di Battaglini*, t. xxii, (1884), pp. 191—200;

Vivanti, *Giornale di Battaglini*, t. xxii, (1884), pp. 243—261, pp. 378—380, ib. t. xxiii, (1885), pp. 96—122, ib. t. xxvi, (1888), pp. 303—314;

Hermite, *Cours à la faculté des Sciences* (4[me] éd., 1891), pp. 91—93.

Ex. 1. The function

$$\sum_{i=1}^{n} e^{c_i z} f_i(z),$$

where the quantities c are constants, n is a finite integer, and the functions $f_i(z)$ are algebraical polynomials, is of class unity.

Ex. 2. If a simple function be of class n, its derivative is also of class n.

Ex. 3. Discuss the conditions under which the sum of two functions, each of class n, is also of class n.

Ex. 4. Examine the following test for the class of a function, due to Poincaré.

Let a be any number, no matter how small provided its argument be such that $e^{az^{n+1}}$ vanishes when z tends towards infinity. Then $f(z)$ is of class n, if the limit of

$$e^{az^{n+1}} f(z)$$

vanish with indefinite increase of z.

A possible value of a is $\sum\limits_{i=1}^{\infty} c_i a_i^{-n-1}$, where c_i is a constant of modulus unity.

Ex. 5. Verify the following test for the class of a function, due to de Sparre*.

Let λ be any positive non-infinitesimal quantity; then the function $f(z)$ is of class n, if the limit, for $m=\infty$, of

$$|a_m|^{n-1}\{|a_{m+1}|-|a_m|\}$$

be not less than λ. Thus $\sin z$ is of class unity.

Ex. 6. Let the roots of $\theta^{n+1}=1$ be $1, a, a^2, \ldots\ldots, a^n$; and let $f(z)$ be a function of class n. Then forming the product

$$\prod_{s=0}^{n} f(a^s z),$$

we evidently have an integral function of z^{n+1}; let it be denoted by $F(z^{n+1})$. The roots of

* *Comptes Rendus*, t. cii, (1886), p. 741.

$F(z^{n+1})=0$ are $a_i a^s$ for $i=1, 2, \ldots\ldots$ and $s=0, 1, \ldots\ldots, n$; and therefore, replacing z^{n+1} by z, the roots of $F(z)=0$ are a_i^{n+1} for $i=1, 2, \ldots\ldots$.

Since $f(z)$ is of class n, the series

$$\sum_{i=1}^{\infty} \frac{1}{a_i^{n+1}}$$

converges uniformly and unconditionally. This series is the sum of the first powers of the reciprocals of the roots of $F(z)=0$; hence, according to the definition (p. 89), $F(z)$ is of class zero.

It therefore follows that *from a function of any class a function of class zero with a modified variable can be deduced. Conversely, by appropriately modifying the variable of a given function of class zero, it is possible to deduce functions of any required class.*

Ex. 7. If all the zeros of the function

$$\prod_{n=1}^{\infty} \left\{ \left(1 - \frac{z}{a_n}\right) e^{\sum_{r=1}^{k-1} \frac{1}{r} \frac{z^r}{a_n^r}} \right\}$$

be real, then all the zeros of its derivative are also real. (Witting.)

CHAPTER VI.

Functions with a Limited Number of Essential Singularities.

62. Some indications regarding the character of a function at an essential singularity have already been given. Thus, though the function is regular in the vicinity of such a point a, it may, like $\operatorname{sn}\frac{1}{z}$ at the origin, have a zero of unlimited multiplicity or an infinity of unlimited multiplicity at the point; and in either case the point is such that there is no factor of the form $(z-a)^\lambda$ which can be associated with the function so as to make the point an ordinary point for the modified function. Moreover, even when the path of approach to the essential singularity is specified, the value acquired is not definite: thus, as z approaches the origin along the axis of x, so that its value may be taken to be $1 \div (4mK+x)$, the value of $\operatorname{sn}\frac{1}{z}$ is not definite in the limit when m is made infinite. One characteristic of the point is the indefiniteness of value of the function there, though in the vicinity the function is uniform.

A brief statement and a proof of this characteristic were given in § 33; the theorem there proved—that a uniform analytical function can assume any value at an essential singularity—may also be proved as follows. The essential singularity will be taken at infinity—a supposition that will be found not to detract from generality.

Let $f(z)$ be a function having any number of zeros and any number of accidental singularities and $z=\infty$ for its sole essential singularity; then it can be expressed in the form

$$f(z)=\frac{G_1(z)}{G_2(z)}e^{g(z)},$$

where $G_1(z)$ is algebraical or transcendental according as the number of zeros is finite or infinite and $G_2(z)$ is algebraical or transcendental according as the number of accidental singularities is finite or infinite.

If $G_2(z)$ be transcendental, we can omit the generalising factor $e^{g(z)}$. Then $f(z)$ has an infinite number of accidental singularities; each of them in the finite part of the plane is of only finite multiplicity and therefore some of them must be at infinity. At each such point, the function $G_2(z)$ vanishes and $G_1(z)$ does not vanish; and so $f(z)$ has infinite values for $z=\infty$.

If $G_2(z)$ be algebraical and $G_1(z)$ be also algebraical, then the factor $e^{g(z)}$ may not be omitted, for its omission would make $f(z)$ an algebraical function. Now $z=\infty$ is either an ordinary point or an accidental singularity of

$$G_1(z)/G_2(z);$$

hence as $g(z)$ is integral there are infinite values of z which make

$$\frac{G_1(z)}{G_2(z)}e^{g(z)}$$

infinite.

If $G_2(z)$ be algebraical and $G_1(z)$ be transcendental, the factor $e^{g(z)}$ may be omitted. Let $a_1, a_2, \ldots, a_n$ be the roots of $G_2(z)$: then taking

$$f(z)=\sum_{r=1}^{n}\frac{A_r}{z-a_r}+G_n(z),$$

we have

$$A_r=\frac{G_1(a_r)}{G_2'(a_r)},$$

a non-vanishing constant; and so

$$f(z)=\frac{G_3(z)}{G_2(z)}+G_n(z),$$

where $G_n(z)$ is a transcendental integral function. When $z=\infty$, the value of $G_3(z)/G_2(z)$ is zero, but $G_n(z)$ is infinite; hence $f(z)$ has infinite values for $z=\infty$.

Similarly it may be shewn, as follows, that $f(z)$ has zero values for $z=\infty$.

In the first of the preceding cases, if $G_1(z)$ be transcendental, so that $f(z)$ has an infinite number of zeros, then some of them must be at an infinite distance; $f(z)$ has a zero value for each such point. And if $G_1(z)$ be algebraical, then there are infinite values of z which, not being zeros of $G_2(z)$, make $f(z)$ vanish.

In the second case, when z is made infinite with such an argument as to make the highest term in $g(z)$ a real negative quantity, then $f(z)$ vanishes for that infinite value of z.

In the third case, $f(z)$ vanishes for a zero of $G_1(z)$ that is at infinity.

Hence the value of $f(z)$ for $z=\infty$ is not definite. If, moreover, there be any value neither zero nor infinity, say C, which $f(z)$ cannot acquire for $z=\infty$, then

$$f(z)-C$$

is a function which cannot be zero at infinity and therefore all its zeros are in the finite part of the plane: no one of them is an essential singularity, for $f(z)$ has only a single value at any point in the finite part of the plane; hence they are finite in number and are isolated points. Let $H_1(z)$ be the algebraical polynomial having them for its zeros. The accidental singularities of $f(z)-C$ are the accidental singularities of $f(z)$; hence

$$f(z)-C=\frac{H_1(z)}{G_2(z)}e^{h(z)},$$

where, if $G_2(z)$ be algebraical, the exponential $h(z)$ must occur, since $f(z)$, and therefore $f(z) - C$, is transcendental. The function

$$F(z) = \frac{1}{f(z) - C} = \frac{G_2(z)}{H_1(z)} e^{-h(z)}$$

evidently has $z = \infty$ for an essential singularity, so that, by the second or the third case above, it certainly has an infinite value for $z = \infty$, that is, $f(z)$ certainly acquires the value C for $z = \infty$.

Hence the function can acquire any value at an essential singularity.

63. We now proceed to obtain the character of the expression of a function at a point z which, lying in the region of continuity, is in the vicinity of an essential singularity b in the finite part of the plane.

With b as centre describe two circles, so that their circumferences and the whole area between them lie entirely within the region of continuity. The radius of the inner circle is to be as small as possible consistent with this condition; and therefore, as it will be assumed that b is the only singularity in its own immediate vicinity, this radius may be made very small.

The ordinary point z of the function may be taken as lying within the circular ring-formed part of the region of continuity. At all such points in this band, the function is holomorphic; and therefore, by Laurent's Theorem (§ 28), it can be expanded in a converging series of positive and negative integral powers of $z - b$ in the form

$$u_0 + u_1(z-b) + u_2(z-b)^2 + \dots$$
$$+ v_1(z-b)^{-1} + v_2(z-b)^{-2} + \dots,$$

where the coefficients u_n are determined by the equation

$$u_n = \frac{1}{2\pi i}\int \frac{f(t)}{(t-b)^{n+1}}\,dt, \quad (n = 0, 1, 2, \dots),$$

the integrals being taken positively round the outer circle, and the coefficients v_n are determined by the equation

$$v_n = \frac{1}{2\pi i}\int (t'-b)^{n-1} f(t')\,dt',$$

the integrals being taken positively round the inner circle.

The series of positive powers converges everywhere within the outer circle of centre b, and so (§ 26) it may be denoted by $P(z-b)$; and the function P may be either algebraical or transcendental.

The series of negative powers converges everywhere without the inner circle of centre b; and, since b is not an accidental but an essential singularity of the function, the series of negative powers contains an infinite number of terms. It may be denoted by $G\left(\frac{1}{z-b}\right)$, a series converging for all points in the plane except $z=b$ and vanishing when $z-b=\infty$.

Thus $$f(z)=G\left(\frac{1}{z-b}\right)+P(z-b)$$

is the analytical representation of the function in the vicinity of its essential singularity b; the function G is transcendental and converges everywhere in the plane except at $z=b$, and the function P, if transcendental, converges uniformly and unconditionally for sufficiently small values of $|z-b|$.

Had the singularity at b been accidental, the function G would have been algebraical.

Corollary I. If the function have any essential singularity other than b, it is an essential singularity of $P(z-b)$ continued outside the outer circle; but it is not an essential singularity of $G\left(\frac{1}{z-b}\right)$, for the latter function converges everywhere in the plane outside the inner circle.

Corollary II. Suppose the function has no singularity in the plane except at the point b; then the outer circle can have its radius made infinite. In that case, all positive powers except the constant term u_0 disappear: and even this term survives only in case the function have a finite value at infinity. The expression for the function is

$$u_0+\frac{v_1}{z-b}+\frac{v_2}{(z-b)^2}+\dots,$$

and the transcendental series converges everywhere outside the infinitesimal circle round b, that is, everywhere in the plane except at the point b. Hence the function can be represented by

$$G\left(\frac{1}{z-b}\right).$$

This special result is deduced by Weierstrass from the earlier investigations*, as follows. If $f(z)$ be such a function with an essential singularity at b, and if we change the independent variable by the relation

$$z'=\frac{1}{z-b},$$

then $f(z)$ changes into a function of z', the only essential singularity of which is at $z'=\infty$. It has no other singularity in the plane; and the form of the function is therefore $G(z')$, that is, *a function having an essential singularity at b but no other singularity in the plane is*

$$G\left(\frac{1}{z-b}\right).$$

* Weierstrass (l.c.), p. 27.

COROLLARY III. *The most general expression of a function having its sole essential singularity at b a point in the finite part of the plane and any number of accidental singularities is*

$$\frac{G_1\left(\frac{1}{z-b}\right)}{G_2\left(\frac{1}{z-b}\right)} e^{g\left(\frac{1}{z-b}\right)},$$

where the zeros of the function are the zeros of G_1, the accidental singularities of the function are the zeros of G_2, and the function g in the exponential is a function which is finite everywhere except at b.

This can be derived in the same way as before; or it can be deduced from the corresponding theorem relating to transcendental integral functions, as above. It would be necessary to construct an integral function $G_2(z')$ having as its zeros

$$\frac{1}{a_1 - b}, \quad \frac{1}{a_2 - b}, \ldots,$$

and then to replace z' by $\frac{1}{z-b}$; and G_2 is algebraical or transcendental, according as the number of zeros is finite or infinite.

Similarly we obtain the following result:

COROLLARY IV. *A uniform function of z, which has its sole essential singularity at b a point in the finite part of the plane and no accidental singularities, can be represented in the form of an infinite product of primary factors of the form*

$$\left(\frac{k}{z-b} + l\right) e^{g\left(\frac{1}{z-b}\right)},$$

which converges uniformly and unconditionally everywhere in the plane except at $z = b$.

The function $g\left(\frac{1}{z-b}\right)$ is an integral function of $\frac{1}{z-b}$ vanishing when $\frac{1}{z-b}$ vanishes; and k and l are constants. In particular factors, $g\left(\frac{1}{z-b}\right)$ may vanish; and either k or l (but not both k and l) may vanish with or without a vanishing exponent $g\left(\frac{1}{z-b}\right)$.

If a_i be any zero, the corresponding primary factor may evidently be expressed in the form

$$\left(\frac{z-a_i}{z-b}\right) e^{g_i\left(\frac{1}{z-b}\right)}.$$

Similarly, for a uniform function of z with its sole essential singularity at b and any number of accidental singularities, the product-form is at once derivable by applying the result of the present Corollary to the result given in Corollary III.

These results, combined with the results of Chapter V., give the complete general theory of uniform functions with only one essential singularity.

64. We now proceed to the consideration of functions, which have a limited number of assigned essential singularities.

The theorem of § 63 gives an expression for the function at any point in the band between the two circles there drawn.

Let c be such a point, which is thus an ordinary point for the function; then in the domain of c, the function is expansible in a form $P_1(z-c)$. This domain may extend to an essential singularity b, or it may be limited by a pole d which is nearer to c than b is, or it may be limited by an essential singularity f which is nearer to c than b is. In the first case, we form a continuation of the function in a direction away from b; in the second case, we continue the function by associating with the function a factor $(z-d)^n$ which takes account of the accidental singularity; in the third case, we form a continuation of the function towards f. Taking the continuations for successive domains of points in the vicinity of f, we can obtain the value of the function for points on two circles that have f for their common centre. Using these values, as in § 63, to obtain coefficients, we ultimately construct a series of positive and negative powers converging except at f for the vicinity of f. Different expressions in different parts of the plane will thus be obtained, each being valid only in a particular portion: the aggregate of all of them is the analytical expression of the function for the whole of the region of the plane where the function exists.

We thus have one mode of representation of the function; its chief advantage is that it indicates the form in the vicinity of any point, though it gives no suggestion of the possible modification of character elsewhere. This deficiency renders the representation insufficiently precise and complete; and it is therefore necessary to have another mode of representation.

65. Suppose that the function has n essential singularities $a_1, a_2, \ldots, a_n$ and that it has no other singularity. Let a circle, or any simple closed curve, be drawn enclosing them all, every point of the boundary as well as the included area (with the exception of the n singularities) lying in the region of continuity of the function.

Let z be any ordinary point in the interior of the circle or curve; and consider the integral

$$\int \frac{f(t)}{t-z}\,dt,$$

taken round the curve. If we surround z and each of the n singularities by small circles with the respective points for centres, then the integral round the outer curve is equal to the sum of the values of the integral taken round the $n+1$ circles. Thus

$$\frac{1}{2\pi i}\int_s \frac{f(t)}{t-z}\,dt = \frac{1}{2\pi i}\int_z \frac{f(t)}{t-z}\,dt + \frac{1}{2\pi i}\Sigma\int_{a_r} \frac{f(t)}{t-z}\,dt,$$

and therefore

$$\frac{1}{2\pi i}\int_z \frac{f(t)}{t-z}\,dt = \frac{1}{2\pi i}\int_s \frac{f(t)}{t-z}\,dt - \frac{1}{2\pi i}\Sigma\int_{a_r} \frac{f(t)}{t-z}\,dt.$$

The left-hand side of the equation is $f(z)$.

Evaluating the integrals, we have

$$\frac{1}{2\pi i}\int_{a_r} \frac{f(t)}{t-z}\,dt = -G_r\left(\frac{1}{z-a_r}\right),$$

where G_r is, as before, a transcendental function of $\frac{1}{z-a_r}$ vanishing when $\frac{1}{z-a_r}$ is zero.

Now, of these functions, $G_r\left(\frac{1}{z-a_r}\right)$ converges everywhere in the plane except at a_r: and therefore, as n is finite,

$$\sum_{r=1}^{n} G_r\left(\frac{1}{z-a_r}\right)$$

is a function which converges everywhere in the plane except at the n points $a_1, \ldots, a_n$.

Because $z=\infty$ is not an essential singularity of $f(z)$, the radius of the circle in the integral $\frac{1}{2\pi i}\int_s \frac{f(t)}{t-z}\,dt$ may be indefinitely increased. The value of $f(t)$ tends, with unlimited increase of t, to some determinate value C which is not infinite; hence, as in § 24, II., Corollary, the value of the integral is C. We therefore have the result that $f(z)$ can be expressed in the form

$$C + \sum_{r=1}^{n} G_r\left(\frac{1}{z-a_r}\right),$$

or, absorbing the constant C into the functions G and replacing the limitation, that the function $G_r\left(\frac{1}{z-a_r}\right)$ shall vanish for $\frac{1}{z-a_r}=0$, by the limitation that, for the same value $\frac{1}{z-a_r}=0$, it shall be finite, we have the theorem*:—

If a given function $f(z)$ have n singularities $a_1, \ldots, a_n$, all of which are in the finite part of the plane and are essential singularities, it can be expressed in the form

$$\sum_{r=1}^{n} G_r\left(\frac{1}{z-a_r}\right),$$

* The method of proof, by an integration, is used for brevity: the theorem can be established by purely algebraical reasoning.

where G_r is a transcendental function converging everywhere in the plane except at a_r and having a determinate finite value g_r for $\frac{1}{z-a_r}=0$, such that $\sum_{r=1}^{n} g_r$ is the finite value of the given function at infinity.

COROLLARY. If the given function have a singularity at ∞, and n singularities in the finite part of the plane, then the function can be expressed in the form

$$G(z)+\sum_{r=1}^{n} G_r\left(\frac{1}{z-a_r}\right),$$

where G_r is a transcendental or an algebraic polynomial function, according as a_r is an essential or an accidental singularity: and so also for $G(z)$, according to the character of the singularity at infinity.

66. Any uniform function, which has an essential singularity at $z=a$, can (§ 63) be expressed in the form

$$g\left(\frac{1}{z-a}\right)+p(z-a)$$

for points z in the vicinity of a. Suppose that, for points in this vicinity, the function $f(z)$ has no zero; that it has no accidental singularity; and therefore, among such points z, the function

$$\frac{1}{f(z)}\frac{df(z)}{dz}$$

has no pole, and therefore no singularity except that at a which is essential. Hence it can be expanded in the form

$$G\left(\frac{1}{z-a}\right)+P(z-a),$$

where G converges everywhere in the plane except at a, and vanishes for $\frac{1}{z-a}=0$. Let

$$G\left(\frac{1}{z-a}\right)=\frac{c}{z-a}+\frac{d}{dz}\left\{G_1\left(\frac{1}{z-a}\right)\right\},$$

where $G_1\left(\frac{1}{z-a}\right)$ converges everywhere in the plane except at a, and vanishes for $\frac{1}{z-a}=0$.

Then c, evidently not an infinite quantity, is an integer. To prove this, describe a small circle of radius ρ round a: then taking $z-a=\rho e^{\theta i}$ so that $\frac{dz}{z-a}=id\theta$, we have

$$\frac{1}{f(z)}\frac{df(z)}{dz}dz=P(z-a)\,dz+cid\theta+\frac{d}{dz}\left\{G_1\left(\frac{1}{z-a}\right)\right\}dz,$$

and therefore

$$f(z) = C e^{ci\theta + \int P(z-a)\,dz + G_1\left(\frac{1}{z-a}\right)}.$$

Now $\int P(z-a)\,dz$ is a uniform function: and so is $f(z)$. But a change of θ into $\theta + 2\pi$ does not alter z or any of the functions: thus

$$e^{ci2\pi} = 1;$$

and therefore c is an integer.

67. If the function $f(z)$ have essential singularities $a_1, \ldots, a_n$ and no others, then it can be expressed in the form

$$C + \sum_{r=1}^{n} g_r\left(\frac{1}{z-a_r}\right).$$

If there be no zeros for this function $f(z)$ anywhere (except of course such as may enter through the indeterminateness at the essential singularities), then

$$\frac{1}{f(z)} \frac{df(z)}{dz}$$

has n essential singularities $a_1, \ldots, a_n$ and no other singularities of any kind. Hence it can be expressed in the form

$$C + \sum_{r=1}^{n} G_r\left(\frac{1}{z-a_r}\right),$$

where the function G_r vanishes with $\dfrac{1}{z-a_r}$. Let

$$G_r\left(\frac{1}{z-a_r}\right) = \frac{c_r}{z-a_r} + \frac{d}{dz}\left\{G_r\left(\frac{1}{z-a_r}\right)\right\},$$

where $G_r\left(\dfrac{1}{z-a_r}\right)$ is a function of the same kind as $G_r\left(\dfrac{1}{z-a_r}\right)$.

Then all the coefficients c_r, evidently not infinite quantities, are integers. For, let a small circle of radius ρ be drawn round a_r: then, if $z - a_r = \rho e^{\theta i}$, we have

$$\frac{c_r dz}{z-a_r} = c_r i\theta,$$

and
$$\frac{c_s dz}{z-a_s} = dP_s(z-a_r).$$

We proceed as before: the expression for the function in the former case is changed so that now the sum $\Sigma P_s(z-a_r)$ for $s = 1, \ldots, r-1, r+1, \ldots, n$ is a uniform function; there is no other change. In exactly the same way as before, we shew that every one of the coefficients c_r is an integer.

Hence it appears that if a given function $f(z)$ have, in the finite part of the plane, n essential singularities $a_1, \ldots, a_n$ and no other singularities and if it have no zeros anywhere in the plane, then

$$\frac{1}{f(z)} \frac{df(z)}{dz} = C + \sum_{i=1}^{n} \frac{c_i}{z - a_i} + \sum_{i=1}^{n} \frac{d}{dz} \left\{ \bar{G}_i \left(\frac{1}{z - a_i} \right) \right\},$$

where all the coefficients c_i are integers, and the functions $\bar{G}$ converge everywhere in the plane except at the essential singularities and $\bar{G}_i$ vanishes for

$$\frac{1}{z - a_i} = 0.$$

Now, since $f(z)$ has no singularity at ∞, we have for very large values of z

$$f(z) = u_0 + \frac{v_1}{z} + \frac{v_2}{z^2} + \ldots,$$

and

$$f'(z) = -\frac{v_1}{z^2} - \frac{2v_2}{z^3} - \ldots,$$

and therefore, for very large values of z,

$$\frac{1}{f(z)} \frac{df(z)}{dz} = -\frac{v_1}{u_0} \frac{1}{z^2} + \frac{w_1}{z^3} + \ldots.$$

Thus there is no constant term in $\dfrac{1}{f(z)} \dfrac{df(z)}{dz}$, and there is no term in $\dfrac{1}{z}$. But the above expression for it gives C as the constant term, which must therefore vanish; and it gives Σc_i as the coefficient of $\dfrac{1}{z}$, for $\dfrac{d}{dz} \left\{ \bar{G}_i \left(\dfrac{1}{z - a_i} \right) \right\}$ will begin with $\dfrac{1}{z^2}$ at least; thus Σc_i must therefore also vanish.

Hence for a function $f(z)$ which has no singularity at $z = \infty$ and no zeros anywhere in the plane and of which the only singularities are the n essential singularities at $a_1, a_2, \ldots, a_n$, we have

$$\frac{1}{f(z)} \frac{df(z)}{dz} = \sum_{i=1}^{n} \frac{c_i}{z - a_i} + \sum_{i=1}^{n} \frac{d}{dz} \left\{ \bar{G}_i \left(\frac{1}{z - a_i} \right) \right\},$$

where the coefficients c_i are integers subject to the condition

$$\sum_{i=1}^{n} c_i = 0.$$

If $a_n = \infty$, so that $z = \infty$ is an essential singularity in addition to $a_1, a_2, \ldots, a_{n-1}$, there is a term $G(z)$ instead of $G_n \left(\dfrac{1}{z - a_n} \right)$; there is no term, that corresponds to $\dfrac{c_n}{z - a_n}$, but there may be a constant C. Writing

$$C + G(z) = \frac{d}{dz} \{ \bar{G}(z) \},$$

with the condition that $\bar{G}(z)$ vanishes when $z = 0$, we then have

$$\frac{1}{f(z)} \frac{df(z)}{dz} = \sum_{i=1}^{n-1} \frac{c_i}{z - a_i} + \frac{d}{dz} \{ \bar{G}(z) \} + \sum_{i=1}^{n-1} \frac{d}{dz} \left\{ \bar{G}_i \left(\frac{1}{z - a_i} \right) \right\},$$

where the coefficients c_i are integers, but are no longer subject to the condition that their sum vanishes.

Let $R^*(z)$ denote the function

$$\prod_{i=1} (z - a_i)^{c_i},$$

the product extending over the factors associated with the essential singularities of $f(z)$ that lie in the finite part of the plane; thus $R^*(z)$ is a rational algebraical meromorphic function. Since

$$\frac{1}{R^*(z)} \frac{dR^*(z)}{dz} = \sum_{i=1} \frac{c_i}{z - a_i},$$

we have

$$\frac{1}{f(z)} \frac{df(z)}{dz} - \frac{1}{R^*(z)} \frac{dR^*(z)}{dz} = \sum_{i=1}^{n} \frac{d}{dz} \left\{ \overline{G}_i \left(\frac{1}{z - a_i} \right) \right\},$$

where $\overline{G}_n \left(\frac{1}{z - a_n} \right)$ is to be replaced by $\overline{G}(z)$ if $a_n = \infty$, that is, if $z = \infty$ be an essential singularity of $f(z)$. Hence, except as to an undetermined constant factor, we have

$$f(z) = R^*(z) \prod_{i=1}^{n} e^{\overline{G}_i \left(\frac{1}{z - a_i} \right)},$$

which is therefore *an analytical representation of a function with n essential singularities, no accidental singularities, and no zeros: and the rational algebraical function $R^*(z)$ becomes zero or ∞ only at the singularities of $f(z)$.*

If $z = \infty$ be not an essential singularity, then $R^*(z)$ for $z = \infty$ is equal to unity because $\sum_{i=1}^{n} c_i = 0$.

COROLLARY. It is easy to see, from § 43, that, if the point a_i be only an accidental singularity, then c_i is a negative integer and $\overline{G}_i \left(\frac{1}{z - a_i} \right)$ is zero: so that the polar property at a_i is determined by the occurrence of a factor $(z - a_i)^{c_i}$ solely in the denominator of the rational meromorphic function $R^*(z)$.

And, in general, each of the integral coefficients c_i is determined from the expansion of the function $f'(z) \div f(z)$ in the vicinity of the singularity with which it is associated.

68. Another form of expression for the function can be obtained from the preceding; and it is valid even when the function has zeros not absorbed into the essential singularities†.

Consider a function with one essential singularity, and let a be the point; and suppose that, within a finite circle of centre a (or within a finite simple curve which encloses a), there are m simple zeros $\alpha, \beta, \ldots, \lambda$ of the

† See Guichard, *Théorie des points singuliers essentiels*, (Thèse, Gauthier-Villars, Paris, 1883), especially the first part.

function $f(z)$—m being assumed to be finite, and it being also assumed that there are no accidental singularities within the circle. Then, if

$$f(z) = (z-\alpha)(z-\beta)\dots(z-\lambda) F(z),$$

the function $F(z)$ has a for an essential singularity and has no zeros within the circle. Hence, for points z within the circle,

$$\frac{F'(z)}{F(z)} = \frac{c}{z-a} + \frac{d}{dz}\left\{G_1\left(\frac{1}{z-a}\right)\right\} + P(z-a),$$

where $G_1\left(\frac{1}{z-a}\right)$ converges everywhere in the plane and vanishes with $\frac{1}{z-a}$, and $P(z-a)$ is an integral function converging uniformly and unconditionally within the circle; moreover, c is an integer. Thus

$$F(z) = A(z-a)^c e^{G_1\left(\frac{1}{z-a}\right)} e^{\int P(z-a)dz}.$$

Let $$(z-\alpha)(z-\beta)\dots(z-\lambda) = (z-a)^m \left\{1 + \frac{p_1}{z-a} + \dots + \frac{p_m}{(z-a)^m}\right\}$$

$$= (z-a)^m g_1\left(\frac{1}{z-a}\right);$$

then $$f(z) = (z-a)^m g_1\left(\frac{1}{z-a}\right) F(z)$$

$$= A(z-a)^{m+c} g_1\left(\frac{1}{z-a}\right) e^{G_1\left(\frac{1}{z-a}\right)} e^{\int P(z-a)\,dz}.$$

Now of this product-expression for $f(z)$ it should be noted:—

(i) That $m+c$ is an integer, finite because m and c are finite:

(ii) The function $e^{G_1\left(\frac{1}{z-a}\right)}$ can be expressed in the form of a series converging uniformly and unconditionally everywhere, except at $z=a$, and proceeding in powers of $\frac{1}{z-a}$ in the form

$$1 + \frac{b_1}{z-a} + \frac{b_2}{(z-a)^2} + \dots.$$

It has no zero within the circle considered, for $F(z)$ has no zero. Also $g_1\left(\frac{1}{z-a}\right)$ is an algebraical function of $\frac{1}{z-a}$, beginning with unity and containing only a finite number of terms: hence, multiplying the two series together, we have as the product a series proceeding in powers of $\frac{1}{z-a}$ in the form

$$1 + \frac{h_1}{z-a} + \frac{h_2}{(z-a)^2} + \dots,$$

which converges uniformly and unconditionally everywhere outside any small circle round a, that is, everywhere except at a. Let this series be denoted by

$H\left(\frac{1}{z-a}\right)$; it has an essential singularity at a and its only zeros are the points $\alpha, \beta, \ldots, \lambda$, for the series multiplied by $g_1\left(\frac{1}{z-a}\right)$ has no zeros:

(iii) The function $\int P(z-a)\,dz$ is a series of positive powers of $z-a$, converging uniformly in the vicinity of a; and therefore $e^{\int P(z-a)dz}$ can be expanded in a series of positive integral powers of $z-a$ which converges in the vicinity of a. Let it be denoted by $Q(z-a)$ which, since it is a factor of $F(z)$, has no zeros within the circle.

Hence we have

$$f(z) = A(z-a)^{\mu}\, Q(z-a)\, H\left(\frac{1}{z-a}\right),$$

where μ is an integer; $H\left(\frac{1}{z-a}\right)$ is a series that converges everywhere except at a, is equal to unity when $\frac{1}{z-a}$ vanishes, and has as its zeros the (finite) number of zeros assigned to $f(z)$ within a finite circle of centre a; and $Q(z-a)$ is a series of positive powers of $z-a$ beginning with unity which converges—but has no zero—within the circle.

The foregoing function $f(z)$ is supposed to have no essential singularity except at a. If, however, a given function have singularities at points other than a, then the circle would be taken of radius less than the distance of a from the nearest essential singularity.

Introducing a new function $f_1(z)$ defined by the equation

$$f(z) = A(z-a)^{\mu}\, H\left(\frac{1}{z-a}\right) f_1(z),$$

the value of $f_1(z)$ is $Q(z-a)$ within the circle, but it is not determined by the foregoing analysis for points without the circle. Moreover, as $(z-a)^{\mu}$ and also $H\left(\frac{1}{z-a}\right)$ are finite everywhere except possibly at a, it follows that essential singularities of $f(z)$ other than a must be essential singularities of $f_1(z)$. Also since $f_1(z)$ is $Q(z-a)$ in the immediate vicinity of a, this point is not an essential singularity of $f_1(z)$.

Thus $f_1(z)$ is a function of the same kind as $f(z)$; it has all the essential singularities of $f(z)$ except a, but it has fewer zeros, on account of the m zeros of $f(z)$ possessed by $H\left(\frac{1}{z-a}\right)$. The foregoing expression for $f(z)$ is the one referred to at the beginning of the section.

If we choose to absorb into $f_1(z)$ the factors $e^{G_1\left(\frac{1}{z-a}\right)}$ and $e^{\int P(z-a)\,dz}$, which occur in

$$A(z-a)^{m+c}\, g_1\left(\frac{1}{z-a}\right) e^{G_1\left(\frac{1}{z-a}\right)} e^{\int P(z-a)\,dz},$$

an expression that is valid within the circle considered, then we obtain a result that is otherwise obvious, by taking

$$f(z) = (z-a)^{\mu} g_1\left(\frac{1}{z-a}\right) f_1(z),$$

where now $g_1\left(\frac{1}{z-a}\right)$ is algebraical and has for its zeros all the zeros within the circle; μ is an integer; and $f_1(z)$ is a function of the same kind as $f(z)$, which now possesses all the essential singularities of $f(z)$ but has zeros fewer by the m zeros that are possessed by $g_1\left(\frac{1}{z-a}\right)$.

69. Next, consider a function $f(z)$ with n essential singularities $a_1, a_2, \dots, a_n$ but without accidental singularities; and let it have any number of zeros.

When the zeros are limited in number, they may be taken to be isolated points, distinct in position from the essential singularities.

When the zeros are unlimited in number, then at least one of the singularities must be such that an infinite number of the zeros lie within a circle of finite radius, described round it as centre and containing no other singularity. For if there be not an infinite number in such a vicinity of some one point (which can only be an essential singularity, otherwise the function would be zero everywhere), then the points are isolated and there must be an infinite number at $z=\infty$. If $z=\infty$ be an essential singularity, the above alternative is satisfied: if not, the function, being continuous save at singularities, must be zero at all other parts of the plane. Hence it follows that if a uniform function have a finite number of essential singularities and an infinite number of zeros, all but a finite number of the zeros lie within circles of finite radii described round the essential singularities as centres; at least one of the circles contains an infinite number of the zeros, and some of the circles may contain only a finite number of them.

We divide the whole plane into regions, each containing one but only one singularity and containing also the circle round the singularity; let the region containing a_i be denoted by C_i, and let the region C_n be the part of the plane other than $C_1, C_2, \dots\dots, C_{n-1}$.

If the region C_1 contain only a limited number of the zeros, then, by § 68, we can choose a new function $f_1(z)$ such that, if

$$f(z) = (z-a_1)^{\mu_1} G_1\left(\frac{1}{z-a_1}\right) f_1(z),$$

the function $f_1(z)$ has a_1 for an ordinary point, has no zeros within the region C_1, and has $a_2, a_3, \dots\dots, a_n$ for its essential singularities.

If the region C_1 contain an unlimited number of the zeros, then, as in Corollaries II. and III. of § 63, we construct any transcendental function

$\bar{G}_1\left(\frac{1}{z-a_1}\right)$, having a_1 for its sole essential singularity and the zeros in C_1 for all its zeros. When we introduce a function $g_1(z)$, defined by the equation

$$f(z)=\bar{G}_1\left(\frac{1}{z-a_1}\right)g_1(z),$$

the function $g_1(z)$ has no zeros in C_1 and certainly has $a_2, a_3, \ldots\ldots, a_n$ for essential singularities; in the absence of the generalising factor of $\bar{G}_1$, it can have a_1 for an essential singularity. By § 67, the function $\bar{g}_1(z)$, defined by

$$\bar{g}_1(z)=(z-a_1)^{c_1}e^{h_1\left(\frac{1}{z-a_1}\right)},$$

has no zero and no accidental singularity, and it has a_1 as its sole essential singularity: hence, properly choosing c_1 and h_1, we may take

$$g_1(z)=\bar{g}_1(z)f_1(z),$$

so that $f_1(z)$ does not have a_1 as an essential singularity, but it has all the remaining singularities of $g_1(z)$, and it has no zeros within C_1.

In either case, we have a new function $f_1(z)$ given by

$$f(z)=(z-a_1)^{\mu_1}G_1\left(\frac{1}{z-a_1}\right)f_1(z),$$

where μ_1 is an integer, the zeros of $f(z)$ that lie in C_1 are the zeros of G_1; the function $f_1(z)$ has $a_2, a_3, \ldots\ldots, a_n$ (but not a_1) for its essential singularities, and it has the zeros of $f(z)$ in the remaining regions for its zeros.

Similarly, considering C_2, we obtain a function $f_2(z)$, such that

$$f_1(z)=(z-a_2)^{\mu_2}G_2\left(\frac{1}{z-a_2}\right)f_2(z),$$

where μ_2 is an integer, G_2 is a transcendental function finite everywhere except at a_2 and has for its zeros all the zeros of $f_1(z)$—and therefore all the zeros of $f(z)$—that lie in C_2; then $f_2(z)$ possesses all the zeros of $f(z)$ in the regions other than C_1 and C_2, and has $a_3, a_4, \ldots, a_n$ for its essential singularities.

Proceeding in this manner, we ultimately obtain a function $f_n(z)$ which has none of the zeros of $f(z)$ in any of the n regions $C_1, C_2, \ldots, C_n$, that is, has no zeros in the plane, and it has no essential singularities; it has no accidental singularities, and therefore $f_n(z)$ is a constant. Hence, when we substitute, and denote by $S^*(z)$ the product $\prod_{i=1}^{n}(z-a_i)^{\mu_i}$, we have

$$f(z)=S^*(z)\prod_{i=1}^{n}G_i\left(\frac{1}{z-a_i}\right)$$

as the most general form of a function having n essential singularities, no accidental singularities, and any number of zeros. The function $S^*(z)$ is a rational algebraical function of z, usually meromorphic in form, and it has the essential singularities of $f(z)$ as its zeros and poles; and the zeros of $f(z)$ are distributed among the functions G_i.

As however the distribution of the zeros by the regions C and therefore the functions $G\left(\frac{1}{z-a}\right)$ are somewhat arbitrary, the above form though general is not unique.

If any one of the singularities, say a_m, had been accidental and not essential, then in the corresponding form the function $G_m\left(\frac{1}{z-a_m}\right)$ would be algebraical and not transcendental.

70. A function $f(z)$, which has *any finite number of accidental singularities in addition to n assigned essential singularities and any number of assigned zeros,* can be constructed as follows.

Let $A(z)$ be the algebraical polynomial which has, for its zeros, the accidental singularities of $f(z)$, each in its proper multiplicity. Then the product

$$f(z)\,A(z)$$

is a function which has no accidental singularities; its zeros and its essential singularities are the assigned zeros and the assigned essential singularities of $f(z)$ and therefore it is included in the form

$$S^*(z)\prod_{i=1}^{n}\left\{G_i\left(\frac{1}{z-a_i}\right)\right\},$$

where $S^*(z)$ is a rational algebraical meromorphic function having the points $a_1, a_2, \ldots, a_n$ for zeros and poles. The form of the function $f(z)$ is therefore

$$\frac{S^*(z)}{A(z)}\prod_{i=1}^{n}\left\{G_i\left(\frac{1}{z-a_i}\right)\right\}.$$

71. A function $f(z)$, which has *an unlimited number of accidental singularities in addition to n assigned essential singularities and any number of assigned zeros,* can be constructed as follows.

Let the accidental singularities be $\alpha', \beta', \ldots$. Construct a function $f_1(z)$, having the n essential singularities assigned to $f(z)$, no accidental singularities, and the series $\alpha', \beta', \ldots$ of zeros. It will, by § 69, be of the form of a product of n transcendental functions $G_{n+1}, \ldots, G_{2n}$ which are such that a function G has for its zeros the zeros of $f_1(z)$ lying within a region of the plane, divided as in § 69; and the function G_{n+i} is associated with the point a_i.

Thus $$f_1(z) = T^*(z)\prod_{i=1}^{n} G_{n+i}\left(\frac{1}{z-a_i}\right),$$

where $T^*(z)$ is a rational algebraical meromorphic function having its zeros and its poles, each of finite multiplicity, at the essential singularities of $f(z)$.

Because the accidental singularities of $f(z)$ are the same points and have the same multiplicity as the zeros of $f_1(z)$, the function $f(z)\,f_1(z)$ has no accidental singularities. This new function has all the zeros of $f(z)$, and $a_1, \ldots, a_n$ are its essential singularities; moreover, it has no accidental singularities. Hence the product $f(z)\,f_1(z)$ can be represented in the form

$$S^*(z) \prod_{i=1}^{n} G_i\left(\frac{1}{z-a_i}\right),$$

and therefore we have

$$f(z) = \frac{S^*(z)}{T^*(z)} \prod_{i=1}^{n} \frac{G_i\left(\dfrac{1}{z-a_i}\right)}{G_{n+i}\left(\dfrac{1}{z-a_i}\right)}$$

as an expression of the function.

But, as by their distribution through the n selected regions of the plane in § 69, the zeros can to some extent be arbitrarily associated with the functions $G_1, G_2, \ldots, G_n$ and likewise the accidental singularities can to some extent be arbitrarily associated with the functions $G_{n+1}, G_{n+2}, \ldots, G_{2n}$, the product-expression just obtained, though definite in character, is not unique in the detailed form of the functions which occur.

The fraction
$$\frac{S^*(z)}{T^*(z)}$$
is algebraical and rational; and it vanishes or becomes infinite only at the essential singularities $a_1, a_2, \ldots, a_n$, being the product of factors of the form $(z-a_i)^{m_i}$, for $i = 1, 2, \ldots, n$. Let the power $(z-a_i)^{m_i}$ be absorbed into the function G_i/G_{n+i} for each of the n values of i; no substantial change in the transcendental character of G_i and of G_{n+i} is thereby caused, and we may therefore use the same symbol to denote the modified function after the absorption. Hence† *the most general product-expression of a uniform function of z which has n essential singularities $a_1, a_2, \ldots, a_n$, any unlimited number of assigned zeros and any unlimited number of assigned accidental singularities is*

$$\prod_{i=1}^{n} \frac{G_i\left(\dfrac{1}{z-a_i}\right)}{G_{n+i}\left(\dfrac{1}{z-a_i}\right)}.$$

The resolution of a transcendental function with one essential singularity into its primary factors, each of which gives only a single zero of the function, has been obtained in § 63, Corollary IV.

We therefore resolve each of the functions $G_1, \ldots, G_{2n}$ into its primary factors. Each factor of the first n functions will contain one and only one zero of the original functions $f(z)$; and each factor of the second n functions will contain one and only one of the poles of $f(z)$. The sole essential singularity of each primary factor is one of the essential singularities of $f(z)$. Hence we have a method of constructing a uniform function with any finite number of essential singularities as a uniformly converging product of any number of primary factors, each of which has one of the essential singularities as its sole essential singularity and either (i) has as its sole zero either one of the zeros

† Weierstrass, l.c., p. 48.

or one of the accidental singularities of $f(z)$, so that it is of the form

$$\left(\frac{z-\epsilon}{z-c}\right) e^{g\left(\frac{1}{z-c}\right)};$$

or (ii) it has no zero and then it is of the form

$$e^{g\left(\frac{1}{z-c}\right)}.$$

When all the primary factors of the latter form are combined, they constitute a generalising factor in exactly the same way as in § 52 and in § 63, Cor. III., except that now the number of essential singularities is not limited to unity.

Two forms of expression of a function with a limited number of essential singularities have been obtained: one (§ 65) as a sum, the other (§ 69) as a product, of functions each of which has only one essential singularity. Intermediate expressions, partly product and partly sum, can be derived, e.g. expressions of the form

$$\frac{\sum_{i=1}^{n} G_i\left(\frac{1}{z-c_i}\right)}{\sum_{i=1}^{n} G_{n+i}\left(\frac{1}{z-c_i}\right)}.$$

But the pure product-expression is the most general, in that it brings into evidence not merely the n essential singularities but also the zeros and the accidental singularities, whereas the expression as a sum tacitly requires that the function shall have no singularities other than the n which are essential.

Note. The formation of the various elements, the aggregate of which is the complete representation of the function with a limited number of essential singularities, can be carried out in the same manner as in § 34; each element is associated with a particular domain, the range of the domain is limited by the nearest singularities, and the aggregate of the singularities forms the boundary of the region of continuity.

To avoid the practical difficulty of the gradual formation of the region of continuity by the construction of the successive domains when there is a limited number of singularities (and also, if desirable to be considered, of branch-points), Fuchs devised a method which simplifies the process. The basis of the method is an appropriate change of the independent variable. The result of that change is to divide the plane of the modified variable ζ into two portions, one of which, G_2, is finite in area and the other of which, G_1, occupies the rest of the plane; and the boundary, common to G_1 and G_2, is a circle of finite radius, called the *discriminating circle** of the function. In G_2 the modified function is holomorphic; in G_1 the function is holomorphic except at $\zeta=\infty$; and all the singularities (and the branch-points, if any) lie on the discriminating circle.

The theory is given in Fuchs's memoir "Ueber die Darstellung der Functionen complexer Variabeln,......," *Crelle*, t. lxxv, (1872), pp. 176—223. It is corrected in details and is amplified in *Crelle*, t. cvi, (1890), pp. 1—4, and in *Crelle*, t. cviii, (1891), pp. 181—192; see also Nekrassoff, *Math. Ann.*, t. xxxviii, (1891), pp. 82—90, and Anissimoff, *Math. Ann.*, t. xl, (1892), pp. 145—148.

* Fuchs calls it *Grenzkreis*.

CHAPTER VII.

FUNCTIONS WITH UNLIMITED ESSENTIAL SINGULARITIES, AND EXPANSION IN SERIES OF FUNCTIONS.

72. It now remains to consider functions which have an infinite number of essential singularities*. It will, in the first place, be assumed that the essential singularities are isolated points, that is, that they do not form a continuous line, however short, and that they do not constitute a continuous area, however small, in the plane. Since their number is unlimited and their distance from one another is finite, there must be at least one point in the plane (it may be at $z = \infty$) where there is an infinite aggregate of such points. But no special note need be taken of this fact, for the character of an essential singularity has not yet entered into question; the essential singularity at such a point would merely be of a nature different from the essential singularity at some other point.

We take, therefore, an infinite series of quantities a_1, a_2, a_3,... arranged in order of increasing moduli, and such that no two are the same: and so we have infinity as the limit of a_ν when $\nu = \infty$.

Let there be an associated series of uniform functions of z such that for all values of i, the function $G_i\left(\frac{1}{z - a_i}\right)$, vanishing with $\frac{1}{z - a_i}$, has a_i as its

* The results in the present chapter are founded, except where other particular references are given, upon the researches of Mittag-Leffler and Weierstrass. The most important investigations of Mittag-Leffler are contained in a series of short notes, constituting the memoir "Sur la théorie des fonctions uniformes d'une variable," *Comptes Rendus*, t. xciv, (1882), pp. 414, 511, 713, 781, 938, 1040, 1105, 1163, t. xcv, (1882), p. 335; and in a memoir "Sur la représentation analytique des fonctions monogènes uniformes," *Acta Math.*, t. iv, (1884), pp. 1—79. The investigations of Weierstrass referred to are contained in his two memoirs "Ueber einen functionentheoretischen Satz des Herrn G. Mittag-Leffler," (1880), and "Zur Functionenlehre," (1880), both included in the volume *Abhandlungen aus der Functionenlehre*, pp. 53—66, 67—101, 102—104. A memoir by Hermite "Sur quelques points de la théorie des fonctions," *Acta Soc. Fenn.*, t. xii, pp. 67—94, *Crelle*, t. xci, (1881), pp. 54—78 may be consulted with great advantage.

sole singularity; the singularity is essential or accidental according as G_i is transcendental or algebraical. These functions can be constructed by theorems already proved. Then we have the theorem, due to Mittag-Leffler:—*It is always possible to construct a uniform analytical function* $F(z)$, *having no singularities other than* $a_1, a_2, a_3, \ldots$ *and such that for each determinate value of* ν, *the difference* $F(z) - G_\nu\left(\frac{1}{z-a_\nu}\right)$ *is finite for* $z = a_\nu$ *and therefore, in the vicinity of* a_ν, *is expressible in the form* $P(z - a_\nu)$.

73. To prove Mittag-Leffler's theorem, we first form subsidiary functions $F_\nu(z)$, derived from the functions G as follows. The function $G_\nu\left(\frac{1}{z-a_\nu}\right)$ converges everywhere in the plane except at the point a_ν; hence within a circle $|z| < |a_\nu|$ it is a monogenic analytic function of z, and can therefore be expanded in a series of positive powers of z which converges uniformly within the circle, say

$$G_\nu\left(\frac{1}{z-a_\nu}\right) = \sum_{\mu=0}^{\infty} \nu_\mu z^\mu,$$

for values of z such that $|z| < |a_\nu|$. If a_ν be zero, there is evidently no expansion.

Let ϵ be a positive quantity less than 1, and let $\epsilon_1, \epsilon_2, \epsilon_3, \ldots$ be arbitrarily chosen positive decreasing quantities, subject to the single condition that $\Sigma\epsilon$ is a converging series, say of sum Δ: and let ϵ_0 be a positive quantity intermediate between 1 and ϵ. Let g be the greatest value of $\left|G_\nu\left(\frac{1}{z-a_\nu}\right)\right|$ for points on or within the circumference $|z| = \epsilon_0 |a_\nu|$; then, because the series $\sum_{\mu=0}^{\infty} \nu_\mu z^\mu$ is a converging series, we have, by § 29,

$$|\nu_\mu z^\mu| \leqslant g,$$

or

$$|\nu_\mu| \leqslant \frac{g}{\epsilon_0{}^\mu |a_\nu|^\mu}.$$

Hence, with values of z satisfying the condition $|z| \leqslant \epsilon |a_\nu|$, we have, for any value of m,

$$\left|\sum_{\mu=m}^{\infty} \nu_\mu z^\mu\right| \leqslant \sum_{\mu=m}^{\infty} |\nu_\mu| |z|^\mu,$$

$$\leqslant \sum_{\mu=m}^{\infty} g \frac{\epsilon^\mu}{\epsilon_0{}^\mu} \leqslant \frac{g}{1 - \frac{\epsilon}{\epsilon_0}} \left(\frac{\epsilon}{\epsilon_0}\right)^m,$$

since $\epsilon < \epsilon_0$. Take the smallest integral value of m such that

$$\frac{g}{1 - \frac{\epsilon}{\epsilon_0}} \left(\frac{\epsilon}{\epsilon_0}\right)^m \leqslant \epsilon_\nu;$$

it will be finite and may be denoted by m_ν: and thus we have

$$\left|\sum_{\mu=m_\nu}^{\infty} \nu_\mu z^\mu\right| \leqslant \epsilon_\nu,$$

for values of z satisfying the condition $|z| \leqslant \epsilon\,|a_\nu|$.

We now construct a subsidiary function $F_\nu(z)$ such that, for all values of z,

$$F_\nu(z) = G_\nu\left(\frac{1}{z-a_\nu}\right) - \sum_{\mu=0}^{m_\nu-1} \nu_\mu z^\mu;$$

then for values of $|z|$, which are $\leqslant \epsilon\,|a_\nu|$,

$$|F_\nu(z)| \leqslant \epsilon_\nu.$$

Moreover, the function $\sum\limits_{\mu=0}^{m_\nu-1} \nu_\mu z^\mu$ is finite for all finite values of z so that, if we take

$$\phi_\nu(z) = z^{-m_\nu} G_\nu\left(\frac{1}{z-a_\nu}\right) - \sum_{\mu=0}^{m_\nu-1} \frac{\nu_\mu}{z^{m_\nu-\mu}},$$

then $\phi_\nu(z)$ is zero at infinity, because, when $z=\infty$, $G_\nu\left(\frac{1}{z-a_\nu}\right)$ is finite by hypothesis. Evidently $\phi_\nu(z)$ is infinite only at $z=a_\nu$, and its singularity is of the same kind as that of $G_\nu\left(\frac{1}{z-a_\nu}\right)$.

74. Now let c be any point in the plane, which is not one of the points $a_1, a_2, a_3, \ldots$; it is possible to choose a positive quantity ρ such that no one of the points a is included within the circle

$$|z-c| = \rho.$$

Let a_ν be the singularity, which is the point nearest to the origin satisfying the condition $|a_\nu| > |c| + \rho$; then, for points within or on the circle, we have

$$\left|\frac{z}{a_s}\right| \leqslant \epsilon,$$

when s has the values $\nu, \nu+1, \nu+2, \ldots$. Introducing the subsidiary functions $F_\nu(z)$, we have, for such values of z,

$$|F_s(z)| \leqslant \epsilon_s,$$

and therefore

$$\left|\sum_{s=\nu}^{\infty} F_s(z)\right| \leqslant \sum_{s=\nu}^{\infty} |F_s(z)|$$

$$\leqslant \sum_{s=\nu}^{\infty} \epsilon_s$$

$$\leqslant \Delta,$$

a finite quantity. It therefore follows that the series $\sum\limits_{s=\nu}^{\infty} F_s(z)$ converges uniformly and unconditionally for all values of z which satisfy the condition

$|z-c| \leqslant \rho$. Moreover, all the functions $F_1(z), F_2(z), \ldots, F_{r-1}(z)$ are finite for such values of z, because their singularities lie without the circle $|z-c|=\rho$; and therefore the series

$$\sum_{r=1}^{\infty} F_r(z)$$

converges uniformly and unconditionally for all points z within or on the circle $|z-c|=\rho$, where ρ is chosen so that the circle encloses none of the points a.

The function, represented by the series, can therefore be expanded in the form $P(z-c)$, in the domain of the point c.

If a_m denote any one of the points $a_1, a_2, \ldots$, and we take ρ' so small that all the points, other than a_m, lie without the circle

$$|z-a_m|=\rho',$$

then, since $F_m(z)$ is the only one of the functions F which has a singularity at a_m, the series

$$\sum_{r=1}^{\infty} \{F_r(z)\} - F_m(z)$$

converges regularly in the vicinity of a, and therefore it can be expressed in the form $P(z-a_m)$. Hence

$$\begin{aligned}\sum_{r=1}^{\infty} F_r(z) &= F_m(z) + P(z-a_m)\\ &= G_m\left(\frac{1}{z-a_m}\right) + P_1(z-a_m),\end{aligned}$$

the difference of F_m and G_m being absorbed into the series P to make P_1. It thus appears that the series $\sum_{r=1}^{\infty} F_r(z)$ is a function which has infinities only at the points $a_1, a_2, \ldots$, and is such that

$$\sum_{r=1}^{\infty} F_r(z) - G_m\left(\frac{1}{z-a_m}\right)$$

can be expressed in the vicinity of a_m in the form $P(z-a_m)$. Hence $\sum_{r=1}^{\infty} F_r(z)$ *is a function of the required kind.*

75. It may be remarked that the function is by no means unique. As the positive quantities ϵ were subjected to merely the single condition that they form a converging series, there is the possibility of wide variation in their choice: and a difference of choice might easily lead to a difference in the ultimate expression of the function.

This latitude of ultimate expression is not, however, entirely unlimited. For, suppose there are two functions $F(z)$ and $\bar{F}(z)$, enjoying all the assigned properties. Then as any point c, other than $a_1, a_2, \ldots$, is an ordinary point for both $F(z)$ and $\bar{F}(z)$, it is an ordinary point for their difference: and so

$$F(z) - \bar{F}(z) = P(z-c)$$

for points in the immediate vicinity of c. The points a are, however, singularities for each of the functions: in the vicinity of such a point a_i we have

$$F(z) = G_i\left(\frac{1}{z - a_i}\right) + P(z - a_i),$$

$$\bar{F}(z) = G_i\left(\frac{1}{z - a_i}\right) + \bar{P}(z - a_i),$$

since the functions are of the required form: hence

$$F(z) - \bar{F}(z) = P(z - a_i) - \bar{P}(z - a_i),$$

or the point a_i is an ordinary point for the difference of the functions. Hence every finite point in the plane, whether an ordinary point or a singularity for each of the functions, is an ordinary point for the difference of the functions: and therefore that difference is a uniform integral function of z. It thus appears that, *if $F(z)$ be a function with the required properties, then every other function with those properties is of the form*

$$F(z) + G(z),$$

where $G(z)$ is a uniform integral function of z either transcendental or algebraical.

The converse of this theorem is also true.

Moreover, the function $G(z)$ can always be expressed in a form $\sum_{\nu=1}^{\infty} g_\nu(z)$, if it be desirable to do so: and therefore it follows that any function with the assigned characteristics can be expressed in the form

$$\sum_{\nu=1}^{\infty} \{F_\nu(z) + g_\nu(z)\}.$$

76. The following applications, due to Weierstrass, can be made so as to give a new expression for functions, already considered in Chapter VI., having $z = \infty$ as their sole essential singularity and an unlimited number of poles at points $a_1, a_2, \ldots$.

If the pole at a_i be of multiplicity m_i, then $(z - a_i)^{m_i} f(z)$ is regular at the point a_i and can therefore be expressed in the form

$$\sum_{\mu=0}^{\infty} c_\mu (z - a_i)^\mu.$$

Hence, if we take $$f_i(z) = \sum_{\mu=0}^{m_i-1} c_\mu (z - a_i)^{-m_i+\mu},$$

we have $$f(z) = f_i(z) + P(z - a_i).$$

Now deduce from $f_i(z)$ a function $F_i(z)$ as in § 73, and let this deduction be effected for each of the functions $f_i(z)$. Then we know that

$$\sum_{i=1}^{\infty} F_i(z)$$

is a uniform function of z having the points $a_1, a_2, \ldots$ for poles in the proper multiplicity and no essential singularity except $z=\infty$. The most general form of the function therefore is

$$\sum_{r=1}^{\infty} \{F_r(z)+g_r(z)\}.$$

Hence *any uniform analytical function which has no essential singularity except at infinity can be expressed as a sum of functions each of which has only one singularity in the finite part of the plane.* The form of $F_r(z)$ is

$$f_r(z)-G_r(z),$$

where $f_r(z)$ is infinite at $z=a_r$ and $G_r(z)$ is a properly chosen integral function.

We pass to the case of a function having a single essential singularity at c and at no other point and any number of accidental singularities, by taking $z'=\dfrac{1}{z-c}$ as in § 63, Cor. II.: and so we obtain the theorem:

Any uniform function which has only one essential singularity, which is at c, can be expressed as a sum of uniform functions each of which has only one singularity different from c.

Evidently the typical summative function $F_r(z)$ for the present case is of the form

$$f_r(z)+G_r\left(\frac{1}{z-c}\right).$$

77. The results, which have been obtained for functions possessed of an infinitude of singularities, are valid on the supposition, stated in § 72, that the limit of a_ν with indefinite increase of ν is infinite; the series $a_1, a_2, \ldots$ tends to one definite limiting point which is $z=\infty$ and, by the substitution $z'(z-c)=1$, can be made any point c in the finite part of the plane.

Such a series, however, does not necessarily tend to one definite limiting point: it may, for instance, tend to condensation on a curve, though the condensation does not imply that all points of the continuous arc of the curve must be included in the series. We shall not enter into the discussion of the most general case, but shall consider that case in which the series of moduli $|a_1|, |a_2|, \ldots$ tends to one definite limiting value so that, with indefinite increase of ν, the limit of $|a_\nu|$ is finite and equal to R; the points $a_1, a_2, \ldots$ tend to condense on the circle $|z|=R$.

Such a series is given by

$$a_{n,k}=\left\{1+\frac{(-1)^{n+1}}{n+1}\right\}e^{\frac{2k\pi i}{n+1}},$$

for $k=0, 1, \ldots, n$, and $n=1, 2, \ldots$ *ad inf.*; and another* by

$$a_n=\{1+(-1)^n c^n\}\, e^{2n\pi i\sqrt{2}},$$

where c is a positive proper fraction.

* The first of these examples is given by Mittag-Leffler, *Acta Math.*, t. iv, p. 11; the second was stated to me by Mr Burnside.

With each point a_m we associate the point on the circumference of the circle, say b_m, to which a_m is nearest: let

$$|a_m - b_m| = \rho_m,$$

so that ρ_m approaches the limit zero with indefinite increase of m. There cannot be an infinitude of points a_p, such that $\rho_p \geqslant \Theta$, any assigned positive quantity; for then either there would be an infinitude of points a within or on the circle $|z| = R - \Theta$, or there would be an infinitude of points a within or on the circle $|z| = R + \Theta$, both of which are contrary to the hypothesis that, with indefinite increase of ν, the limit of $|a_\nu|$ is R. Hence it follows that a finite integer n exists for every assigned positive quantity Θ, such that

$$|a_m - b_m| < \Theta$$

when $m \geqslant n$.

Then the theorem, which corresponds to Mittag-Leffler's as stated in § 72 and which also is due to him, is as follows:—

It is always possible to construct a uniform analytical function of z which exists over the whole plane, except at the points a and b, and which, in the immediate vicinity of each one of the singularities a, can be expressed in the form

$$G_i\left(\frac{1}{z - a_i}\right) + P(z - a_i),$$

where the functions G_i are assigned functions, vanishing with $\dfrac{1}{z - a_i}$ and finite everywhere in the plane except at the single points a_i with which they are respectively associated.

In establishing this theorem, we shall need a positive quantity ϵ less than unity and a converging series $\epsilon_1, \epsilon_2, \epsilon_3, \ldots$ of positive quantities, all less than unity.

Let the expression of the function G_n be

$$G_n\left(\frac{1}{z - a_n}\right) = \frac{c_{n,1}}{z - a_n} + \frac{c_{n,2}}{(z - a_n)^2} + \frac{c_{n,3}}{(z - a_n)^3} + \ldots.$$

Then, since
$$z - a_n = (z - b_n)\left\{1 - \frac{a_n - b_n}{z - b_n}\right\},$$

the function G_n can be expressed* in the form

$$G_n\left(\frac{1}{z - a_n}\right) = \sum_{\mu=1}^{\infty} A_{n,\mu}\left(\frac{a_n - b_n}{z - b_n}\right)^{\mu}$$

for values of z such that

$$\left|\frac{a_n - b_n}{z - b_n}\right| < \epsilon;$$

and the coefficients A are given by the equations

$$A_{n,\mu} = \sum_{r=1}^{\mu} \frac{c_{n,r}}{(a_n - b_n)^r} \frac{(\mu - 1)!}{(\mu - r)!\,(r - 1)!}.$$

* The justification of this statement is to be found in the proposition in § 82.

Now, because G_n is finite everywhere in the plane except at a_n, the series

$$\frac{|c_{n,1}|}{\xi_n} + \frac{|c_{n,2}|}{\xi_n^{\,2}} + \frac{|c_{n,3}|}{\xi_n^{\,3}} + \ldots$$

has a finite value, say g, for any non-zero value of the positive quantity ξ_n; then

$$|c_{n,r}| < g\xi_n^{\,r}.$$

Hence
$$|A_{n,\mu}| \leqslant \sum_{r=1}^{\mu} \frac{|c_{n,r}|}{|a_n - b_n|^r} \frac{(\mu-1)!}{(\mu-r)!\,(r-1)!}$$
$$< \sum_{r=1}^{\mu} g \frac{\xi_n^{\,r}}{|a_n - b_n|^r} \frac{(\mu-1)!}{(\mu-r)!\,(r-1)!}$$
$$< \frac{g\xi_n}{|a_n - b_n|} \left\{1 + \frac{\xi_n}{|a_n - b_n|}\right\}^{\mu-1}.$$

Introducing a positive quantity α such that

$$(1+\alpha)\,\epsilon < 1,$$

we choose ξ_n so that
$$\xi_n < \alpha |a_n - b_n|;$$
and then
$$|A_{n,\mu}| < g\alpha (1+\alpha)^{\mu-1}.$$

Because $(1+\alpha)\,\epsilon$ is less than unity, a quantity θ exists such that

$$(1+\alpha)\,\epsilon < \theta < 1.$$

Then for values of z determined by the condition $\left|\dfrac{a_n - b_n}{z - b_n}\right| < \epsilon$, we have

$$\sum_{\mu=m_n+1}^{\infty} |A_{n,\mu}| \left|\frac{a_n - b_n}{z - b_n}\right|^{\mu} < \frac{g\alpha}{1+\alpha} \frac{\theta^{m_n+1}}{1-\theta}.$$

Let the integer m_n be chosen so that

$$\frac{g\alpha}{1+\alpha} \frac{\theta^{m_n+1}}{1-\theta} \leqslant \epsilon_n;$$

it will be a finite integer, because $\theta < 1$. Then

$$\sum_{\mu=m_n+1}^{\infty} |A_{n,\mu}| \left|\frac{a_n - b_n}{z - b_n}\right|^{\mu} < \epsilon_n.$$

We now construct, as in § 73, a subsidiary function $F_n(z)$, defining it by the equation

$$F_n(z) = G_n\left(\frac{1}{z - a_n}\right) - \sum_{\mu=0}^{m_n} A_{n,\mu} \left(\frac{a_n - b_n}{z - b_n}\right)^{\mu},$$

so that for points z determined by the condition $\left|\dfrac{a_n - b_n}{z - b_n}\right| < \epsilon$, we have

$$|F_n(z)| < \epsilon_n.$$

A function with the required properties is

$$F(z) = \sum_{m=1}^{\infty} F_m(z).$$

To prove it, let c be any point in the plane distinct from any of the points a and b; we can always find a value of ρ such that the circle

$$|z-c|=\rho$$

contains none of the points a and b. Let l be the shortest distance between this circle and the circle of radius R, on which all the points b lie; then for all points z within or on the circle $|z-c|=\rho$ we have

$$|z-b_m| \geqslant l.$$

Now we have seen that, for any assigned positive quantity Θ, there is a finite integer n such that

$$|a_m-b_m|<\Theta$$

when $m \geqslant n$. Taking $\Theta=\epsilon l$, we have

$$\left|\frac{a_m-b_m}{z-b_m}\right|<\epsilon$$

when $m \geqslant n$, n being the finite integer associated with the positive quantity ϵl.

It therefore follows that, for points z within or on the circle $|z-c|=\rho$,

$$|F_m(z)|<\epsilon_m,$$

when m is not less than the finite integer n. Hence

$$\sum_{m=n}^{\infty}|F_m(z)|<\epsilon_n+\epsilon_{n+1}+\epsilon_{n+2}+\ldots,$$

a finite quantity because $\epsilon_1, \epsilon_2, \ldots$ is a converging series; and therefore

$$\sum_{m=n}^{\infty}F_m(z)$$

is a converging series. Each of the functions $F_1(z)$, $F_2(z)$, ..., $F_{n-1}(z)$ is finite when $|z-c| \leqslant \rho$; and therefore

$$\sum_{m=1}^{\infty}F_m(z)$$

is a series which converges uniformly and unconditionally for all values of z included in the region

$$|z-c| \leqslant \rho.$$

Hence the function represented by the series can be expressed in the form $P(z-c)$ for all such values of z. The function therefore exists over the whole plane except at the points a and b.

It may be proved, exactly as in § 74, that, for points z in the immediate vicinity of a singularity a_m,

$$F(z)=G_m\left(\frac{1}{z-a_m}\right)+P(z-a_m).$$

The theorem is thus completely established.

The function thus obtained is not unique, for a wide variation of choice of the converging series $\epsilon_1+\epsilon_2+\ldots$ is possible. But, in the same way as in the corresponding case in § 75, it is proved that, *if $F(z)$ be a function with the required properties, every other function with those properties is of the form*

$$F(z) + G(z),$$

where $G(z)$ behaves regularly in the immediate vicinity of every point in the plane except the points b.

78. The theorem just given regards the function in the light of an infinite converging series of functions of the variable: it is natural to suppose that a corresponding theorem holds when the function is expressed as an infinite converging product. With the same series of singularities as in § 77, when the limit of $|a_\nu|$ with indefinite increase of ν is finite and equal to R, the theorem* is:—

It is always possible to construct a uniform analytical function which behaves regularly everywhere in the plane except at the points a and b and which in the vicinity of any one of the points a_ν can be expressed in the form

$$(z - a_\nu)^{n_\nu} e^{P(z-a_\nu)},$$

where the numbers $n_1, n_2, \ldots$ are any assigned integers.

The proof is similar in details to proofs of other propositions and it will therefore be given only in outline. We have

$$\frac{n_\nu}{z - a_\nu} = \frac{n_\nu}{z - b_\nu} + \frac{n_\nu}{z - b_\nu} \sum_{\mu=1}^{\infty} \left(\frac{a_\nu - b_\nu}{z - b_\nu}\right)^\mu,$$

provided $\left|\dfrac{a_\nu - b_\nu}{z - b_\nu}\right| < \epsilon$, the notation being the same as in § 77. Hence, for such values of z,

$$\left(\frac{z - a_\nu}{z - b_\nu}\right)^{n_\nu} = e^{-n_\nu \sum\limits_{\mu=1}^{\infty} \frac{1}{\mu}\left(\frac{a_\nu - b_\nu}{z - b_\nu}\right)^\mu}.$$

If we denote

$$\left(1 - \frac{a_\nu - b_\nu}{z - b_\nu}\right)^{n_\nu} e^{n_\nu \sum\limits_{\mu=1}^{m_\nu} \frac{1}{\mu}\left(\frac{a_\nu - b_\nu}{z - b_\nu}\right)^\mu}$$

by $E_\nu(z)$, we have

$$E_\nu(z) = e^{-n_\nu \sum\limits_{\mu=m_\nu+1}^{\infty} \frac{1}{\mu}\left(\frac{a_\nu - b_\nu}{z - b_\nu}\right)^\mu}.$$

Hence, if $F(z)$ denote the infinite product

$$\prod_{\nu=1}^{\infty} E_\nu(z),$$

we have

$$F(z) = e^{-\sum\limits_{\nu=1}^{\infty}\left\{n_\nu \sum\limits_{\mu=m_\nu+1}^{\infty} \frac{1}{\mu}\left(\frac{a_\nu - b_\nu}{z - b_\nu}\right)^\mu\right\}};$$

and $F(z)$ is a determinate function provided the double series in the index of the exponential converge.

* Mittag-Leffler, *Acta Math.*, t. iv, p. 32; it may be compared with Weierstrass's theorem in § 67.

Because n_ν is a finite integer and because

$$\sum_{\mu=1}^{\infty} \frac{1}{\mu} \left(\frac{a_\nu - b_\nu}{z - b_\nu}\right)^\mu$$

is a converging series, it is possible to choose an integer m_ν so that

$$\left| n_\nu \sum_{\mu=m_\nu+1}^{\infty} \frac{1}{\mu} \left(\frac{a_\nu - b_\nu}{z - b_\nu}\right)^\mu \right| < \eta_\nu,$$

where η_ν is any assigned positive quantity. We take a converging series of positive quantities η_ν: and then the moduli of the terms in the double series form a converging series. The double series itself therefore converges uniformly and unconditionally; and then the infinite product $F(z)$ converges uniformly and unconditionally for points z such that

$$\left| \frac{a_\nu - b_\nu}{z - b_\nu} \right| < \epsilon.$$

As in § 77, let c be any point in the plane, distinct from any of the points a and b. We take a finite value of ρ such that the circle $|z - c| = \rho$ contains none of the points a and b; and then, for all points within or on this circle,

$$\left| \frac{a_m - b_m}{z - b_m} \right| < \epsilon$$

when $m \geqslant n$, n being the finite integer associated with the positive quantity ϵl. The product

$$\prod_{\nu=n}^{\infty} E_\nu(z)$$

is therefore finite, for its modulus is less than

$$e^{\sum\limits_{\nu=n}^{\infty} \eta_\nu};$$

the product

$$\prod_{\nu=1}^{n-1} E_\nu(z)$$

is finite, because the circle $|z - c| = \rho$ contains none of the points a and b; and therefore the function $F(z)$ is finite for all points within or on the circle. Hence in the vicinity of c, the function can be expanded in the form $P(z - c)$; and therefore the function exists everywhere in the plane except at the points a and b.

The infinite product converges; it can be zero only at points which make one of the factors zero and, from the form of the factors, this can take place only at the points a_ν with positive integers n_ν. In the vicinity of a_ν all the factors of $F(z)$ except $E_\nu(z)$ are regular; hence $F(z)/E_\nu(z)$ can be expressed as a function of $z - a_\nu$ in the vicinity. But the function has no zeros there, and therefore the form of the function is

$$e^{P_1(z-a_\nu)}.$$

Hence in the vicinity of a_ν, we have

$$F(z) = E_\nu(z)\, e^{P_1(z-a_\nu)}$$
$$= (z-a_\nu)^{n_\nu} e^{P(z-a_\nu)},$$

on combining with $P_1(z-a_\nu)$ the exponential index in $E_\nu(z)$. This is the required property.

Other general theorems will be found in Mittag-Leffler's memoir just quoted.

79. The investigations in §§ 72—75 have led to the construction of a function with assigned properties. It is important to be able to change, into the chosen form, the expression of a given function, having an infinite series of singularities tending to a definite limiting point, say to $z=\infty$. It is necessary for this purpose to determine (i) the functions $F_r(z)$ so that the series $\sum_{r=1}^{\infty} F_r(z)$ may converge uniformly and (ii) the function $G(z)$.

Let $\Phi(z)$ be the given function, and let S be a simple contour embracing the origin and μ of the singularities, viz., $a_1, \ldots\ldots, a_\mu$: then, if t be any point, we have

$$\int^S \frac{\Phi(t)}{t-z}\left(\frac{z}{t}\right)^m dt = \int^{(0)} \frac{\Phi(t)}{t-z}\left(\frac{z}{t}\right)^m dt + \int^{(z)} \frac{\Phi(t)}{t-z}\left(\frac{z}{t}\right)^m dt$$
$$+ \sum_{\nu=1}^{\mu} \int^{(a_\nu)} \frac{\Phi(t)}{t-z}\left(\frac{z}{t}\right)^m dt,$$

where $\int^{(a)}$ implies an integral taken round a very small circle centre a.

If the origin be one of the points $a_1, a_2, \ldots\ldots$, then the first term will be included in the summation.

Assuming that z is neither the origin nor any one of the points $a_1, \ldots, a_\mu$, we have

$$\int^{(z)} \frac{\Phi(t)}{t-z}\left(\frac{z}{t}\right)^m dt = 2\pi i\, \Phi(z),$$

so that

$$\Phi(z) = \frac{1}{2\pi i}\int^S \frac{\Phi(t)}{t-z}\left(\frac{z}{t}\right)^m dt - \frac{1}{2\pi i}\int^{(0)} \frac{\Phi(t)}{t-z}\left(\frac{z}{t}\right)^m dt$$
$$- \frac{1}{2\pi i}\sum_{\nu=1}^{\mu} \int^{(a_\nu)} \frac{\Phi(t)}{t-z}\left(\frac{z}{t}\right)^m dt.$$

Now $\frac{1}{2\pi i}\int^{(0)} \frac{\Phi(t)}{t-z}\left(\frac{z}{t}\right)^m dt$

$$= \frac{z^m}{(m-1)!}\left[\frac{d^{m-1}}{dt^{m-1}}\frac{\Phi(t)}{t-z}\right]_{t=0}$$
$$= -\frac{z^m}{(m-1)!}\left[\frac{d^{m-1}}{dt^{m-1}}\left\{\frac{\Phi(t)}{z} + \frac{t\Phi(t)}{z^2} + \ldots\right\}\right]_{t=0}$$
$$= -\frac{z^m}{(m-1)!}\left[\frac{1}{z}\Phi^{m-1}(0) + \frac{m-1}{z^2}\Phi^{m-2}(0) + \ldots\right]$$
$$= -\left[\Phi(0) + \frac{z}{1}\Phi'(0) + \ldots + \frac{z^{m-1}\Phi^{m-1}(0)}{(m-1)!}\right] = -G(z),$$

unless $z=0$ be a singularity and then there will be no term $G(z)$. Similarly, it can be shewn that

$$-\frac{1}{2\pi i}\int^{(a_\nu)}\frac{\Phi(t)}{t-z}\left(\frac{z}{t}\right)^m dt$$

is equal to
$$G_\nu\left(\frac{1}{z-a_\nu}\right)-\sum_{\lambda=0}^{m-1}\nu_\lambda\left(\frac{z}{a_\nu}\right)^\lambda=F_\nu(z),$$

where
$$G_\nu\left(\frac{1}{z-a_\nu}\right)=-\frac{1}{2\pi i}\int^{(a_\nu)}\frac{\Phi(t)}{t-z}dt$$

and the subtractive sum of m terms is the sum of the first m terms in the development of G_ν in ascending powers of z. Hence

$$\Phi(z)=G(z)+\sum_{\nu=1}^{\mu}F_\nu(z)+\frac{1}{2\pi i}\int^S\frac{\Phi(t)}{t-z}\left(\frac{z}{t}\right)^m dt.$$

If, for an infinitely large contour, m can be chosen so that the integral

$$\frac{1}{2\pi i}\int\frac{\Phi(t)}{t-z}\left(\frac{z}{t}\right)^m dt$$

diminishes indefinitely with increasing contours enclosing successive singularities, then

$$\Phi(z)=G(z)+\sum_{\nu=1}^{\infty}F_\nu(z).$$

The integer m may be called the *critical integer*.

If the origin be a singularity, we take

$$F_0(z)=G_0\left(\frac{1}{z}\right),$$

and there is then no term $G(z)$: hence, including the origin in the summation, we then have

$$\Phi(z)=\sum_{\nu=0}^{\infty}F_\nu(z)+\frac{1}{2\pi i}\int^S\frac{\Phi(t)}{t-z}\left(\frac{z}{t}\right)^m dt;$$

so that if, for this case also, there be some finite value of m which makes the integral vanish, then

$$\Phi(z)=\sum_{\nu=0}^{\infty}F_\nu(z).$$

Other expressions can be obtained by choosing for m a value greater than the critical integer; but it is usually most advantageous to take m equal to its least lawful value.

Ex. 1. The singularities of the function $\pi\cot\pi z$ are given by $z=\lambda$, for all integer values of λ from $-\infty$ to $+\infty$ including zero, so that the origin is a singularity.

The integral to be considered is

$$J=\frac{1}{2\pi i}\int^{(S)}\frac{\pi\cot\pi t}{t-z}\left(\frac{z}{t}\right)^m dt.$$

We take the contour to be a circle of very large radius R chosen so that the circumference does not pass infinitesimally near any one of the singularities of $\pi\cot\pi t$ at infinity; this is, of course, possible because there is a finite distance between any two of them. Then, round the circumference so taken, $\pi \cot \pi t$ is never infinite: hence its modulus is never greater than some finite quantity M.

Let $t = Re^{\theta i}$, so that $\frac{dt}{t} = i d\theta$; then

$$J = \frac{1}{2\pi} \int_0^{2\pi} \pi \cot \pi t \frac{z}{t-z} \left(\frac{z}{t}\right)^{m-1} d\theta,$$

and therefore

$$|J| \leqslant M \frac{|z|}{|t-z|} \left|\frac{z}{t}\right|^{m-1},$$

for some point t on the circle. Now, as the circle is very large, we have $|t-z|$ infinite: hence $|J|$ can be made zero merely by taking m unity.

Thus, for the function $\pi \cot \pi z$, the critical integer is unity.

Hence from the general theorem we have the equation

$$\pi \cot \pi z = -\frac{1}{2\pi i} \Sigma \int \frac{\pi \cot \pi t}{t-z} \frac{z}{t} dt,$$

the summation extending to all the points λ for integer values of $\lambda = -\infty$ to $+\infty$, and each integral being taken round a small circle centre λ.

Now if, in

$$\frac{1}{2\pi i} \int^{(\lambda)} \frac{\pi \cot \pi t}{t-z} \frac{z}{t} dt,$$

we take $t = \lambda + \zeta$, we have

$$\pi \cot \pi t = \frac{1}{\zeta} + P(\zeta),$$

where $P(\zeta) = 0$ when $\zeta = 0$; and therefore the value of the integral is

$$= \frac{1}{2\pi i} \int \frac{\frac{1}{\zeta} + P(\zeta)}{\lambda - z + \zeta} \frac{z}{\lambda + \zeta} d\zeta$$

$$= \frac{1}{2\pi i} \int \frac{\{1 + \zeta P(\zeta)\} z}{(\lambda - z + \zeta)(\lambda + \zeta)} \frac{d\zeta}{\zeta}.$$

In the limit when $|\zeta|$ is infinitesimal, this integral

$$= \frac{z}{(\lambda - z)\lambda}$$

$$= \frac{1}{\lambda - z} - \frac{1}{\lambda},$$

and therefore

$$F_\lambda(z) = \frac{1}{z - \lambda} + \frac{1}{\lambda},$$

if λ be not zero.

And for the zero of λ, the value of the integral is

$$-\frac{1}{2\pi i} \int \{1 + \zeta P(\zeta)\} \frac{z}{z - \zeta} \frac{d\zeta}{\zeta^2}$$

$$= -\frac{d}{d\zeta}\left[\{1 + \zeta P(\zeta)\} \frac{z}{z-\zeta}\right]_{\zeta=0}$$

$$= -\left[\frac{z}{(z-\zeta)^2} \{1 + \zeta P(\zeta)\} + \frac{z}{z-\zeta} \{P(\zeta) + \zeta P'(\zeta)\}\right]_{\zeta=0}$$

$$= -\frac{1}{z},$$

so that $F_0(z)$ is $\frac{1}{z}$. In fact, in the notation of § 72, we have

$$G_0\left(\frac{1}{z}\right)=\frac{1}{z},$$

$$G_\lambda\left(\frac{1}{z-\lambda}\right)=\frac{1}{z-\lambda},$$

and the expansion of G_λ needs to be carried only to one term.

We thus have
$$\pi\cot\pi z=\frac{1}{z}+\sum_{\lambda=-\infty}^{\lambda=\infty}{}'\left(\frac{1}{z-\lambda}+\frac{1}{\lambda}\right),$$
the summation not including the zero value of λ.

Ex. 2. Obtain, ab initio, the relation

$$\frac{1}{\sin^2 z}=\sum_{\lambda=-\infty}^{\lambda=\infty}\frac{1}{(z-\lambda\pi)^2}.$$

Ex. 3. Shew that, if

$$R(z)=\left\{1-\left(\frac{2z}{1}\right)^2\right\}\left\{1-\left(\frac{2z}{3}\right)^2\right\}\ldots\ldots\left\{1-\left(\frac{2z}{2n+1}\right)^2\right\},$$

then
$$\frac{\pi\cot\pi z}{R(z)}=\frac{1}{z}+2z\sum_{\lambda=1}^{\infty}\frac{1}{R(\lambda)}\frac{1}{z^2-\lambda^2}.$$

(Gyldén, Mittag-Leffler.)

Ex. 4. Obtain an expression, in the form of a sum, for

$$\frac{\pi\cot\pi z}{Q(z)},$$

where $Q(z)$ denotes
$$(1-z)\left(1-\frac{z}{2}\right)^2\left(1-\frac{z}{3}\right)^3\ldots\ldots\left(1-\frac{z}{n}\right)^n.$$

80. The results obtained in the present chapter relating to functions which have an unlimited number of singularities, whether distributed over the whole plane or distributed over only a finite portion of it, shew that analytical functions can be represented, not merely as infinite converging series of powers of the variable, but also as infinite converging series of functions of the variable. The properties of functions when represented by series of powers of the variable depended in their proof on the condition that the series proceeded in powers; and it is therefore necessary at least to revise those properties in the case of functions when represented as series of functions of the variable.

Let there be a series of uniform functions $f_1(z)$, $f_2(z)$, ...; then the aggregate of values of z, for which the series

$$\sum_{i=1}^{\infty} f_i(z)$$

has a finite value, is the region of continuity of the series. If a positive quantity ρ can be determined such that, for all points z within the circle

$$|z-a|=\rho,$$

the series $\sum_{i=1}^{\infty} f_i(z)$ converges uniformly and unconditionally*, the series is said to converge in the vicinity of a. If R be the greatest value of ρ for which this holds, then the area within the circle

$$|z - a| = R$$

is called the domain of a; and the series converges uniformly and unconditionally in the vicinity of any point in the domain of a.

It will be proved in § 82 that the function can be represented by power-series, each such series being equivalent to the function within the domain of some one point. In order to be able to obtain all the power-series, it is necessary to distribute the region of continuity of the function into domains of points where it has a uniform, finite value. We therefore form the domain of a point b in the domain of a from a knowledge of the singularities of the function, then the domain of a point c in the domain of b, and so on; the aggregate of these domains is a continuous part of the plane which has isolated points and which has one or several lines for its boundaries. Let this part be denoted by A_1.

For most of the functions, which have already been considered, the region A_1, thus obtained, is the complete region of continuity. But examples will be adduced almost immediately to shew that A_1 does not necessarily include all the region of continuity of the series under consideration. Let a' be a point not in A_1 within whose vicinity the function has a uniform, finite value; then a second portion A_2 can be separated from the whole plane, by proceeding from a' as before from a. The limits of A_1 and A_2 may be wholly or partially the same, or may be independent of one another: but no point within either can belong to the other. If there be points in the region of continuity which belong to neither A_1 nor A_2, then there must be at least another part of the plane A_3 with properties similar to A_1 and A_2. And so on. The series $\sum_{i=1}^{\infty} f_i(z)$ converges uniformly and unconditionally in the vicinity of every point in each of the separate portions of its region of continuity.

It was proved that a function represented by a series of powers has a definite finite derivative at every point lying actually within the circle of convergence of the series, but that this result cannot be affirmed for a point on the boundary of the circle of convergence even though the value of the series itself should be finite at the point, an illustration being provided by the hypergeometric series at a point on the circumference of its circle of

* In connection with most of the investigations in the remainder of this chapter, Weierstrass's memoir "Zur Functionenlehre" already quoted (p. 112, note) should be consulted.

It may be convenient to give here Weierstrass's definition (l.c., p. 70) of *uniform, unconditional convergence*. A series $\sum_{n=1}^{\infty} f_n$ converges uniformly, if an integer m can be determined so that $\left|\sum_{n=m}^{\infty} f_n\right|$ can be made less than any arbitrary positive quantity, however small; and it converges unconditionally, if the uniform convergence of the series be independent of any special arrangement of order or combination of the terms.

convergence. It will appear that a function represented by a series of functions has a definite finite derivative at every point lying actually within its region of continuity, but that the result cannot be affirmed for a point on the boundary; and an example will be given (§ 83) in which the derivative is indefinite.

Again, it has been seen that a function, initially defined by a given power-series, is, in most cases, represented by different analytical expressions in different parts of the plane, each of the elements being a valid expression of the function within a certain region. The questions arise whether a given analytical expression, either a series of powers or a series of functions: (i) can represent different functions in the same continuous part of its region of continuity, (ii) can represent different functions in distinct (that is, non-continuous) parts of its region of continuity.

81. Consider first a function defined by a given series of powers.

Let there be a region A' in the plane and let the region of continuity of the function, say $g(z)$, have parts common with A'. Then if a_0 be any point in one of these common parts, we can express $g(z)$ in the form $P(z - a_0)$ in the domain of a_0.

As already explained, the function can be continued from the domain of a_0 by a series of elements, so that the whole region of continuity is gradually covered by domains of successive points; to find the value in the domain of any point a, it is sufficient to know any one element, say, the element in the domain of a_0. The function is *the same* through its region of continuity.

Two distinct cases may occur in the continuations.

First, it may happen that the region of continuity of the function $g(z)$ extends beyond A'. Then we can obtain elements for points outside A', their aggregate being a uniform analytical function. The aggregate of elements then represents within A' a single analytical function: but as that function has elements for points without A', the aggregate within A' does not completely represent the function. Hence

If a function be defined within a continuous region of a plane by an aggregate of elements in the form of power-series, which are continuations of one another, the aggregate represents in that part of the plane one (and only one) analytical function: but if the power-series can be continued beyond the boundary of the region, the aggregate of elements within the region is not the complete representation of the analytical function.

This is the more common case, so that examples need not be given.

Secondly, it may happen that the region of continuity of the function does not extend beyond A' in any direction. There are then no elements of the function for points outside A' and the function cannot be continued beyond the boundary of A'. The aggregate of elements is then the complete representation of the function and therefore:

If a function be defined within a continuous region of a plane by an aggregate of elements in the form of power-series, which are continuations of one another, and if the power-series cannot be continued across the boundary of that region, the aggregate of elements in the region is the complete representation of a single uniform monogenic function which exists only for values of the variable within the region.

The boundary of the region of continuity of the function is, in the latter case, called the *natural limit* of the function*, as it is a line beyond which the function cannot be continued. Such a line arises for the series

$$1 + 2z + 2z^4 + 2z^9 + \dots ,$$

in the circle $|z| = 1$, a remark due to Kronecker; other illustrations occur in connection with the modular functions, the axis of real variables being the natural limit, and in connection with the automorphic functions (see Chapter XXII.) when the fundamental circle is the natural limit. A few examples will be given at the end of the present Chapter.

It appears that Weierstrass was the first to announce the existence of natural limits for analytic functions, *Berlin Monatsber.* (1866), p. 617; see also Schwarz, *Ges. Werke*, t. ii, pp. 240—242, who adduces other illustrations and gives some references; Klein and Fricke, *Vorl. über die Theorie der elliptischen Modulfunctionen*, t. i, (1890), p. 110; Jordan, *Cours d'Analyse*, t. iii, pp. 609, 610. Some interesting examples and discussions of functions, which have the axis of real variables for a natural limit, are given by Hankel, "Untersuchungen über die unendlich oft oscillirenden und unstetigen Functionen," *Math. Ann.*, t. xx, (1870), pp. 63—112.

82. Consider next a series of functions of the variable; let it be

$$\sum_{i=1}^{\infty} f_i(z).$$

The region of continuity may be supposed to consist of several distinct parts, in the most general case; let one of them be denoted by A. Take some point in A, say the origin, which is either an ordinary point or an isolated singularity; and let two concentric circles of radii R and R' be drawn in A, so that

$$R < |z| = r < R',$$

and the space between these circles lies within A. In this space, each term of the series is finite and the whole series converges uniformly and unconditionally.

Now let $f_i(z)$ be expanded in a series of powers of z, which series converges within the space assigned, and in that expansion let i_μ be the coefficient of z^μ; then we can prove *that* $\sum_{i=0}^{\infty} i_\mu$ *is finite and that the series*

$$\sum_{\mu} \left\{ \left(\sum_{i=0}^{\infty} i_\mu \right) z^\mu \right\}$$

* *Die natürliche Grenze*, according to German mathematicians.

converges uniformly and unconditionally within this space, so that

$$\sum_{i=1}^{\infty} f_i(z) = \sum_{\mu} \left\{ \left(\sum_{i=0}^{\infty} i_\mu \right) z^\mu \right\}.$$

Because the infinite series $\sum_{i=1}^{\infty} f_i(z)$ converges uniformly and unconditionally, a number n can be chosen so that

$$\left| \sum_{i=n}^{\infty} f_i(z) \right| < \tfrac{1}{2}k,$$

where k is an arbitrary finite quantity, ultimately made infinitesimal; and therefore also

$$\left| \sum_{i=n}^{n'} f_i(z) \right| < k,$$

where $n' > n$ and is infinite in the limit. Now since the number of terms in the series

$$\sum_{i=n}^{n'} f_i(z)$$

is not infinite before the limit, we have

$$\sum_{i=n}^{n'} f_i(z) = \sum_{\mu} \left\{ \left(\sum_{i=n}^{n'} i_\mu \right) z^\mu \right\}.$$

But the original series converges unconditionally, and therefore k is not less than the greatest value of the modulus of $\sum_{i=n}^{n'} f_i(z)$ for points within the region; hence, by § 29, we have

$$\left| \sum_{i=n}^{n'} i_\mu \right| < kr^{-\mu}.$$

Moreover, k is not less than the greatest value of the modulus of $\sum_{i=n}^{\infty} f_i(z)$ in the given region; and so

$$\left| \sum_{i=n}^{\infty} i_\mu \right| < kr^{-\mu}.$$

Now, by definition, k can be made as small as we desire by choice of n; hence the series

$$\sum_{i=1}^{\infty} i_\mu$$

is a converging series. Let it be denoted by A_μ.

Let $$\sum_{i=1}^{n-1} i_\mu = A_\mu', \qquad \sum_{i=n}^{\infty} i_\mu = A_\mu'';$$

then, by the above suppositions, we can always choose n so that

$$|A_\mu''| < kr^{-\mu},$$

k being any assignable small quantity.

When two new quantities r_1 and r_2 are introduced, as in § 28, satisfying the inequalities

$$R < r_1 < |z| < r_2 < R',$$

the integer n can be chosen so that

$$|A_\mu''| < kr_1^{-\mu} < kr_2^{-\mu}.$$

Then
$$\sum_{\mu=-1}^{-\infty} |A_\mu'' z^\mu| < k \sum_{\mu=-1}^{-\infty} \left(\frac{r}{r_1}\right)^\mu < k \frac{r_1}{r - r_1},$$

and
$$\sum_{\mu=0}^{\infty} |A_\mu'' z^\mu| < k \sum_{\mu=0}^{\infty} \left(\frac{r}{r_2}\right)^\mu < k \frac{r_2}{r_2 - r},$$

so that
$$\sum_{\mu=-\infty}^{\infty} |A_\mu'' z^\mu| < k \frac{r_1}{r - r_1} + k \frac{r_2}{r_2 - r}.$$

Hence the series $\sum_{\mu=-\infty}^{\mu=\infty} A_\mu'' z^\mu$ can by choice of n be made to have a modulus less than any finite quantity; and therefore, since

$$\sum_{\mu=-\infty}^{\mu=\infty} A_\mu' z^\mu = \sum_{i=0}^{n-1} f_i(z),$$

(for there is a finite number of terms in the coefficients on each side, the expansions are converging series, and the sum on the right-hand side is a finite quantity), it follows that the series

$$\sum_{\mu=-\infty}^{\mu=\infty} A_\mu z^\mu$$

converges uniformly.

Finally, we have

$$\sum_{i=1}^{\infty} f_i(z) - \Sigma A_\mu z^\mu = \sum_{i=1}^{\infty} f_i(z) - \Sigma A_\mu' z^\mu - \Sigma A_\mu'' z^\mu$$
$$= \sum_{i=n}^{\infty} f_i(z) - \Sigma A_\mu'' z^\mu,$$

and therefore
$$\left|\sum_{i=1}^{\infty} f_i(z) - \Sigma A_\mu z^\mu\right| = \left|\sum_{i=n}^{\infty} f_i(z) - \Sigma A_\mu'' z^\mu\right|$$
$$< \sum_{i=n}^{\infty} |f_i(z)| + \Sigma |A_\mu'' z^\mu|$$
$$< k + k \frac{r_1}{r - r_1} + k \frac{r_2}{r_2 - r},$$

which, as k can be diminished indefinitely, can be made less than any finite quantity. Hence the series $\sum_{\mu=-\infty}^{\mu=\infty} A_\mu z^\mu$ converges unconditionally, and therefore we have

$$\sum_{i=1}^{\infty} f_i(z) = \sum_{\mu=-\infty}^{\mu=\infty} A_\mu z^\mu,$$

provided
$$R < |z| < R'.$$

When we take into account all the parts of the region of continuity of the series, constituted by the sum of the functions, we have similar expansions in the form of successive series of powers of $z-c$, converging uniformly and unconditionally in the vicinities of the successive points c. But, in forming the domains of these points c, the boundary of the region of continuity of the function must not be crossed; and a new series of powers is required when the circle of convergence of any one series (lying within the region of continuity) is crossed.

It therefore appears that a converging series of functions of a variable can be expressed in the form of series of powers of the variable which converge within the parts of the plane where the series of functions converges uniformly and unconditionally; but the equivalence of the two expressions is limited to such parts of the plane and cannot be extended beyond the boundary of the region of continuity of the series of functions.

If the region of continuity of a series of functions consist of several parts of the plane, then the series of functions can in each part be expressed in the form of a set of converging series of powers: but the sets of series of powers are not necessarily the same for the different parts, and they are not necessarily continuations of one another, regarded as power-series.

Suppose, then, that the region of continuity of a series of functions

$$F(z) = \sum_{i=1}^{\infty} f_i(z)$$

consists of several parts $A_1, A_2, \ldots$. Within the part A_1 let $F(z)$ be represented, as above, by a set of power-series. At every point within A_1, the values of $F(z)$ and of its derivatives are each definite and unique; so that, at every point which lies in the regions of convergence of two of the power-series, the values which the two power-series, as the equivalents of $F(z)$ in their respective regions, furnish for $F(z)$ and for its derivatives must be the same. Hence the various power-series, which are the equivalents of $F(z)$ in the region A_1, are continuations of one another: and they are sufficient to determine a uniform monogenic analytic function, say $F_1(z)$. The functions $F(z)$ and $F_1(z)$ are equivalent in the region A_1; and therefore, by § 81, *the series of functions represents one and the same function for all points within one continuous part of its region of continuity.* It may (and frequently does) happen that the region of continuity of the analytical function $F_1(z)$ extends beyond A_1; and then $F_1(z)$ can be continued beyond the boundary of A_1 by a succession of elements. Or it may happen that the region of continuity of $F_1(z)$ is completely bounded by the boundary of A_1; and then the function cannot be continued across that boundary. In either case, the equivalence of $F_1(z)$ and $\sum_{i=1}^{\infty} f_i(z)$ does not extend beyond the boundary of A_1, one complete and distinct part of the region of continuity of $\sum_{i=1}^{\infty} f_i(z)$; and therefore, by using the theorem proved in § 81, it follows that:

A series of functions of a variable, which converges within a continuous part of the plane of the variable z, is either a partial or a complete representation of a single uniform, analytic function of the variable in that part of the plane.

83. Further, it has just been proved that the converging series of functions can, in any of the regions A, be changed into an equivalent uniform, analytic function, the equivalence being valid for all points in that region, say

$$\sum_{i=1}^{\infty} f_i(z) = F_1(z).$$

But for any point within A, the function $F_1(z)$ has a uniform finite derivative (§ 21); and therefore also $\sum_{i=1}^{\infty} f_i(z)$ has a uniform finite derivative. The equivalence of the analytic function and the series of functions has not been proved for points on the boundary; even if they are equivalent there, the function $F_1(z)$ cannot be proved to have a uniform finite derivative at every point on the boundary of A, and therefore *it cannot be affirmed that* $\sum_{i=1}^{\infty} f_i(z)$ *has, of necessity, a uniform, finite derivative at points on the boundary of* A, *even though the value of* $\sum_{i=1}^{\infty} f_i(z)$ *be uniform and finite at every point on the boundary**.

Ex. In illustration of the inference just obtained, regarding the derivative of a function at a point on the boundary of its region of continuity, consider the series

$$g(z) = \sum_{n=0}^{\infty} b^n z^{a^n},$$

where b is a positive quantity less than unity, and a is a positive quantity which will be taken to be an odd integer.

For points within and on the circumference of the circle $|z|=1$, the series converges uniformly and unconditionally; and for all points without the circle the series diverges. It thus defines a function for points within the circle and on the circumference, but not for points without the circle.

Moreover for points actually within the circle the function has a first derivative and consequently has any number of derivatives. But it cannot be declared to have a derivative for points on the circle: and it will in fact now be proved that, if a certain condition be satisfied, the derivative for variations at any point on the circle is not merely infinite but that the sign of the infinite value depends upon the direction of the variation, so that the function is not monogenic for the circumference†.

* It should be remarked here, as at the end of § 21, that the result in itself does not contravene Riemann's definition of a function, according to which (§ 8) $\frac{dw}{dz}$ must have the same value whatever be the direction of the vanishing quantity dz; at a point on the boundary of the region there are outward directions for which dw is not defined.

† The following investigation is due to Weierstrass, who communicated it to Du Bois-Reymond: see *Crelle*, t. lxxix, (1875), pp. 29—31.

Let $z=e^{\theta i}$: then, as the function converges unconditionally for all points along the circle, we take

$$f(\theta)=\sum_{n=0}^{\infty} b^n e^{a^n\theta i},$$

where θ is a real variable. Hence

$$\frac{f(\theta+\phi)-f(\theta)}{\phi}=\sum_{n=0}^{\infty}\frac{b^n}{\phi}\{e^{a^n(\theta+\phi)i}-e^{a^n\theta i}\}$$

$$=\sum_{n=0}^{m-1} a^n b^n\left\{\frac{e^{a^n(\theta+\phi)i}-e^{a^n\theta i}}{a^n\phi}\right\}$$

$$+\sum_{n=0}^{\infty} b^{m+n}\left\{\frac{e^{a^{m+n}(\theta+\phi)i}-e^{a^{m+n}\theta i}}{\phi}\right\},$$

assuming m, in the first place, to be any positive integer. To transform the first sum on the right-hand side, we take

$$e^{a^n(\theta+\phi)i}-e^{a^n\theta i}=2ie^{a^n(\theta+\frac{1}{2}\phi)i}\sin\left(\tfrac{1}{2}a^n\phi\right),$$

and therefore

$$\left|\sum_{n=0}^{m-1}(ab)^n\frac{e^{a^n(\theta+\phi)i}-e^{a^n\theta i}}{a^n\phi}\right|$$

$$<\sum_{n=0}^{m-1}(ab)^n\left|\frac{\sin\left(\frac{1}{2}a^n\phi\right)}{\frac{1}{2}a^n\phi}\right|$$

$$<\sum_{n=0}^{m-1}(ab)^n<\frac{(ab)^m}{ab-1},$$

if $ab>1$. Hence, on this hypothesis, we have

$$\sum_{n=0}^{m-1}(ab)^n\left\{\frac{e^{a^n(\theta+\phi)i}-e^{a^n\theta i}}{a^n\phi}\right\}=\gamma\frac{(ab)^m}{ab-1},$$

where γ is a complex quantity with modulus <1.

To transform the second sum on the right-hand side, let the integer nearest to $a^m\dfrac{\theta}{\pi}$ be α_m, so that

$$\tfrac{1}{2}\geqslant a^m\frac{\theta}{\pi}-\alpha_m>-\tfrac{1}{2}$$

for any value of m: then taking

$$x=a^m\theta-\pi\alpha_m,$$

we have

$$\tfrac{1}{2}\pi\geqslant x>-\tfrac{1}{2}\pi,$$

and $\cos x$ is not negative. We choose the quantity ϕ so that

$$\frac{\theta+\phi}{\pi}=\frac{\alpha_m+1}{a^m},$$

and therefore

$$\phi=\frac{\pi-x}{a^m},$$

which, by taking m sufficiently large (a is >1), can be made as small as we please. We now have

$$e^{a^{m+n}(\theta+\phi)i}=e^{a^n\pi i(1+\alpha_m)}=-(-1)^{\alpha_m},$$

if a be an odd integer, and

$$e^{a^{m+n}\theta i}=e^{a^n i(x+\pi\alpha_m)}=(-1)^{\alpha_m}e^{a^n xi}.$$

Hence

$$\frac{e^{a^{m+n}(\theta+\phi)i}-e^{a^{m+n}\theta i}}{\phi}=-(-1)^{\alpha_m}\frac{1+e^{a^n xi}}{\pi-x}a^m,$$

and therefore

$$\sum_{n=0}^{\infty} b^{m+n}\left\{\frac{e^{a^{m+n}(\theta+\phi)i}-e^{a^{m+n}\theta i}}{\phi}\right\}=-(-1)^{\alpha_m}\frac{a^m b^m}{\pi-x}\sum_{n=0}^{\infty} b^n(1+e^{a^n xi}).$$

The real part of the series on the right-hand side is

$$\sum_{n=0}^{\infty} b^n \{1 + \cos a^n x\};$$

every term of this is positive and therefore, as the first term is $1 + \cos x$, the real part

$$> 1 + \cos x$$

$$> 1$$

for $\cos x$ is not negative; and it is finite, for it is

$$< 2 \sum_{n=0}^{\infty} b^n$$

$$< \frac{2}{1-b}.$$

Moreover $$\tfrac{1}{2}\pi < \pi - x < \tfrac{3}{2}\pi,$$

so that $\dfrac{\pi}{\pi - x}$ is positive and $> \dfrac{2}{3}$. Hence

$$\sum_{n=0}^{\infty} b^{m+n} \left\{ \frac{e^{a^{m+n}(\theta+\phi)i} - e^{a^{m+n}\theta i}}{\phi} \right\} = -(-1)^{\alpha_m} \frac{a^m b^m}{\pi} \frac{2}{3} \eta,$$

where η is a finite complex quantity, the real part of which is positive and greater than unity. We thus have

$$\frac{f(\theta+\phi) - f(\theta)}{\phi} = -(-1)^{\alpha_m} (ab)^m \left[\frac{2}{3} \frac{\eta}{\pi} + \gamma' \frac{1}{ab-1} \right],$$

where $|\gamma'| < 1$, and the real part of η is positive and > 1.

Proceeding in the same way and taking

$$\frac{\theta - \chi}{\pi} = \frac{\alpha_m - 1}{a^m},$$

so that $$\chi = \frac{\pi + x}{a^m},$$

we find $$\frac{f(\theta-\chi) - f(\theta)}{\chi} = -(-1)^{\alpha_m} (ab)^m \left[\frac{2}{3} \frac{\eta_1}{\pi} + \gamma_1' \frac{1}{ab-1} \right],$$

where $|\gamma_1'| < 1$ and the real part of η_1, a finite complex quantity, is positive and greater than unity.

If now we take $$ab - 1 > \tfrac{3}{2}\pi,$$

the real parts of $$\frac{2}{3} \frac{\eta}{\pi} + \gamma' \frac{1}{ab-1}, \text{ say of } \zeta,$$

and of $$\frac{2}{3} \frac{\eta_1}{\pi} + \gamma_1' \frac{1}{ab-1}, \text{ say of } \zeta_1,$$

are both positive and different from zero. Then, since

$$\frac{f(\theta+\phi) - f(\theta)}{\phi} = -(-1)^{\alpha_m} (ab)^m \zeta,$$

and $$\frac{f(\theta-\chi) - f(\theta)}{-\chi} = (-1)^{\alpha_m} (ab)^m \zeta_1,$$

m being at present any positive integer, we have the right-hand sides essentially different quantities, because the real part of the first is of sign opposite to the real part of the second.

Now let m be indefinitely increased; then ϕ and χ are infinitesimal quantities which ultimately vanish; and the limit of $\dfrac{1}{\phi}[f(\theta+\phi) - f(\theta)]$ for $\phi = 0$ is a complex infinite quantity with its real part opposite in sign to the real part of the complex infinite quantity which is the limit of $\frac{1}{-\chi}[f(\theta-\chi)-f(\theta)]$ for $\chi=0$. If $f(\theta)$ had a differential coefficient these two limits would be equal: hence $f(\theta)$ *has not, for any value of* θ, *a determinate differential coefficient.*

From this result, a remarkable result relating to real functions may be at once derived. The real part of $f(\theta)$ is

$$\sum_{n=0}^{\infty} b^n \cos(a^n\theta),$$

which is a series converging uniformly and unconditionally. The real parts of

$$-(-1)^{\alpha_m}(ab)^m\zeta$$

$$\text{and of } +(-1)^{\alpha_m}(ab)^m\zeta_1$$

are the corresponding magnitudes for the series of real quantities: and they are of opposite signs. Hence for no value of θ has the series

$$\sum_{n=0}^{\infty} b^n \cos(a^n\theta)$$

a determinate differential coefficient, that is, we can choose an increase ϕ and a decrease χ of θ, both being made as small as we please and ultimately zero, such that the limits of the expressions

$$\frac{f(\theta+\phi)-f(\theta)}{\phi}, \quad \frac{f(\theta-\chi)-f(\theta)}{-\chi}$$

are different from one another, provided a be an odd integer and $ab > 1+\frac{3}{2}\pi$.

The chief interest of the above investigation lies in its application to functions of real variables, continuity in the value of which is thus shewn not necessarily to imply the existence of a determinate differential coefficient defined in the ordinary way. The application is due to Weierstrass, as has already been stated. Further discussions will be found in a paper by Wiener, *Crelle*, t. xc, (1881), pp. 221—252, in a remark by Weierstrass, *Abh. aus der Functionenlehre*, (1886), p. 100, and in a paper by Lerch, *Crelle*, t. ciii, (1888), pp. 126—138, who constructs other examples of continuous functions of real variables; and an example of a continuous function without a derivative is given by Schwarz, *Ges. Werke*, t. ii, pp. 269—274.

The simplest classes of ordinary functions are characterised by the properties:—

(i) Within some region of the plane of the variable they are uniform, finite and continuous:

(ii) At all points within that region (but not necessarily on its boundary) they have a differential coefficient:

(iii) When the variable is real, the number of maximum values and the number of minimum values within any given range is finite.

The function $\sum_{n=0}^{\infty} b^n \cos(a^n\theta)$, suggested by Weierstrass, possesses the first but not the second of these properties. Köpcke (*Math. Ann.*, t. xxix, pp. 123—140) gives an example of a function which possesses the first and the second but not the third of these properties.

84. In each of the distinct portions $A_1, A_2, \ldots$ of the complete region of continuity of a series of functions, the series can be represented by a monogenic analytic function, the elements of which are converging power-series. But the equivalence of the function-series and the monogenic analytic function for any portion A_1 is limited to that region. When the monogenic analytic function can be continued from A_1 into A_2, the continuation is not necessarily the same as the monogenic analytic function which is the equivalent of the series $\sum_{i=1}^{\infty} f_i(z)$ in A_2. Hence, if the monogenic analytic functions for the two portions A_1 and A_2 be different, the function-series represents different functions in the distinct parts of its region of continuity.

A simple example will be an effective indication of the actual existence of such variety of representation in particular cases; that, which follows, is due to Tannery*.

Let a, b, c be any three constants; then the fraction

$$\frac{a + bcz^m}{1 + bz^m},$$

when m is infinite, is equal to a if $|z| < 1$, and is equal to c if $|z| > 1$.

Let $m_0, m_1, m_2, \ldots$ be any set of positive integers arranged in ascending order and be such that the limit of m_n, when $n = \infty$, is infinite. Then, since

$$\begin{aligned}\frac{a + bcz^{m_n}}{1 + bz^{m_n}} &= \frac{a + bcz^{m_0}}{1 + bz^{m_0}} + \sum_{i=1}^{n} \left\{\frac{a + bcz^{m_i}}{1 + bz^{m_i}} - \frac{a + bcz^{m_{i-1}}}{1 + bz^{m_{i-1}}}\right\} \\ &= \frac{a + bcz^{m_0}}{1 + bz^{m_0}} + b(c-a) \sum_{i=1}^{n} \left\{\frac{(z^{m_i - m_{i-1}} - 1) z^{m_{i-1}}}{(1 + bz^{m_i})(1 + bz^{m_{i-1}})}\right\},\end{aligned}$$

the function $\phi(z)$, *defined by the equation*

$$\phi(z) = \frac{a + bcz^{m_0}}{1 + bz^{m_0}} + b(c-a) \sum_{i=1}^{\infty} \left\{\frac{(z^{m_i - m_{i-1}} - 1) z^{m_{i-1}}}{(1 + bz^{m_i})(1 + bz^{m_{i-1}})}\right\},$$

converges uniformly and unconditionally to a value a *if* $|z| < 1$, *and converges uniformly and unconditionally to a value* c *if* $|z| > 1$. But it does not converge uniformly and unconditionally if $|z| = 1$.

The simplest case occurs when $b = -1$ and $m_i = 2^i$; then, denoting the function by $\phi(z)$, we have

$$\begin{aligned}\phi(z) &= \frac{a - cz}{1 - z} + (a - c) \sum_{i=0}^{\infty} \frac{z^{2^i}}{z^{2^{i+1}} - 1} \\ &= \frac{a - cz}{1 - z} + (a - c) \left\{\frac{z}{z^2 - 1} + \frac{z^2}{z^4 - 1} + \frac{z^4}{z^8 - 1} + \ldots\right\},\end{aligned}$$

that is, the function $\phi(z)$ is equal to a if $|z| < 1$, and it is equal to c if

$$|z| > 1.$$

* It is contained in a letter of Tannery's to Weierstrass, who communicated it to the Berlin Academy in 1881, *Abh. aus der Functionenlehre*, pp. 103, 104. A similar series, which indeed is equivalent to the special form of $\phi(z)$, was given by Schröder, *Schlöm. Zeitschrift*, t. xxii, (1876), p. 184; and Pringsheim, *Math. Ann.*, t. xxii, (1883), p. 110, remarks that it can be deduced, without material modifications, from an expression given by Seidel, *Crelle*, t. lxxiii, (1871), pp. 297–299.

When $|z| = 1$, the function can have any value whatever. Hence a circle of radius unity is a line of singularities, that is, it is a line of discontinuity for the series. The circle evidently has the property of dividing the plane into two parts such that *the analytical expression represents different functions in the two parts.*

If we introduce a new variable ζ connected with z by the relation*

$$\zeta = \frac{1+z}{1-z},$$

then, if $\zeta = \xi + i\eta$ and $z = x + iy$, we have

$$\xi = \frac{1 - x^2 - y^2}{(1-x)^2 + y^2},$$

so that ξ is positive when $|z| < 1$, and ξ is negative when $|z| > 1$. If then

$$\phi(z) = \chi(\zeta),$$

the function $\chi(\zeta)$ is equal to a or to c according as the real part of ζ is positive or negative.

And, generally, if we take ζ a rational function of z and denote the modified form of $\phi(\zeta)$, which will be a sum of rational functions of z, by $\phi_1(z)$, then $\phi_1(z)$ will be equal to a in some parts of the plane and to c in other parts of the plane. The boundaries between these parts are lines of singular points: and they are constituted by the z-curves which correspond to $|\zeta| = 1$.

85. Now let $F(z)$ and $G(z)$ be two functions of z with any number of singularities in the plane: it is possible to construct a function which shall be equal to $F(z)$ within a circle centre the origin and to $G(z)$ without the circle, the circumference being a line of singularities. For, when we make $a = 1$ and $c = 0$ in $\phi(z)$ of § 84, the function

$$\theta(z) = \frac{1}{1-z} + \frac{z}{z^2-1} + \frac{z^2}{z^4-1} + \frac{z^4}{z^8-1} + \dots$$

is unity for all points within the circle and is zero for all points without it: and therefore

$$G(z) + \{F(z) - G(z)\}\,\theta(z)$$

is a function which has the required property.

Similarly $$F_3(z) + \{F_1(z) - F_2(z)\}\,\theta(z) + \{F_2(z) - F_3(z)\}\,\theta\left(\frac{z}{r}\right)$$

is a function which has the value $F_1(z)$ within a circle of radius unity, the value $F_2(z)$ between a circle of radius unity and a concentric circle of radius r greater than unity, and the value $F_3(z)$ without the latter circle. All the singularities of the functions F_1, F_2, F_3 are singularities of the function thus represented; and it has, in addition to these, the two lines of singularities given by the circles.

* The significance of a relation of this form will be discussed in Chapter XIX.

Again,
$$G(z)+\{F(z)-G(z)\}\,\theta\left(\frac{z-1}{z+1}\right)$$
is a function of z, which is equal to $F(z)$ on the positive side of the axis of y, and is equal to $G(z)$ on the negative side of that axis.

Also, if we take
$$\zeta e^{-i\alpha_1}-p_1=\frac{1+z}{1-z},$$
where α_1 and p_1 are real constants, as an equation defining a new variable $\xi+i\eta$, we have
$$\xi\cos\alpha_1+\eta\sin\alpha_1-p_1=\frac{1-x^2-y^2}{(1-x)^2+y^2}$$
so that the two regions of the z-plane determined by $|z|<1$ and $|z|>1$ correspond to the two regions of the ζ-plane into which the line $\xi\cos\alpha_1+\eta\sin\alpha_1-p_1=0$ divides it. Let
$$\theta(z)=\theta\left(\frac{\zeta e^{-i\alpha_1}-p_1-1}{\zeta e^{-i\alpha_1}-p_1+1}\right)$$
$$=\theta_1(\zeta),$$
so that on the positive side of the line $\xi\cos\alpha_1+\eta\sin\alpha_1-p_1=0$ the function θ_1 is unity and on the negative side of that line it is zero. Take any three lines defined by α_1, p_1; α_2, p_2; α_3, p_3 respectively; then
$$\tfrac{1}{2}\{-F+F(\theta_1+\theta_2+\theta_3)\}$$
is a function which has the value F within the triangle, the value $-F$ in three of the spaces without it, and the value zero in the remaining three spaces without it, as indicated in the figure (fig. 13).

And for every division of the plane by lines, into which a circle can be transformed by rational equations, as will be explained when conformal representation is discussed hereafter, there is a possibility of representing discontinuous functions, by expressions similar to those just given.

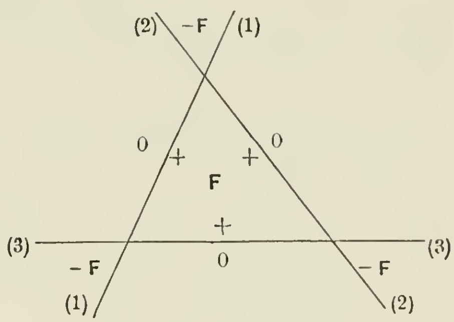

Fig. 13.

These examples are sufficient to lead to the following result*, which is complementary to the theorem of § 82:

When the region of continuity of an infinite series of functions consists of several distinct parts, the series represents a single function in each part but it does not necessarily represent the same function in different parts.

It thus appears that an analytical expression of given form, which converges uniformly and unconditionally in different parts of the plane separated from one another, can represent different functions of the variable in those different parts; and hence *the idea of monogenic functionality of a complex variable is not coextensive with the idea of functional dependence expressible through arithmetical operations*, a distinction first established by Weierstrass.

86. We have seen that an analytic function has not a definite value at an essential singularity and that, therefore, every essential singularity is excluded from the region of definition of the function.

* Weierstrass, l.c., p. 90.

Again, it has appeared that not merely must single points be on occasion excluded from the region of definition but also that functions exist with continuous lines of essential singularities which must therefore be excluded. One method for the construction of such functions has just been indicated: but it is possible to obtain other analytical expressions for functions which possess what may be called a *singular line*. Thus let a function have a circle of radius c as a line of essential singularity*; let it have no other singularities in the plane and let its zeros be $a_1, a_2, a_3, \ldots$, supposed arranged in such order that, if $\rho_n e^{i\theta_n} = a_n$, then

$$|\rho_n - c| \geqslant |\rho_{n+1} - c|,$$

so that the limit of ρ_n, when n is infinite, is c.

Let $c_n = ce^{i\theta_n}$, a point on the singular circle, corresponding to a_n which is assumed not to lie on it. Then, proceeding as in Weierstrass's theory in § 51, if

$$G(z) = \prod_{n=1}^{n=\infty} \left\{ \frac{z - a_n}{z - c_n} e^{g_n(z)} \right\},$$

where $$g_n(z) = \frac{a_n - c_n}{z - c_n} + \frac{1}{2}\left(\frac{a_n - c_n}{z - c_n}\right)^2 + \ldots + \frac{1}{m_n - 1}\left(\frac{a_n - c_n}{z - c_n}\right)^{m_n - 1},$$

$G(z)$ is a uniform function, continuous everywhere in the plane except along the circumference of the circle which may be a line of essential singularities.

Special simpler forms can be derived according to the character of the series of quantities constituted by $|a_n - c_n|$. If there be a finite integer m, such that $\sum\limits_{n=1}^{\infty} |a_n - c_n|^m$ is a converging series, then in $g_n(z)$ only the first $m - 1$ terms need be retained.

Ex. Construct the function when

$$a_n = \left(1 - \frac{1}{n^r}\right) e^{\frac{2m\pi i}{n}},$$

m being a given positive integer and r a positive quantity.

Again, the point c_n was associated with a_n so that they have the same argument: but this distribution of points on the circle is not necessary and can be made in any manner which satisfies the condition that in the limited case just quoted the series $\sum\limits_{n=1}^{\infty} |a_n - c_n|^m$ is a converging series.

Singular lines of other classes, for example, *sections*† in connection with functions defined by integrals, arise in connection with analytical functions. They are discussed by Painlevé, "Sur les lignes singulières des fonctions analytiques," (Thèse, Gauthier-Villars, Paris, 1887).

Ex. Shew that, if the zeros of a function be the points

$$A = \frac{b + c - (a - d)\, i}{a + d + (b - c)\, i},$$

* This investigation is due to Picard, *Comptes Rendus*, t. xci, (1881), pp. 690—692.

† Called *coupures* by Hermite; see § 103.

where a, b, c, d are integers satisfying the condition $ad-bc=1$, so that the function has a circle of radius unity for an essential singular line, then if

$$B=\frac{b+di}{d+bi},$$

the function
$$\Pi\left\{\frac{z-A}{z-B}e^{\frac{A-B}{z-B}}\right\},$$

where the product extends to all positive integers subject to the foregoing condition $ad-bc=1$, is a uniform function finite for all points in the plane not lying on the circle of radius unity. (Picard.)

87. In the earlier examples, instances were given of functions which have only isolated points for their essential singularities: and, in the later examples, instances have been given of functions which have lines of essential singularities, that is, there are continuous lines for which the functions do not exist. We now proceed to shew how functions can be constructed which do not exist in assigned continuous spaces in the plane, these spaces being aggregates of essential singularities. Weierstrass was the first to draw attention to *lacunary* functions, as they may be called; the following investigation in illustration of Weierstrass's theorem is due to Poincaré*.

Take any convex curve in the plane, say C; and consider the function

$$\Sigma\frac{A}{z-b},$$

where the quantities A are constants, subject to the conditions

(i) The series $\Sigma|A|$ converges uniformly and unconditionally:

(ii) Each of the points b is either within or on the curve C:

(iii) The points b are the aggregate of all rational‡ points within and on C: then the function is a uniform analytical function for all points without C and it has the area of C for a lacunary space.

First, it is evident that, if $z=b$, then the series does not converge. Moreover as the points b are the aggregate of all the rational points within or on C, there will be an infinite number of singularities in the immediate vicinity of b: we shall thus have an unlimited number of terms each infinite of the first order, and thus (§ 42) the point b will be an essential singularity. As this is true of all points z within or on C, it follows that the area C is a lacunary space for the function, if the function exist at all.

Secondly, let z be a point without C; and let d be the distance of z from the nearest point of the boundary of C†, so that d is not a vanishing quantity.

* *Acta Soc. Fenn.*, t. xii, (1883), pp. 341—350.

‡ *Rational* points within or on C are points whose positions can be determined rationally in terms of the coordinates of assigned points on C; examples will be given.

† This will be either the shortest normal from z to the boundary or the distance of z from some point of abrupt change of direction, as for instance at the angular point of a polygon.

Then $|z-b| \geqslant d$; and therefore

$$\left|\frac{A}{z-b}\right| = \frac{|A|}{|z-b|} \leqslant \frac{|A|}{d},$$

so that

$$\left|\Sigma \frac{A}{z-b}\right| < \Sigma \left|\frac{A}{z-b}\right|$$

$$< \Sigma \frac{|A|}{d}$$

$$< \frac{1}{d} \Sigma |A|.$$

Now $\Sigma |A|$ converges uniformly and unconditionally and therefore, as d does not vanish,

$$\left|\Sigma \frac{A}{z-b}\right|$$

converges uniformly and unconditionally, that is,

$$\Sigma \frac{A}{z-b}$$

is a function of z which converges uniformly and unconditionally for every point without C. Let it be denoted by $\phi(z)$.

Let c be any point without C, and let r be the radius of the greatest circle centre c which can be drawn so as to have no point of C within itself or on its circumference, so that r is the radius of the domain of c; then $|b-c| > r$, for all points b.

If we take a point z within this circle, we have $|z-c| = \theta r$, where $\theta < 1$.

Now for all points within this circle the function $\phi(z)$ converges uniformly, and every term $\dfrac{A}{z-b}$ of $\phi(z)$ is finite. Also, for points within the circle, we can expand $\dfrac{A}{z-b}$ in powers of $z-c$ in the form

$$-\frac{A}{z-b} = A \sum_{m=0}^{\infty} \frac{(z-c)^m}{(b-c)^{m+1}}$$

of a converging series. Hence, by § 82, we have

$$\phi(z) = \sum_{m=0}^{\infty} B_m (z-c)^m,$$

a series converging uniformly and unconditionally for all points within the circle centre c and radius r, which circle is the circle of convergence of the series. The function can be expressed in the usual manner over the whole of the region of continuity, which is the part of the plane without the curve C.

Thus $\phi(z)$ is a uniform analytical function, having the area of C for a lacunary space.

As an example, take a convex polygon having $a_1, \ldots\ldots, a_p$ for its angular points; then any point

$$\frac{m_1 a_1 + \ldots\ldots + m_p a_p}{m_1 + \ldots\ldots + m_p},$$

where $m_1, \ldots\ldots, m_p$ are positive integers or zero (simultaneous zeros being excluded), is either within the polygon or on its boundary: and any rational point within the polygon or on its boundary can be represented by

$$\frac{\sum_{r=1}^{p} m_r a_r}{\sum_{r=1}^{p} m_r},$$

by proper choice of $m_1, \ldots\ldots, m_p$, a choice which can be made in an infinite number of ways.

Let $u_1, \ldots\ldots, u_p$ be given quantities, the modulus of each of which is less than unity: then the series

$$\sum_0^\infty u_1^{m_1} \ldots\ldots u_p^{m_p}$$

converges uniformly and unconditionally. Then all the assigned conditions are satisfied for the function

$$\Sigma \left\{ \frac{u_1^{m_1} \ldots\ldots u_p^{m_p}}{z - \dfrac{m_1 a_1 + \ldots\ldots + m_p a_p}{m_1 + \ldots\ldots + m_p}} \right\},$$

and therefore it is a function which converges uniformly and unconditionally everywhere outside the polygon and which has the polygonal space (including the boundary) for a lacunary space.

If, in particular, $p = 2$, we obtain a function which has the straight line joining a_1 and a_2 as a line of essential singularity. When we take $a_1 = 0$, $a_2 = 1$ and slightly modify the summation, we obtain the function

$$\sum_{n=1}^{\infty} \sum_{m=0}^{n} \frac{u_1^m u_2^{n-m}}{z - \dfrac{m}{n}},$$

which, when $|u_1| < 1$ and $|u_2| < 1$, converges uniformly and unconditionally everywhere in the plane except at points between 0 and 1 on the axis of real quantities, this part of the axis being a line of essential singularity.

For the general case, the following remarks may be made:

(i) The quantities $u_1, u_2, \ldots$ need not be the same for every term; a numerator, quite different in form, might be chosen, such as $(m_1^2 + \ldots + m_p^2)^{-\mu}$ where $2\mu > p$; all that is requisite is that the series, made up of the numerators, should converge uniformly and unconditionally.

(ii) The preceding is only a particular illustration and is not necessarily the most general form of function having the assigned lacunary space.

It is evident that the first step in the construction of a function, which shall have any assigned lacunary space, is the formation of some expression which, by the variation of the constants it contains, can be made to represent indefinitely nearly any point within or on the contour of the space. Thus for the space between two concentric circles of radii a and c and centre the origin we should take

$$\frac{m_1 a + (n - m_1) b}{n} e^{\frac{m_2}{n} 2\pi i},$$

which, by giving m_1 all values from 0 to n, m_2 all values from 0 to $n-1$ and n all values from 1 to infinity will represent all rational points in the space: and a function, having the space between the circles as lacunary, would be given by

$$\sum_{n=1}^{\infty} \sum_{m_1=0}^{n} \sum_{m_2=0}^{n-1} \left[\frac{u^n u_1^{m_1} u_2^{m_2}}{\left\{ z - \dfrac{m_1 a + (n - m_1) b}{n} e^{\frac{m_2}{n} 2\pi i} \right\}} \right],$$

provided $|u| < 1$, $|u_1| < 1$, $|u_2| < 1$.

In particular, if $a = b$, then the common circumference is a line of essential singularity for the corresponding function. It is easy to see that the function

$$\sum_{n=0}^{\infty} \sum_{m=0}^{2n-1} \frac{u^{m}_{m,n} \, v^{n}_{m,n}}{z - a e^{\frac{m}{n} \pi i}},$$

provided the series $\sum_{n=1}^{\infty} \sum_{m=0}^{2n-1} u^{m}_{m,n} \, v^{n}_{m,n}$

converges uniformly and unconditionally, is a function having the circle $|z| = a$ as a line of essential singularity.

Other examples will be found in memoirs by Goursat*, Poincaré†, and Homén‡.

Ex. 1. Shew that the function

$$\sum_{m=-\infty}^{m=\infty} \sum_{n=-\infty}^{n=\infty} (m + nz)^{-2-r},$$

where r is a real positive quantity and the summation is for all integers m and n between the positive and the negative infinities, is a uniform function in all parts of the plane except the axis of real quantities which is a line of essential singularity.

Ex. 2. Discuss the region in which the function

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sum_{p=1}^{\infty} \frac{n^{-2} m^{-2} p^{-2}}{z - \left(\dfrac{p}{n} + \dfrac{m}{n} i \right)}$$

is definite. (Homén.)

Ex. 3. Prove that the function

$$\sum_{n=0}^{\infty} 2^{-n} x^{3^n}$$

exists only within a circle of radius unity and centre the origin. (Poincaré.)

Ex. 4. An infinite number of points $a_1, a_2, a_3, \ldots\ldots$ are taken on the circumference of a given circle, centre the origin, so that they form the aggregate of rational points on the circumference. Shew that the series

$$\sum_{n=1}^{\infty} \frac{1}{n^3} \frac{z}{a_n - z}$$

can be expanded in a series of ascending powers of z which converges for points within the circle, but that the function cannot be continued across the circumference of the circle. (Stieltjes.)

* *Comptes Rendus*, t. xciv, (1882), pp. 715—718; *Bulletin de Darboux*, 2^{me} Sér., t. xi, (1887), pp. 109—114.

† In the memoir, quoted p. 138, and *Comptes Rendus*, t. xcvi, (1883), pp. 1134—1136.

‡ *Acta Soc. Fenn.*, t. xii, (1883), pp. 445—464.

Ex. 5. Prove that the series

$$\frac{2}{\pi}(z+z^{-1})+\frac{2}{\pi}\sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty}\left\{\frac{z}{(1-2m-2nzi)(2m+2nzi)^2}\right\}$$
$$+\frac{2}{\pi}\sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty}\left\{\frac{z^{-1}}{(1-2m-2nz^{-1}i)(2m+2nz^{-1}i)^2}\right\},$$

where the summation extends over all positive and negative integral values of m and of n except simultaneous zeros, is a function which converges uniformly and unconditionally for all points in the finite part of plane which do not lie on the axis of y; and that it has the value $+1$ or -1 according as the real part of z is positive or negative.

(Weierstrass.)

Ex. 6. Prove that the region of continuity of the series

$$\sum_{n=0}^{\infty}\frac{1}{z^n+z^{-n}}$$

consists of two parts, separated by the circle $|z|=1$ which is a line of infinities for the series: and that, in these two parts of the plane, it represents two different functions.

If two complex quantities ω and ω' be taken, such that $z=e^{-\frac{\omega'\pi}{\omega i}}$ and the real part of $\frac{\omega'}{\omega i}$ is positive, and if they be associated with the elliptic function $\wp(u)$ as its half-periods, then for values of z which lie within the circle $|z|=1$

$$\sum_{n=0}^{\infty}\frac{1}{z^n+z^{-n}}=\frac{\omega}{2\pi}\frac{\sigma_3(\omega)}{\sigma(\omega)}+\tfrac{1}{4},$$

in the usual notation of Weierstrass's theory of elliptic functions.

Find the function which the series represents for values of z without the circle $|z|=1$.

(Weierstrass.)

Ex. 7. Four circles are drawn each of radius $\frac{1}{\sqrt{2}}$ having their centres at the points $1, i, -1, -i$ respectively; the two parts of the plane, excluded by the four circumferences, are denoted the interior and the exterior parts. Shew that the function

$$\sum_{n=1}^{n=\infty}\frac{\sin\frac{1}{4}n\pi}{2^{\frac{1}{2}n}n}\left\{\frac{1}{(1-z)^n}+\frac{1}{(1+iz)^n}+\frac{1}{(1+z)^n}+\frac{1}{(1-iz)^n}\right\}$$

is equal to π in the interior part and is zero in the exterior part. (Appell.)

Ex. 8. Obtain the values of the function

$$\sum_{n=1}^{n=\infty}\frac{1-(-1)^n}{n}\left\{(\tfrac{1}{2}z)^n-\frac{1}{(z+1)^n}-\frac{1}{(z-1)^n}\right\}$$

in the two parts of the area within a circle centre the origin and radius 2 which lie without two circles of radius unity, having their centres at the points 1 and -1 respectively.

(Appell.)

Ex. 9. If
$$f(z)=U_1+U_2+\ldots\ldots+U_n,$$
and
$$U_m=F_m(z)-\frac{1}{z-a_m}+(z-a_m-1)\left\{\frac{1}{(z-a_m)^2}+\frac{1}{(z-a_m)^3}+\ldots\ldots\right\}$$

where the regions of continuity of the functions F extend over the whole plane, then $f(z)$ is a function existing everywhere except within the circles of radius unity described round the points $a_1, a_2, \ldots\ldots, a_n$.

(Teixeira.)

Ex. 10. Let there be n circles having the origin for a common centre, and let $C_1, C_2, \ldots\ldots, C_n, C_{n+1}$ be $n+1$ arbitrary constants; also let $a_1, a_2, \ldots\ldots, a_n$ be any n points lying respectively on the circumferences of the first, the second,......, the nth circles. Shew that the expression

$$\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{C_1}{ze^{i\theta}}+\frac{C_2-C_1}{ze^{i\theta}-a_1}+\ldots\ldots+\frac{C_{n+1}-C_n}{ze^{i\theta}-a_n}\right)ze^{i\theta}d\theta$$

has the value C_m for points z lying between the $(m-1)$th and the mth circles and the value C_{n+1} for points lying without the nth circle.

Construct a function which shall have any assigned values in the various bands into which the plane is divided by the circles. (Pincherle.)

88. In § 32 it was remarked that the discrimination of the various species of essential singularities could be effected by means of the properties of the function in the immediate vicinity of the point.

Now it was proved, in § 63, that in the vicinity of an isolated essential singularity b the function could be represented by an expression of the form

$$G\left(\frac{1}{z-b}\right)+P(z-b)$$

for all points in the space without a circle centre b of small radius and within a concentric circle of radius not large enough to include singularities at a finite distance from b. Because the essential singularity at b is isolated, the radius of the inner circle can be diminished to be all but infinitesimal: the series $P(z-b)$ is then unimportant compared with $G\left(\frac{1}{z-b}\right)$, which can be regarded as characteristic for the singularity of the function.

Another method of obtaining a function, which is characteristic of the singularity, is provided by § 68. It was there proved that, in the vicinity of an essential singularity a, the function could be represented by an expression of the form

$$(z-a)^n H\left(\frac{1}{z-a}\right)Q(z-a),$$

where, within a circle of centre a and radius not sufficiently large to include the nearest singularity at a finite distance from a, the function $Q(z-a)$ is finite and has no zeros: all the zeros of the given function within this circle (except such as are absorbed into the essential singularity at a) are zeros of the factor $H\left(\frac{1}{z-a}\right)$, and the integer-index n is affected by the number of these zeros. When the circle is made small, the function

$$(z-a)^n H\left(\frac{1}{z-a}\right)$$

can be regarded as characteristic of the immediate vicinity of a or, more briefly, as characteristic of a.

It is easily seen that the two characteristic functions are distinct. For if F and F_1 be two functions, which have essential singularities at a of the same kind as determined by the first characteristic, then

$$\begin{aligned} F(z) - F_1(z) &= P(z-a) - P_1(z-a) \\ &= P_2(z-a), \end{aligned}$$

while if their singularities at a be of the same kind as determined by the second characteristic, then

$$\frac{F(z)}{F_1(z)} = \frac{Q(z-a)}{Q_1(z-a)} = Q_2(z-a)$$

in the immediate vicinity of a, since Q_1 has no zeros. Two such equations cannot subsist simultaneously, except in one instance.

Without entering into detailed discussion, the results obtained in the preceding chapters are sufficient to lead to an indication of the classification of singularities*.

Singularities are said to be of the *first class* when they are accidental; and a function is said to be of the first class when all its singularities are of the first class. It can, by § 48, have only a finite number of such singularities, each singularity being isolated.

It is for this case alone that the two characteristic functions are in accord.

When a function, otherwise of the first class, fails to satisfy the last condition, solely owing to failure of finiteness of multiplicity at some point, say at $z = \infty$, then that point ceases to be an accidental singularity. It has been called (§ 32) an essential singularity; it belongs to the simplest kind of essential singularity; and it is called a singularity of the *second class*.

A function is said to be of the second class when it has some singularities of the second class; it may possess singularities of the first class. By an argument similar to that adopted in § 48, a function of the second class can have only a limited number of singularities of the second class, each singularity being isolated.

When a function, otherwise of the second class, fails to satisfy the last condition solely owing to unlimited condensation at some point, say at $z = \infty$, of singularities of the second class, that point ceases to be a singularity of the second class: it is called a singularity (necessarily essential) of the *third class*.

* For a detailed discussion, reference should be made to Guichard, "Théorie des points singuliers essentiels" (Thèse, Gauthier-Villars, Paris, 1883), who gives adequate references to the investigations of Mittag-Leffler in the introduction of the classification and to the researches of Cantor. See also Mittag-Leffler, *Acta Math.*, t. iv, (1884), pp. 1—79; Cantor, *Crelle*, t. lxxxiv, (1878), pp. 242—258, *Acta Math.*, t. ii, (1883), pp. 311—328.

A function is said to be of the third class when it has some singularities of the third class; it may possess singularities of the first and the second classes. But it can have only a limited number of singularities of the third class, each singularity being isolated.

Proceeding in this gradual sequence, we obtain an unlimited number of classes of singularities: and functions of the various classes can be constructed by means of the theorems which have been proved. A function of class n has a limited number of singularities of class n, each singularity being isolated, and any number of singularities of lower classes which, except in so far as they are absorbed in the singularities of class n, are isolated points.

The effective limit of this sequence of classes is attained when the number of the class increases beyond any integer, however large. When once such a limit is attained, we have functions with essential singularities of unlimited class, each singularity being isolated; when we pass to functions which have their essential singularities no longer isolated but, as in previous class-developments, of infinite condensation, it is necessary to add to the arrangement in classes an arrangement in a wider group, say, in species*.

Calling, then, all the preceding classes of functions functions of the first species, we may, after Guichard (l.c.), construct, by the theorems already proved, a function which has at the points $a_1, a_2, \ldots$ singularities of classes $1, 2, \ldots$, both series being continued to infinity. Such a function is called a function of the second species.

By a combination of classes in species, this arrangement can be continued indefinitely; each species will contain an infinitely increasing number of classes; and when an unlimited number of species is ultimately obtained, another wider group must be introduced.

This gradual construction, relative to essential singularities, can be carried out without limit; the singularities are the characteristics of the functions.

* Guichard (l.c.) uses the term *genre*.

CHAPTER VIII.

Multiform Functions.

89. Having now discussed some of the more important general properties of uniform functions, we proceed to discuss some of the properties of multiform functions.

Deviations from uniformity in character may arise through various causes: the most common is the existence of those points in the z-plane, which have already (§ 12) been defined as branch-points.

As an example, consider the two power-series

$$w_1 = 1 - \tfrac{1}{2}z' - \tfrac{1}{8}z'^2 - \ldots \quad , \qquad w_2 = -(1 - \tfrac{1}{2}z' - \tfrac{1}{8}z'^2 - \ldots),$$

which, for points in the plane such that $|z'|$ is less than unity, are the two values of $(1-z')^{\frac{1}{2}}$; they may be regarded as two branches of the function w defined by the equation

$$w^2 = 1 - z' = z.$$

Let z' describe a small curve (say a circle of radius r) round the point $z'=1$, beginning on the axis of x; the point 1 is the origin for z. Then z is r initially, and at the end of the first description of the circle z is $re^{2\pi i}$; hence initially w_1 is $+r^{\frac{1}{2}}$ and w_2 is $-r^{\frac{1}{2}}$, and at the end of the description w_1 is $+r^{\frac{1}{2}}e^{\pi i}$ and w_2 is $-r^{\frac{1}{2}}e^{\pi i}$, that is, w_1 is $-r^{\frac{1}{2}}$ and w_2 is $+r^{\frac{1}{2}}$. Thus the effect of the single circuit is to change w_1 into w_2 and w_2 into w_1, that is, the effect of a circuit round the point, at which w_1 and w_2 coincide in value, is to interchange the values of the two branches.

If, however, z describe a circuit which does not include the branch-point, w_1 and w_2 return each to its initial value.

Instances have already occurred, e.g. integrals of uniform functions, in which a variation in the path of the variable has made a difference in the result; but this interchange of value is distinct from any of the effects produced by points belonging to the families of critical points which have been considered. The critical point is of a new nature; it is, in fact, a characteristic of multiform functions at certain associated points.

We now proceed to indicate more generally the character of the relation of such points to functions affected by them.

The method of constructing a monogenic analytic function, described in § 34, by forming all the continuations of a power-series, regarded as a given initial element of the function, leads to the aggregate of the elements of the function and determines its region of continuity. When the process of continuation has been completely carried out, two distinct cases may occur.

In the first case, the function is such that any and every path, leading from one point a to another point z by the construction of a series of successive domains of points along the path, gives a single value at z as the continuation of one initial value at a. When, therefore, there is only a single value of the function at a, the process of continuation leads to only a single value of the function at any other point in the plane. The function is uniform throughout its region of continuity. The detailed properties of such functions have been considered in the preceding chapters.

In the second case, the function is such that different paths, leading from a to z, do not give a single value at z as the continuation of one and the same initial value at a. There are different sets of elements of the function, associated with different sets of consecutive domains of points on paths from a to z, which lead to different values of the function at z; but any change in a path from a to z does not necessarily cause a change in the value of the function at z. The function is multiform in its region of continuity. The detailed properties of such functions will now be considered.

90. In order that the process of continuation may be completely carried out, continuations must be effected, beginning at the domain of any point a and proceeding to the domain of any other point b by all possible paths in the region of continuity, and they must be effected for all points a and b. Continuations must be effected, beginning in the domain of every point a and returning to that domain by all possible closed paths in the region of continuity. When they are effected from the domain of one point a to that of another point b, all the values at any point z in the domain of a (and not merely a single value at such points) must be continued: and similarly when they are effected, beginning in the domain of a and returning to that domain. The complete region of the plane will then be obtained in which the function can be represented by a series of positive integral powers: and the boundary of that region will be indicated.

In the first instance, let the boundary of the region be constituted by a number, either finite or infinite, of isolated points, say L_1, L_2, L_3, ... Take any point A in the region, so that its distance from any of the points L is not infinitesimal; and in the region draw a closed path $ABC...EFA$ so as to enclose one point, say L_1, but only one point, of the boundary and to have no point of the curve at a merely infinitesimal distance from L_1. Let such curves be drawn, beginning and ending at A, so that each of them encloses one and only one of the points of the boundary: and let K_r be the curve which encloses the point L_r.

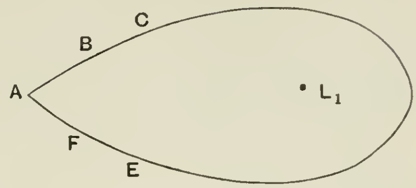

Fig. 14.

Let w_1 be one of the power-series defining the function in a domain with its centre at A: let this series be continued along each of the curves K_s by successive domains of points along the curve returning to A. The result of the description of all the curves will be that the series w_1 cannot be reproduced at A for all the curves though it may be reproduced for some of them; otherwise, w_1 would be a uniform function. Suppose that $w_2, w_3, \ldots$, each in the form of a power-series, are the aggregate of new distinct values thus obtained at A; let the same process be effected on $w_2, w_3, \ldots$ as has been effected on w_1, and let it further be effected on any new distinct values obtained at A through $w_2, w_3, \ldots$, and so on. When the process has been carried out so far that all values obtained at A, by continuing any series round any of the curves K back to A, are included in values already obtained, the aggregate of the values of the function at A is complete: they are the values at A of the *branches* of the function.

We shall now assume that the number of values thus obtained is finite, say n, so that the function has n branches at A: if their values be denoted by $w_1, w_2, \ldots, w_n$, these n quantities are all the values of the function at A. Moreover, n is the same for all points in the plane, as may be seen by continuing the series at A to any other point and taking account of the corollaries at the end of the present section.

The boundary-points L may be of two kinds. It may (and not infrequently does) happen that a point L_s is such that, whatever branch is taken at A as the initial value for the description of the circuit K_s, that branch is reproduced at the end of the circuit. Let the aggregate of such points be $I_1, I_2, \ldots$. Then each of the remaining points L is such that a description of the circuit round it effects a change on at least one of the branches, taken as an initial value for the description; let the aggregate of these points be $B_1, B_2, \ldots$. They are the branch-points; their association with the definition in § 12 will be made later.

When account is taken of the continuations of the function from a point A to another point B, we have n values at B as the continuations of n values at A. The selection of the individual branch at B, which is the continuation of a particular branch at A, depends upon the path of z between A and B; it is governed by the following fundamental proposition:—

The final value of a branch of a function for two paths of variation of the independent variable from one point to another will be the same, if one path can be deformed into the other without passing over a branch-point.

Let the initial and the final points be a and b, and let one path of variation be acb. Let another path of variation be aeb, both paths lying in the region in which the function can be expressed by series of positive integral powers: the two paths are assumed to have no point within an infinitesimal distance of any of the boundary-points L and to be taken so close together that the circles of convergence of pairs of points (such as c_1 and e_1, c_2 and e_2, and so on) along the two paths have common areas. When we begin at a with a branch of the function, values at c_1 and at e_1 are obtained, depending upon the values of the branch and its derivatives at a and upon the positions of c_1 and e_1; hence, at any point in the area common to the circles of convergence of these two points, only a single value arises as derived through the initial value at a. Proceeding in this way, only a single value is obtained at any point in an area common to the circles of convergence of points in the two paths. Hence ultimately one and the same value will be obtained at b as the continuation of the value of the one branch at a by the two different paths of variation which have been taken so that no boundary-point L lies between them or infinitesimally near to them.

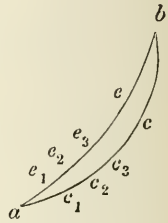

Fig. 15.

Now consider any two paths from a to b, say acb and adb, such that neither of them is near a boundary-point and that the contour they constitute does not enclose a boundary-point. Then by a series of successive infinitesimal deformations we can change the path acb to adb; and as at b the same value of w is obtained for variations of z from a to b along the successive deformations, it follows that the same value of w is obtained at b for variations of z along acb as for variations along adb.

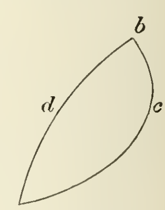

Fig. 16.

Next, let there be two paths acb, adb constituting a closed contour, enclosing one (but not more than one) of the points I and none of the points B. When the original curve K which contains the point I is described, the initial value is restored: and hence the branches of the function obtained at any point of K by the two paths from any point, taken as initial point, are the same. By what precedes, the parts of this curve K can be deformed into the parts of *acbda* without affecting the branches of the function: hence the value obtained at *b*, by continuation along *acb*, is the same as the value there obtained by continuation along *adb*. It therefore follows that a path between two points *a* and *b* can be deformed over any point *I* without affecting the value of the function at *b*; so that, when the preceding results are combined, the proposition enunciated is proved.

By the continued application of the theorem, we are led to the following results:—

Corollary I. *Whatever be the effect of the description of a circuit on the initial value of a function, a reversal of the circuit restores the original value of the function.*

For the circuit, when described positively and negatively, may be regarded as the contour of an area of infinitesimal breadth, which encloses no branch-point within itself and the description of the contour of which therefore restores the initial value of the function.

Corollary II. *A circuit can be deformed into any other circuit without affecting the final value of the function, provided that no branch-point be crossed in the process of deformation.*

It is thus justifiable, and it is often convenient, to deform a path containing a single branch-point into a loop round the point. A *loop** consists of a line nearly to the point, nearly the whole of a very small circle round the point, and a line back to the initial point; see figure 17.

Fig. 17.

Corollary III. *The value of a function is unchanged when the variable describes a closed circuit containing no branch-point; it is likewise unchanged when the variable describes a closed circuit containing all the branch-points.*

The first part is at once proved by remarking that, without altering the value of the function, the circuit can be deformed into a point.

For the second part, the simplest plan is to represent the variable on Neumann's sphere. The circuit is then a curve on the sphere enclosing all the branch-points: the effect on the value of the function is unaltered by any deformation of this curve which does make it cross a branch-point. The curve can, without crossing a branch-point, be deformed into a point in that other part of the area of the sphere which contains none of the branch-points; and the point, which is the limit of the curve, is not a branch-point. At such a point, the value of the function is unaltered; and therefore the description of a circuit, which encloses all the branch-points, restores the initial value of the function.

Corollary IV. *If the values of w at b for variations along two paths*

* French writers use the word *lacet*, German writers the word *Schleife*.

acb, adb be not the same, then a description of acbda will not restore the initial value of w at a.

In particular, let the path be the loop $O\epsilon c\epsilon O$ (fig. 17), and let it change w at O into w'. Since the values of w at O are different and because there is no branch-point in $O\epsilon$ (or in the evanescent circuit $O\epsilon O$), the values of w at ϵ cannot be the same: that is, the value with which the infinitesimal circle round a begins to be described is changed by the description of that circle. Hence *the part of the loop that is effective for the change in the value of w is the small circle round the point;* and it is because the description of a small circle changes the value of w that the value of w is changed at O after the description of a loop.

If $f(z)$ be the value of w which is changed into $f_1(z)$ by the description of the loop, so that $f(z)$ and $f_1(z)$ are the values at O, then the foregoing explanation shews that $f(\epsilon)$ and $f_1(\epsilon)$ are the values at ϵ, the branch $f(\epsilon)$ being changed by the description of the circle into the branch $f_1(\epsilon)$.

From this result the inference can be derived that the points B_1, B_2, ... are branch-points as defined in § 12. Let a be any one of the points, and let $f(z)$ be the value of w which is changed into $f_1(z)$ by the description of a very small circle round a. Then as the branch of w is monogenic, the difference between $f(z)$ and $f_1(z)$ is an infinitesimal quantity of the same order as the length of the circumference of the circle: so that, as the circle is infinitesimal and ultimately evanescent, $|f(z)-f_1(z)|$ can be made as small as we please with decrease of $|z-a|$ or, in the limit, the values of $f(a)$ and $f_1(a)$ at the branch-point are equal. Hence each of the points B is such that *two or more branches of the function have the same value at the point and there is interchange among these branches when the variable describes a small circuit round the point:* which affords a definition of a branch-point, more complete than that given in § 12.

Corollary V. *If a closed circuit contain several branch-points, the effect which it produces can be obtained by a combination of the effects produced in succession by a set of loops each going round only one of the branch-points.*

If the circuit contain several branch-points, say three as at a, b, c, then a path such as $AEFD$, in fig. 18, can without crossing any branch-point, be deformed into the loops AaB, BbC, CcD; and therefore the complete circuit $AEFDA$ can be deformed validly into $AaBbCcDA$, and the same effect will be produced by the two forms of circuit. When D is made practically to coincide with A, the whole of the second circuit is composed of the three loops. Hence the corollary.

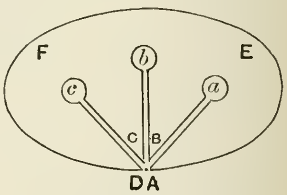

Fig. 18.

This corollary is of especial importance in the consideration of integrals of multiform functions.

Corollary VI. *In a continuous part of the plane where there are no branch-points, each branch of a multiform function is uniform.*

Each branch is monogenic and, except at isolated points, continuous; hence, in such regions of the plane, all the propositions which have been proved for monogenic analytic functions can be applied to each of the branches of a multiform function.

91. If there be a branch-point within the circuit, then the value of the function at b consequent on variations along acb may, but will not necessarily, differ from its value at the same point consequent on variations along adb. Should the values be different, then the description of the whole curve $acbda$ will lead at a not to the initial value of w, but to a different value. The test as to whether such a change is effected by the description is immediately derivable from the foregoing proposition; and as in Corollary IV., § 90, it is proved that the value is or is not changed by the loop, according as the value of w for a point near the circle of the loop is or is not changed by the description of that circle. Hence it follows that, *if there be a branch-point which affects the branch of the function, a path of variation of the independent variable cannot be deformed across the branch-point without a change in the value of w at the extremity of the path.*

And it is evident that *a point can be regarded as a branch-point for a function only if a circuit round the point interchange some (or all) of the branches of the function which are equal at the point.* It is not necessary that all the branches of the function should be thus affected by the point: it is sufficient that some should be interchanged*.

Further, *the change in the value of w for a single description of a circuit enclosing a branch-point is unique.*

For, if a circuit could change w into w' or w'', then, beginning with w'' and describing it in the negative sense we should return to w and afterwards describing it in the positive sense with w as the initial value we should obtain w'. Hence the circuit, described and then reversed, does not restore the original value w'' but gives a different branch w'; and no point on the circuit is a branch-point. This result is in opposition to Corollary I., of § 90; and therefore the hypothesis of alternative values at the end of the circuit is not valid, that is, the change for a single description is unique.

But repetitions of the circuit may, of course, give different values at the end of successive descriptions.

* In what precedes, certain points were considered which were regular singularities (see p. 163, note) and certain which were branch-points. Frequently points will occur which are at once branch-points and infinities; proper account must of course be taken of them.

92. Let O be any ordinary point of the function; join it to all the branch-points (generally assumed finite in number) in succession by lines which do not meet each other: then each branch is uniform for each path of variation of the variable which meets none of these lines. The effects produced by the various branch-points and their relations on the various branches can be indicated by describing curves, each of which begins at a point indefinitely near O and returns to another point indefinitely near it after passing round one of the branch-points, and by noting the value of each branch of the function after each of these curves has been described.

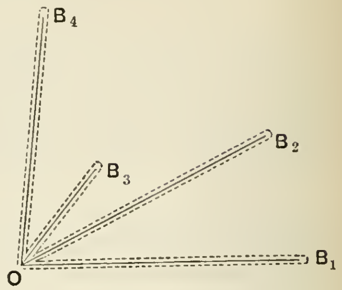

Fig. 19.

The law of interchange of branches of a function after description of a circuit round a branch-point is as follows:—

All the branches of a function, which are affected by a branch-point as such, can either be arranged so that the order of interchange (for description of a path round the point) is cyclical, or be divided into sets in each of which the order of interchange is cyclical.

Let $w_1, w_2, w_3, \ldots$ be the branches of a function for values of z near a branch-point a which are affected by the description of a small closed curve C round a: they are not necessarily all the branches of the function, but only those affected by the branch-point.

The branch w_1 is changed after a description of C; let w_2 be the branch into which it is changed. Then w_2 cannot be unchanged by C; for a reversed description of C, which ought to restore w_1, would otherwise leave w_2 unchanged. Hence w_2 is changed after a description of C; it may be changed either into w_1 or into a new branch, say w_3. If into w_1, then w_1 and w_2 form a cyclical set.

If the change be into w_3, then w_3 cannot remain unchanged after a description of C, for reasons similar to those that before applied to the change of w_2; and it cannot be changed into w_2, for then a reversed description of C would change w_2 into w_3, and it ought to change w_2 into w_1. Hence, after a description of C, w_3 is changed either into w_1 or into a new branch, say w_4. If into w_1, then w_1, w_2, w_3 form a cyclical set.

If the change be into w_4, then w_4 cannot remain unchanged after a description of C; and it cannot be changed into w_2 or w_3, for by a reversal of the circuit that earlier branch would be changed into w_4 whereas it ought to be changed into the branch, which gave rise to it by the forward description—a branch which is not w_4. Hence, after a description of C, w_4 is changed either into w_1 or into a new branch. If into w_1, then w_1, w_2, w_3, w_4 form a cyclical set.

If w_4 be changed into a new branch, we proceed as before with that new branch and either complete a cyclical set or add one more to the set. By repetition of the process, we complete a cyclical set sooner or later.

If all the branches be included, then evidently their complete system taken in the order in which they come in the foregoing investigation is a system in which the interchange is cyclical.

If only some of the branches be included, the remark applies to the set constituted by them. We then begin with one of the branches not included in that set and evidently not inclusible in it, and proceed as at first, until we complete another set which may include all the remaining branches or only some of them. In the latter case, we begin again with a new branch and repeat the process; and so on, until ultimately all the branches are included. The whole system is then arranged in sets, in each of which the order of interchange is cyclical.

93. *The analytical test of a branch-point* is easily obtained by constructing the general expression for the branches of a function which are interchanged there.

Let $z = a$ be a branch-point where n branches $w_1, w_2, \ldots, w_n$ are cyclically interchanged. Since by a first description of a small curve round a, the branch w_1 changes into w_2, the branch w_2 into w_3, and so on, it follows that by r descriptions w_1 is changed into w_{r+1} and by n descriptions w_1 reverts to its initial value. Similarly for each of the branches. Hence *each branch returns to its initial value after n descriptions of a circuit round a branch-point where n branches of the function are interchangeable.*

Now let
$$z - a = Z^n;$$
then, when z describes circles round a, Z moves in a circular arc round its origin. For each circumference described by z, the variable Z describes $\frac{1}{n}$th part of its circumference; and the complete circle is described by Z round its origin when n complete circles are described by z round a. Now the substitution changes w_r as a function of z into a function of Z, say into W_r; and, after n complete descriptions of the z-circle round a, w_r returns to its initial value. Hence, after the description of a Z-circle round its origin, W_r returns to its initial value, that is, $Z = 0$ ceases to be a branch-point for W_r. Similarly for all the branches W.

But no other condition has been associated with a as a point for the function w; and therefore $Z = 0$ may be any point for the function W, that is, it may be an ordinary point, or a singularity. In every case we have W a uniform function of Z in the immediate vicinity of the origin; and therefore in that vicinity it can be expressed in the form
$$G\left(\frac{1}{Z}\right) + P(Z),$$

with the significations of P and G already adopted. When Z is an ordinary point, G is a constant or zero; when Z is an accidental singularity, G is an algebraical function; and, when Z is an essential singularity, G is a transcendental function.

The simpler cases are, of course, those in which the form of G is algebraical or constant or zero; and then W can be put into the form

$$Z^m \bar{P}(Z),$$

where $\bar{P}$ is an infinite series of positive powers and m is an integer. As this is the form of W in the vicinity of $Z=0$, it follows that the form of w in the vicinity of $z=a$ is

$$(z-a)^{\frac{m}{n}} \bar{P}\{(z-a)^{\frac{1}{n}}\}$$

and the various n branches of the function are easily seen to be given by substituting in the above for $(z-a)^{\frac{1}{n}}$ the values

$$e^{\frac{2\pi si}{m}} (z-a)^{\frac{1}{n}},$$

where $s=0, 1, \ldots, n-1$. We therefore infer that *the general expression for the n branches of a function, which are interchanged by circuits round a branch-point $z=a$, assumed not to be an essential singularity, is*

$$(z-a)^{\frac{m}{n}} \bar{P}\{(z-a)^{\frac{1}{n}}\},$$

where m is an integer, and where to $(z-a)^{\frac{1}{n}}$ its n values are in turn assigned to obtain the different branches of the function.

There may be, however, more than one cyclical set of branches. If there be another set of r branches, then it may similarly be proved that their general expression is

$$(z-a)^{\frac{m_1}{r}} \bar{Q}\{(z-a)^{\frac{1}{r}}\},$$

where m_1 is an integer, and $\bar{Q}$ is an integral function; the various branches are obtained by assigning to $(z-a)^{\frac{1}{r}}$ its r values in turn.

And so on, for each of the sets, the members of which are cyclically interchangeable at the branch-point.

When the branch-point is at infinity, a different form is obtained. Thus in the case of a set of n cyclically interchangeable branches we take

$$z=u^{-n},$$

so that n negative descriptions of a closed z-curve, excluding infinity and no other branch-point, requires a single positive description of a closed curve round the u-origin. These n descriptions restore the value of w as a function of z to its initial value; and therefore the single description of the u-curve round the origin restores the value of U—the equivalent of w after the change of the independent variable—as a function of u. Thus $u = 0$ ceases to be a branch-point for the function U; and therefore the form of U is

$$G\left(\frac{1}{u}\right) + P(u),$$

where the symbols have the same general signification as before.

If, in particular, $z = \infty$ be a branch-point but not an essential singularity, then G is either a constant or an algebraical function; and then U can be expressed in the form

$$u^{-m}\bar{P}(u),$$

where m is an integer. When the variable is changed from u to z, then *the general expression for the n branches of a function which are interchangeable at* $z = \infty$, *assumed not to be an essential singularity, is*

$$z^{\frac{m}{n}} P(z^{-\frac{1}{n}}),$$

where m is an integer and where to $z^{\frac{1}{n}}$ *its n values are assigned to obtain the different branches of the function.*

If, however, the branch-point $z = a$ in the former case or $z = \infty$ in the latter be an essential singularity, the forms of the expressions in the vicinity of the point are

$$G\{(z-a)^{-\frac{1}{n}}\} + P\{(z-a)^{\frac{1}{n}}\},$$

and

$$G(z^{\frac{1}{n}}) + P(z^{-\frac{1}{n}}),$$

respectively.

Note. When a multiform function is defined, either explicitly or implicitly, it is practically always necessary to consider the relations of the branches of the function for $z = \infty$ as well as their relations for points that are infinities of the function. The former can be determined by either of the processes suggested in § 4 for dealing with $z = \infty$; the latter can be determined as in the present article.

Moreover, the total number of branches of the function has been assumed to be finite. The cases, in which the number of branches is unlimited, need not be discussed in general: it will be sufficient to consider them when they arise, as they do arise, e.g., when the function is of the form of an algebraical irrational with an irrational index such as $z^{\sqrt{2}}$—hardly a function in the ordinary sense—, or when the function is the logarithm of a function of z, or is the inverse of a periodic function. In the nature of their multiplicity of branching and of their sequence of interchange, they are for the most part distinct from the multiform functions with only a finite number of branches.

Ex. The simplest illustrations of multiform functions are furnished by functions defined by algebraical equations, in particular, by algebraic irrationals.

The general type of the algebraical irrational is the product of a number of functions of the form $w=\{A(z-a_1)(z-a_2)......(z-a_n)\}^{\frac{1}{m}}$, m and n being integers.

This particular function has m branches; the points $a_1, a_2, ..., a_n$ are branch-points. To find the law of interchange, we take $z-a_r=\rho e^{\theta i}$; then when a small circle of radius ρ is described round a_r, so that z returns to its initial position, the value of θ increases by 2π and the new value of w is aw, where a is the mth root of unity defined by $e^{\frac{1}{m}2\pi i}$. Taking then the various branches as given by $w, aw, a^2w, ..., a^{m-1}w$, we have the law of interchange for description of a small curve round any one-branch point as given by this succession in cyclical order. The law of succession for a circuit enclosing more than one of the branch-points is derivable by means of Corollary V, § 90.

To find the relation of $z=\infty$ to w, we take $zz'=1$ and consider the new function W in the vicinity of the z'-origin. We have

$$W=\{A(1-a_1z')(1-a_2z')......(1-a_nz')\}^{\frac{1}{m}}z'^{-\frac{n}{m}}.$$

If the variable z' describe a very small circle round the origin in the negative sense, then z' is multiplied by $e^{-2\pi i}$ and so W acquires a factor $e^{2\pi i\frac{n}{m}}$, that is, W is changed unless this acquired factor is unity. It can be unity only when n/m is an integer; and therefore, except when n/m is an integer, $z=\infty$ is a branch-point of the function. The law of succession is the same as that for negative description of the z'-circle, viz., $w, a^nw, a^{2n}w, ...$; the m values form a single cycle only if n be prime to m, and a set of cycles if n be not prime to m.

Thus $z=\infty$ is a branch-point for $w=(4z^3-g_2z-g_3)^{-\frac{1}{2}}$; it is not a branch-point for $w=\{(1-z^2)(1-k^2z^2)\}^{-\frac{1}{2}}$; and $z=b$ is a branch-point for the function defined by

$$(z-b)w^2=z-a,$$

but $z=b$ is not a branch-point for the function defined by $(z-b)^2w^2=z-a$.

Again, if p denote a particular value of $z^{\frac{1}{2}}$, when z has a given value, and q similarly denote a particular value of $\left(\frac{z-1}{z+1}\right)^{\frac{1}{3}}$, then $w=p+q$ is a six-valued function, the values being

$$w_1=p+q, \quad w_3=p+aq, \quad w_5=p+a^2q,$$
$$w_2=-p+q, \quad w_4=-p+aq, \quad w_6=-p+a^2q,$$

where a is a primitive cube root of unity. The branch-points are $-1, 0, 1, \infty$; and the orders of change for small circuits round one (and only one) of these points are as follows:

For a small circuit round	-1	0	1	∞
w_1 changes to	w_5	w_2	w_3	w_2
w_2 ,,	w_6	w_1	w_4	w_1
w_3 ,,	w_1	w_4	w_5	w_4
w_4 ,,	w_2	w_3	w_6	w_3
w_5 ,,	w_3	w_6	w_1	w_6
w_6 ,,	w_4	w_5	w_2	w_5

Combinations can at once be effected; thus, for a positive circuit enclosing both 1 and ∞ but* not -1 or 0, the succession is

$$w_1, w_4, w_5, w_2, w_3, w_6$$

in cyclical order.

94. It has already been remarked that algebraic irrationals are a special class of functions defined by algebraical equations. Functions thus generally defined by equations, which are algebraical so far as concerns the dependent variable but need not be so in reference to the independent variable, are often called *algebraical.* The term, in one sense, cannot be strictly applied to the roots of an equation of every degree, seeing that the solution of equations of the fifth and higher degrees can be effected only by transcendental functions; but what is implied is that a finite number of determinations of the dependent variable is given by the equation†.

The equation is algebraical in relation to the dependent variable w, that is, it will be taken to be of finite degree n in w. The coefficients of the different powers will be supposed to be rational uniform functions of z: were they irrational in any given equation, the equation could be transformed into another, the coefficients of which are rational uniform functions. And the equation is supposed to be irreducible, that is, if the equation be taken in the form

$$f(w, z) = 0,$$

the left-hand member $f(w, z)$ cannot be resolved into factors of a form and character as regards w and z similar to f itself.

The existence of equal roots of the equation for general values of z requires that

$$f(w, z) \text{ and } \frac{\partial f(w, z)}{\partial w}$$

shall have a common factor, which will be rational owing to the form of $f(w, z)$. This form of factor is excluded by the irreducibility of the equation; so that $f = 0$, as an equation in w, has not equal roots for *general* values of z. But though the two equations are not both satisfied in virtue of a simpler equation, they are two equations determining values of w and z; and their form is such that they will give equal values of w for *special* values of z.

Since the equation is of degree n, it may be taken to be

$$w^n + w^{n-1} F_1(z) + w^{n-2} F_2(z) + \ldots + wF_{n-1}(z) + F_n(z) = 0,$$

where the functions $F_1, F_2, \ldots$ are rational and uniform. If all their singu-

* Such a circuit, if drawn on the Neumann's sphere, may be regarded as excluding -1 and 0, or taking account of the other portion of the surface of the sphere, it may be regarded as a negative circuit including -1 and 0, the cyclical interchange for which is easily proved to be $w_1, w_4, w_5, w_2, w_3, w_6$ as in the text.

† Such a function is called *bien défini* by Liouville.

larities be accidental, they are meromorphic algebraical functions of z (unless $z=\infty$ is the only singularity, in which case they are holomorphic); and the equation can then be replaced by one which is equivalent and has all its coefficients holomorphic, the coefficient of w^n being the least common multiple of all the denominators of the meromorphic functions in the first form. This form cannot however be deduced, if any of the singularities be essential.

The equation, as an equation in w, has n roots, all functions of z; let these be denoted by $w_1, w_2, \ldots, w_n$, which are the n branches of the function w. When the geometrical interpretation is associated with the analytical relation, there are n points in the w-plane, say $\alpha_1, \ldots, \alpha_n$, which correspond with a point in the z-plane, say with a_1; and in general these n points are distinct. As z varies so as to move in its own plane from a, then each of the w-points moves in their common plane; and thus there are n w-paths corresponding to a given z-path. These n curves may or may not meet one another.

If they do not, there are n distinct w-paths, leading from $\alpha_1, \ldots, \alpha_n$ to $\beta_1, \ldots, \beta_n$, respectively corresponding to the single z-path leading from a to b.

If two or more of the w-paths do meet one another, and if the describing w-points coincide at their point of intersection, then at such a point of intersection in the w-plane, the associated branches w are equal; and therefore the point in the z-plane is a point that gives equal values for w. It is one of the roots of the equation obtained by the elimination of w between

$$f(w, z) = 0, \quad \frac{\partial f(w, z)}{\partial w} = 0;$$

the analytical test as to whether the point is a branch-point will be considered later. The march of the concurrent w-branches from such a point of intersection of two w-paths depends upon their relations in its immediate vicinity.

When no such point lies on a z-path from a to b, no two of the w-points coincide during the description of their paths. By § 90, the z-path can be deformed (provided that, in the deformation, it does not cross a branch-point) without causing any two of the w-points to coincide. Further, if z describe a closed curve which includes none of the branch-points, then each of the w-branches describes a closed curve and no two of the tracing points ever coincide.

Note. The limitation for a branch-point, that the tracing w-points coincide at the point of intersection of the w-curves, is of essential importance.

What is required to establish a point in the z-plane as a branch-point, is not a mere geometrical intersection of a couple of completed w-paths but the coincidence of the w-points as those paths are traced, together with interchange of the branches for a small circuit round the point. Thus let there be such a geometrical intersection of two w-curves, without coincidence of the tracing points. There are two points in the z-plane corresponding to the geometrical intersection; one belongs to the intersection as a point of the w-path which first passed through it, and the other to the intersection as a point of the w-path which was the second to pass through it. The two branches of w for the respective values of z are undoubtedly equal; but the equality would not be for the same value of z. And unless the equality of branches subsists for the same value of z, the point is not a branch-point.

A simple example will serve to illustrate these remarks. Let w be defined by the equation

$$f = c^2(w^2 - 2zw) - z^4 = 0,$$

so that the branches w_1 and w_2 are given by

$$cw_1 = cz + z(z^2+c^2)^{\frac{1}{2}}, \qquad cw_2 = cz - z(z^2+c^2)^{\frac{1}{2}};$$

it is easy to prove that the equation resulting from the elimination of w between $f=0$ and $\frac{\partial f}{\partial w}=0$ is

$$z^2(z^2+c^2)=0,$$

and that only the two points $z = \pm ic$ are branch-points.

The values of z which make w_1 equal to the value of w_2 for $z=a$ (supposed not equal to either 0, ci or $-ci$) are given by

$$cz + z(z^2+c^2)^{\frac{1}{2}} = ca - a(a^2+c^2)^{\frac{1}{2}},$$

which evidently has not $z=a$ for a root. Rationalising the equation so far as concerns z and removing the factor $z-a$, as it has just been seen not to furnish a root, we find that z is determined by

$$z^3 + z^2a + za^2 + a^3 + 2ac^2 - 2ac(a^2+c^2)^{\frac{1}{2}} = 0,$$

the three roots of which are distinct from a, the assumed point, and from $\pm ci$, the branch-point. Each of these three values of z will make w_1 equal to the value of w_2 for $z=a$: we have geometrical intersection without coincidence of the tracing points.

95. When the characteristics of a function are required, the most important class are its infinities: these must therefore now be investigated. It is preferable to obtain the infinities of the function rather than the singularities alone, in the vicinity of which each branch of the function is uniform*: for the former will include these singularities as well as those branch-points which, giving infinite values, lead to regular singularities when the variables are transformed as in § 93. The theorem which determines them is:—

The infinities of a function determined by an algebraical equation are the singularities of the coefficients of the equation.

Let the equation be

$$w^n + w^{n-1}F_1(z) + w^{n-2}F_2(z) + \ldots + wF_{n-1}(z) + F_n(z) = 0,$$

* These singularities will, for the sake of brevity, be called *regular*.

and let w' be any branch of the function; then, if the equation which determines the remaining branches be

$$w^{n-1} + w^{n-2} G_1(z) + w^{n-3} G_2(z) + \dots + wG_{n-2}(z) + G_{n-1}(z) = 0,$$

we have

$$F_n(z) = - w'G_{n-1}(z),$$

$$F_{n-1}(z) = - w'G_{n-2}(z) + G_{n-1}(z),$$

$$F_{n-2}(z) = - w'G_{n-3}(z) + G_{n-2}(z),$$

$$\vdots$$

$$F_1(z) = - w' + G_1(z).$$

Now suppose that a is an infinity of w'; then, unless it be a zero of order at least equal to that of $G_{n-1}(z)$, a is an infinity of $F_n(z)$. If, however, it be a zero of $G_{n-1}(z)$ of sufficient order, then from the second equation it is an infinity of $F_{n-1}(z)$ unless it is a zero of order at least equal to that of $G_{n-2}(z)$; and so on. The infinity must be an infinity of some coefficient not earlier than $F_i(z)$ in the equation, or it must be a zero of all the functions G which are later than $G_{i-1}(z)$. If it be a zero of all the functions G_r, so that we may not, without knowing the order, assert that it is of rank at least equal to its order as an infinity of w', still from the last equation it follows that a must be an infinity of $F_1(z)$. Hence *any infinity of w is an infinity of at least one of the coefficients of the equation.*

Conversely, from the same equations it follows that a singularity of one of the coefficients is an infinity either of w' or of at least one of the coefficients G. Similarly the last alternative leads to an inference that the infinity is either an infinity of another branch w'' or of the coefficients of the (theoretical) equation which survives when the two branches have been removed. Proceeding in this way, we ultimately find that the infinity either is an infinity of one of the branches or is an infinity of the coefficient in the last equation, that is, of the last of the branches. Hence *any singularity of a coefficient is an infinity of at least one of the branches of the function.*

It thus appears that all the infinities of the function are included among, and include, all the singularities of the coefficients; but the order of the infinity for a branch does not necessarily make that point a regular singularity nor, if it be made a regular singularity, is the order necessarily the same as for the coefficient.

96. The following method is effective for the determination of the order of the infinity of the branch.

Let a be an accidental singularity of one or more of the F functions, say of order m_i for the function F_i; and assume that, in the vicinity of a, we have

$$F_i(z) = (z-a)^{-m_i}[c_i + d_i(z-a) + e_i(z-a)^2 + \dots].$$

Then the equation which determines the first term of the expansion of w in a series in the vicinity of a is

$$w^n + c_1(z-a)^{-m_1}w^{n-1} + c_2(z-a)^{-m_2}w^{n-2} + \dots$$
$$+ c_{n-1}(z-a)^{-m_{n-1}}w + c_n(z-a)^{-m_n} = 0.$$

Mark in a plane, referred to two rectangular axes, points n, 0; $n-1$, $-m_1$; $n-2$, $-m_2$; ..., 0, $-m_n$; let these be A_0, A_1, ..., A_n respectively. Any line through A_i has its equation of the form

$$y + m_i = \lambda\{x - (n-i)\},$$

that is,

$$y - \lambda x = -\lambda(n-i) - m_i.$$

If then $w = (z-a)^{-\lambda}f(z)$, where $f(z)$ is finite when $z = a$, the intercept of the foregoing line on the negative side of the axis of y is equal to the order of the infinity in the term

$$w^{n-i}F_i(z).$$

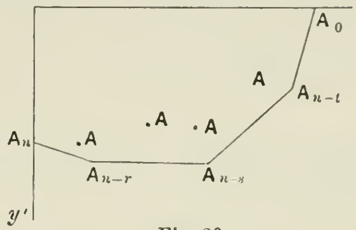

Fig. 20.

This being so, we take a line through A_n coinciding in direction with the negative part of the axis of y and we turn it about A_n in a trigonometrically positive direction until it first meets one of the other points, say A_{n-r}; then we turn it about A_{n-r} until it meets one of the other points, say A_{n-s}; and so on until it passes through A_0. There will thus be a line from A_n to A_0, generally consisting of a number of parts; and none of the points A will be outside it.

The perpendicular from the origin on the line through A_{n-r} and A_{n-s} is evidently greater than the perpendicular on any parallel line through a point A, that is, on any line through a point A with the same value of λ; and, as this perpendicular is

$$\{\lambda(n-i) + m_i\}(1+\lambda^2)^{-\frac{1}{2}},$$

it follows the order of the infinite terms in the equation, when the particular substitution is made for w, is greater for terms corresponding to points lying on the line than it is for any other terms.

If $f(z) = \theta$ when $z = a$, then the terms of lowest order after the substitution of $(z-a)^{-\lambda}f(z)$ for w are

$$(z-a)^{-m_{n-r}-\lambda r}[c_{n-r}\theta^r + \dots + c_{n-s}\theta^s]$$

as many terms occurring in the bracket as there are points A on the line joining A_{n-r} to A_{n-s}. Since the equation determining w must be satisfied, terms of all orders must disappear, and therefore

$$c_{n-s}\theta^{s-r} + \dots + c_{n-r} = 0,$$

an equation determining $s-r$ values of θ, that is, the first terms in the expansions of $s-r$ branches w.

Similarly for each part of the line: for the first part, there are r branches with an associated value of λ; for the second, $s-r$ branches with another associated value; for the third, $t-s$ branches with a third associated value; and so on.

The order of the infinity for the branches is measured by the tangent of the angle which the corresponding part of the broken line makes with the axis of x; thus for the line joining A_{n-r} to A_{n-s} the order of the infinity for the $s-r$ branches is

$$\frac{m_{n-r}-m_{n-s}}{s-r},$$

where m_{n-r} and m_{n-s} are the orders of the accidental singularities of $F_{n-r}(z)$ and $F_{n-s}(z)$.

If any part of the broken line should have its inclination to the axis of x greater than $\frac{1}{2}\pi$ so that the tangent is negative and equal to $-\mu$, then the form of the corresponding set of branches w is $(z-a)^{\mu} g(z)$ for all of them, that is, the point is not an infinity for those branches. But when the inclination of a part of the line to the axis is $< \frac{1}{2}\pi$, so that the tangent is positive and equal to λ, then the form of the corresponding set of branches w is $(z-a)^{-\lambda} f(z)$ for all of them, that is, the point is an infinity of order λ for those branches.

In passing from A_n to A_0 there may be parts of the broken line which have the tangential coordinate negative, implying therefore that a is not an infinity of the corresponding set or sets of branches w. But as the revolving line has to change its direction from $A_n y'$ to some direction through A_0, there must evidently be some part or parts of the broken line which have their tangential coordinate positive, implying therefore that a is an infinity of the corresponding set or sets of branches.

Moreover, the point a is, by hypothesis, an accidental singularity of at least one of the coefficients and it has been supposed to be an essential singularity of none of them; hence the points $A_0, A_1, \ldots, A_n$ are all in the finite part of the plane. And as no two of their abscissæ are equal, no line joining two of them can be parallel to the axis of y, that is, the inclination of the broken line is never $\frac{1}{2}\pi$ and therefore the tangential coordinate is finite, that is, the order of the infinity for the branches is finite for any accidental singularity of the coefficients.

If the singularity at a be essential for some of the coefficients, the corresponding result can be inferred by passing to the limit which is obtained by making the corresponding value or values of m infinite. In that case the corresponding points A move to infinity and then parts of the broken line pass through A_0 (which is always on the axis of x) parallel to the axis of y, that is, the tangential coordinate is infinite and the order of the infinity at a for the corresponding branches is also infinite. The point is then an essential singularity (and it may be also a branch-point).

It has been assumed implicitly that the singularity is at a finite point in the z-plane; if, however, it be at ∞, we can, by using the transformation $zz' = 1$ and discussing as above the function in vicinity of the origin, obtain the relation of the singularity to the various branches. We thus have the further proposition:

The order of the infinity of a branch of an algebraical function at a singularity of a coefficient of the equation, which determines the function, is finite or infinite according as the singularity is accidental or essential.

If the coefficients F_i of the equation be holomorphic functions, then $z = \infty$ is their only singularity and it is consequently the only infinity for branches of the function. If some of or all the coefficients F_i be meromorphic functions, the singularities of the coefficients are the zeros of the denominators and, possibly, $z = \infty$; and, if the functions be algebraical, all such singularities are accidental. In that case, the equation can be modified to

$$h_0(z)\, w^n + h_1(z)\, w^{n-1} + h_2(z)\, w^{n-2} + \ldots = 0,$$

where $h_0(z)$ is the least common multiple of all the denominators of the functions F_i. The preceding results therefore lead to the more limited theorem:

When a function w is determined by an algebraical equation the coefficients of which are holomorphic functions of z, then each of the zeros of the coefficient of the highest power of w is an infinity of some of (and it may be of all) the branches of the function w, each such infinity being of finite order. The point $z = \infty$ may also be an infinity of the function w; the order of that infinity is finite or infinite according as $z = \infty$ is an accidental or an essential singularity of any of the coefficients.

It will be noticed that no precise determination of the forms of the branches w at an infinity has been made. The determination has, however, only been deferred: the infinities of the branches for a singularity of the coefficients are usually associated with a branch-point of the function and therefore the relations of the branches at such a point will be of a general character independent of the fact that the point is an infinity.

If, however, in any case a singularity of a coefficient should prove to be, not a branch-point of w but only a regular singularity, then in the vicinity of that point the branch of w is a uniform function. A necessary (but not sufficient) condition for uniformity is that $(m_{n-r} - m_{n-s}) \div (s - r)$ be an integer.

Note. The preceding method can be applied to determine the leading terms of the branches in the vicinity of a point a which is an ordinary point for each of the coefficients F.

97. There remains therefore the consideration of the branch-points of a function determined by an algebraical equation.

The characteristic property of a branch-point is the equality of branches of the function for the associated value of the variable, coupled with the interchange of some of (or all) the equal branches after description by the variable of a small contour enclosing the point.

So far as concerns the first part, the general indication of the form of the values has already (§ 93) been given. The points, for which values of w determined as a function of z by the equation

$$f(w, z) = 0$$

are equal, are determined by the solution of this equation treated simultaneously with

$$\frac{\partial f(w, z)}{\partial w} = 0;$$

and when a point z is thus determined the corresponding values of w, which are equal there, are obtained by substituting that value of z and taking M, the greatest common measure of f and $\frac{\partial f}{\partial w}$. The factors of M then lead to the value or the values of w at the point; the index m of a linear factor gives at the point the multiplicity of the value which it determines, and shews that $m+1$ values of w have a common value there, though they are distinct at infinitesimal distances from the point. If $m = 1$ for any factor, the corresponding value of w is an isolated value and determines a branch that is uniform at the point.

Let $z = a$, $w = \alpha$ be a value of z and a value of w thus obtained; and suppose that m is the number of values of w that are equal to one another. The point $z = a$ is not a branch-point unless some interchange among the m values of w is effected by a small circuit round a; and it is therefore necessary to investigate the values of the branches* in the vicinity of $z = a$.

Let $w = \alpha + w'$, $z = a + z'$; then we have

$$f(\alpha + w', \ a + z') = 0,$$

that is, on the supposition that $f(w, z)$ has been freed from fractions,

$$f(\alpha, a) + \sum_{r, s} \sum A_{rs} z'^r w'^s = 0,$$

so that, since α is a value of w corresponding to the value a of z, we have w' and z' connected by the relation

$$\sum_r \sum_s A_{rs} z'^r w'^s = 0.$$

* The following investigations are founded on the researches of Puiseux on algebraic functions; they are contained in two memoirs, *Liouville*, 1re Sér., t. xv, (1850), pp. 365—480, ib., t. xvi, (1851), pp. 228—240. See also the chapters on algebraic functions, pp. 19—76, in the second edition of Briot and Bouquet's *Théorie des fonctions elliptiques*.

When z' is 0, the zero value of w' must occur m times, since α is a root m times repeated; hence there are terms in the foregoing equation independent of z', and the term of lowest index among them is w'^m. Also when $w'=0$, $z'=0$ is a possible root; hence there must be a term or terms independent of w' in the equation.

First, suppose that the lowest power of z' among the terms independent of w' is the first. The equation has the form

$$\begin{aligned} &Az' + \text{higher powers of } z' \\ &+ Bw'^m + \text{higher powers of } w' \\ &+ \text{terms involving } z' \text{ and } w' = 0, \end{aligned}$$

where A is the value of $\dfrac{\partial f(w, z)}{\partial z}$ for $w=\alpha$, $z=a$. Let $z'=\zeta^m$, $w'=v\zeta$; the last form changes to

$$(A + Bv^m)\,\zeta^m + \text{terms with } \zeta^{m+1} \text{ as a factor} = 0;$$

and therefore $$A + Bv^m + \text{terms involving } \zeta = 0.$$

Hence in the immediate vicinity of $z=a$, that is, of $\zeta=0$, we have

$$A + Bv^m = 0.$$

Neither A nor B is zero, so that all the m values of v are finite. Let them be $v_1, \ldots, v_m$, so arranged that their arguments increase by $2\pi/m$ through the succession. The corresponding values of w' are

$$\begin{aligned} w_i' &= v_i\zeta \\ &= v_i z'^{\frac{1}{m}}, \end{aligned}$$

for $i=1, \ldots, m$. Now a z-circuit round a, that is, a z'-circuit round its origin, increases the argument of z' by 2π; hence after such a circuit we have the new value of w_i' as $v_i z'^{\frac{1}{m}} e^{\frac{2\pi i}{m}}$, that is, it is $v_{i+1} z'^{\frac{1}{m}}$ which is the value of w'_{i+1}. Hence the set of values $w'_1, w'_2, \ldots, w'_m$ form a complete set of interchangeable values in their cyclical succession; all the m values, which are equal at a, form a single cycle and the point is a branch-point.

Next, suppose that the lowest power of z' among the terms independent of w' is z'^l, where $l>1$. The equation now has the form

$$\begin{aligned} 0 = {} &Az'^l + \text{higher powers of } z' \\ &+ Bw'^m + \text{higher powers of } w' \\ &+ \sum_{r=1}^{l-1}\sum_{s=1}^{m-1} A_{rs} z'^r w'^s + \Sigma\Sigma C_{rs} z'^r w'^s, \end{aligned}$$

where in the last summation r and s are not zero and in every term either (i), r is equal to or greater than l or (ii), s is equal to or greater than m or (iii), both (i) and (ii) are satisfied. As only terms of the lowest orders need be retained for the present purpose, which is the derivation of the first term of w' in its expansion in powers of z', we may use the foregoing equation in the form

$$Az'^l + \sum_{r=1}^{l-1} \sum_{s=1}^{m-1} A_{rs} z'^r w'^s + Bw'^m = 0.$$

To obtain this first term we proceed in a manner similar to that in § 96*. Points $A_0, \ldots, A_m$ are taken in a plane referred to rectangular axes having as coordinates $0, l; \ldots; s, r; \ldots; m, 0$ respectively. A line is taken through A_m and is made to turn round A_m from the position $A_m O$ until it first meets one of the other points; then round the last point which lies in this direction, say round A_j, until it first meets another; and so on.

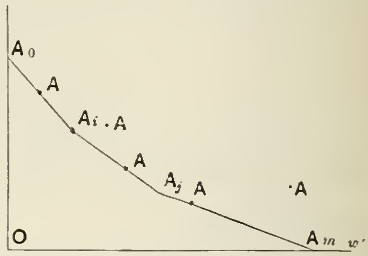

Fig. 21.

Any line through A_i (the point s_i, r_i) is of the form

$$y - r_i = -\lambda (x - s_i).$$

The intercept on the axis of z'-indices is $\lambda s_i + r_i$, that is, the order of the term involving $A_{r_i s_i}$ for a substitution $w' \propto z'^\lambda$. The perpendicular from the origin for a line through A_i and A_j is less than for any parallel line through other points with the same inclination; and, as this perpendicular is

$$(\lambda s_i + r_i)(1 + \lambda^2)^{-\frac{1}{2}},$$

it follows that, for the particular substitution $w' \propto z'^\lambda$, the terms corresponding to the points lying on the line with coordinate λ are the terms of lowest order and consequently they are the terms which give the initial terms for the associated set of quantities w'.

Evidently, from the indices retained in the equation, the quantities λ for the various pieces of the broken line from A_m to A_0 are positive and finite.

Consider the first piece, from A_m to A_j say; then taking the value of λ for that piece as μ_1, so that we write $v_1 z'^{\mu_1}$ as the first term of w', we have as the set of terms involving the lowest indices

$$Bw'^m + \Sigma\Sigma A_{rs} z'^r w'^s + A_{r_j s_j} z'^{r_j} w'^{s_j},$$

s_j being the smallest value of s retained; and then

$$m\mu_1 = s\mu_1 + r = s_j \mu_1 + r_j,$$

so that
$$\mu_1 = \frac{r}{m-s} = \frac{r_j}{m - s_j}.$$

* Reference in this connection may be made to Chrystal's *Algebra*, ch. xxx., with great advantage, as well as the authorities quoted on p. 168, note.

Let p/q be the equivalent value of μ_1 as the fraction in its lowest terms; and write $z' = \zeta^q$. Then $w' = v_1 z'^{\frac{p}{q}} = v_1 \zeta^p$; all the terms except the above group are of order $> mp$ and therefore the equation leads after division by $\zeta^{mp} v^{s_j}$ to

$$Bv_1^{m-s_j} + \Sigma A_{rs} v_1^{s-s_j} + A_{r_j s_j} = 0,$$

an equation which determines $m - s_j$ values for v_1, and therefore the initial terms of $m - s_j$ of the w-branches.

Consider now the second piece, from A_j to A_i say; then taking the value of λ for that piece as μ_2, so that we write $v_2 z'^{\mu_2}$ as the first term of w', we have as the set of terms involving the lowest indices for this value of μ_2

$$A_{r_j s_j} z'^{r_j} w'^{s_j} + \Sigma\Sigma A_{rs} z'^{r} w'^{s} + A_{r_i s_i} z'^{r_i} w'^{s_i},$$

where s_i is the smallest value of s retained. Then

$$s_j \mu_2 + r_j = s\mu_2 + r = s_i \mu_2 + r_i.$$

Proceeding exactly as before, we find

$$A_{r_j s_j} v_2^{s_j - s_i} + \Sigma\Sigma A_{rs} v_2^{s-s_i} + A_{r_i s_i} = 0$$

as the equation determining $s_j - s_i$ values for v_2 and therefore the initial terms of $s_j - s_i$ of the w-branches.

And so on, until all the pieces of the line are used; the initial terms of all the w-branches are thus far determined in groups connected with the various pieces of the line $A_m A_j A_i \ldots A_0$. By means of these initial terms, the m-branches can be arranged for their interchanges, by the description of a small circuit round the branch-point, according to the following theorem:—

Each group can be resolved into systems, the members of each of which are cyclically interchangeable.

It will be sufficient to prove this theorem for a single group, say the group determined by the first piece of broken line: the argument is general.

Since $\frac{p}{q}$ is the equivalent of $\frac{r}{m-s}$ and of $\frac{r_j}{m-s_j}$, and since $s_j < s$, we have

$$m - s = kq, \qquad m - s_j = k_j q, \qquad k_j > k;$$

and then the equation which determines v_1 is

$$Bv_1^{k_j q} + \Sigma A_{rs} v_1^{(k_j - k) q} + A r_j s_j = 0,$$

that is, an equation of degree k_j in v_1^q as its variable. Let U be any root of it; then the corresponding values of v_1 are the values of $U^{\frac{1}{q}}$. Suppose these q values to be arranged so that the arguments increase by $2\pi \frac{p}{q}$, which is possible, because p is prime to q. Then the q values of w' being the values of $v_1 z'^{\mu_1}$ are

$$v_{11} z'^{\frac{p}{q}}, \qquad v_{12} z'^{\frac{p}{q}}, \qquad v_{13} z'^{\frac{p}{q}}, \ldots,$$

where $v_{1\alpha}$ is that value of $U^{\frac{1}{q}}$ which has $\dfrac{2\pi p\alpha}{q}$ for its argument. A circuit round the z'-origin evidently increases the argument of any one of these w'-values by $2\pi p/q$, that is, it changes it into the value next in the succession; and so the set of q values is a system the members of which are cyclically interchangeable.

This holds for each value of U derived from the above equation; so that the whole set of $m - s_j$ branches are resolved into k_j systems, each containing q members with the assigned properties.

It is assumed that the above equation of order k_j in $v_1{}^q$ has its roots unequal. If, however, it should have equal roots, it must be discussed *ab initio* by a method similar to that for the general equation; as the order k_j (being a factor of $m - s_j$) is less than m, the discussion will be shorter and simpler, and will ultimately depend on equations with unequal roots as in the case above supposed.

It may happen that some of the quantities μ are integers, so that the corresponding integers q are unity: a number of the branches would then be uniform at the point.

It thus appears that $z = a$ is a branch-point and that, under the present circumstances, the m branches of the function can be arranged in systems, the members of each one of which are cyclically interchangeable.

Lastly, it has been tacitly assumed in what precedes that the common value of w for the branch-point is finite. If it be infinite, this infinite value can, by § 95, arise only out of singularities of the coefficients of the equation: and there is therefore a reversion to the discussion of §§ 95, 96. The distribution of the various branches into cyclical systems can be carried out exactly as above.

Another method of proceeding for these infinities would be to take $ww' = 1$, $z = c + z'$; but this method has no substantial advantage over the earlier one and, indeed, it is easy to see that there is no substantial difference between them.

Ex. 1. As an example, consider the function determined by the equation

$$8zw^3 + (1 - z)(3w + 1) = 0.$$

The equation determining the values of z which give equal roots for w is

$$8z(z-1)^2 = 4(z-1)^3$$

so that the values are $z = 1$ (repeated) and $z = -1$.

When $z = 1$, then $w = 0$, occurring thrice; and, if $z = 1 + z'$ then

$$8w'^3 = z',$$

that is,

$$w' = \tfrac{1}{2} z'^{\frac{1}{3}}.$$

The three values are branches of one system in cyclical order for a circuit round $z = 1$.

When $z=-1$, the equation for w is

$$4w^3-3w-1=0,$$

that is,

$$(w-1)(2w+1)^2=0,$$

so that $w=1$ or $w=-\frac{1}{2}$, occurring twice.

For the former of these we easily find that, for $z=-1+z'$, the value of w is $1+\frac{2}{9}z'+\ldots\ldots$, an isolated branch as is to be expected, for the value 1 is not repeated.

For the latter we take $w=-\frac{1}{2}+w'$ and find

$$w'^2=\tfrac{1}{24}z'+\ldots\ldots,$$

so that the two branches are

$$w=-\tfrac{1}{2}+\frac{1}{2\sqrt{6}}z'^{\frac{1}{2}}+\ldots\ldots$$

$$w=-\tfrac{1}{2}-\frac{1}{2\sqrt{6}}z'^{\frac{1}{2}}+\ldots\ldots$$

and they are cyclically interchangeable for a small circuit round $z=-1$.

These are the finite values of w at branch-points. For the infinities of w, which may arise in connection with the singularities of the coefficients, we take the zeros of the coefficient of the highest power of w in the integral equation, viz., $z=0$, which is thus the only infinity of w. To find its order we take $w=z^{-n}f(z)=\gamma z^{-n}+\ldots\ldots$, where γ is a constant and $f(z)$ is finite for $z=0$; and then we have

$$-\frac{8z^{1-3n}}{1-z}\gamma^3+\ldots\ldots=3\gamma z^{-n}+\ldots\ldots+1.$$

Thus

$$1-3n=-n,$$

provided both of them be negative; the equality gives $n=\frac{1}{2}$ and satisfies the condition. And $8\gamma^3=-3\gamma$. Of these values one is zero, and gives a branch of the function without an infinity; the other two are $\pm\frac{1}{2}\sqrt{-\frac{3}{2}}$ and they give the initial term of the two branches of w, which have an infinity of order $-\frac{1}{2}$ at the origin and are cyclically interchangeable for a small circuit round it. The three values of w for infinitesimal values of z are

$$w_1=\sqrt{\frac{3}{8}}\,iz^{-\frac{1}{2}}+\frac{1}{6}-\frac{7}{18}\sqrt{\frac{3}{8}}\,iz^{\frac{1}{2}}-\frac{4}{81}z-\frac{275}{1944}\sqrt{\frac{3}{8}}\,iz^{\frac{3}{2}}-\frac{4}{729}z^2+\ldots\ldots$$

$$w_2=-\sqrt{\frac{3}{8}}\,iz^{-\frac{1}{2}}+\frac{1}{6}+\frac{7}{18}\sqrt{\frac{3}{8}}\,iz^{\frac{1}{2}}-\frac{4}{81}z+\frac{275}{1944}\sqrt{\frac{3}{8}}\,iz^{\frac{3}{2}}-\frac{4}{729}z^2-\ldots\ldots$$

$$w_3=-\frac{1}{3}+\frac{8}{81}z+\frac{8}{729}z^2+\ldots\ldots.$$

The first two of these form the system for the branch-point at the origin, which is neither an infinity nor a critical point for the third branch of the function.

Ex. 2. Obtain the branch-points of the functions which are defined by the following equations, and determine the cyclical systems at the branch-points:

(i) $w^3-w+z=0$;

(ii) $w^3-3w^2+z^6=0$;

(iii) $w^3-3w+2z^2(2-z^2)=0$;

(iv) $w^3-3zw+z^3=0$;

(v) $w^5-(1-z^2)w^4-\frac{4^4}{5^5}z^2(1-z^2)^4=0$. (Briot and Bouquet.)

Also discuss the branches, in the vicinity of $z=0$ and of $z=\infty$, of the functions defined by the following equations:

(vi) $aw^7+bw^5z+cw^4z^4+dw^2z^5+ewz^7+fz^9+gw^8+hw^4z^5+kz^{10}=0$;

(vii) $w^mz^n+w^n+z^m=0$.

98. There is one case of considerable importance which, though limited in character, is made the basis of Clebsch and Gordan's investigations* in the theory of Abelian functions—the results being, of course, restricted by the initial limitations. It is assumed that *all the branch-points are simple*, that is, are such that only one pair of branches of w are interchanged by a circuit of the variable round the point; and it is assumed that the equation $f=0$ is algebraical not merely in w but also in z. The equation $f=0$ can then be regarded as the generalised form of the equation of a curve of the nth order, the generalisation consisting in replacing the usual coordinates by complex variables; and it is further assumed, in order to simplify the analysis, that all the multiple points on the curve are (real or imaginary) double-points. But, even with the limitations, the results are of great value: and it is therefore desirable to establish the results that belong to the present section of the subject.

We assume, therefore, that the branch-points are such that only one pair of branches of w are interchanged by a small closed circuit round any one of the points. The branch-points are among the values of z determined by the equations

$$f(w, z)=0, \qquad \frac{\partial f(w, z)}{\partial w}=0.$$

When $f=0$ has the most general form consistent with the assigned limitations, $f(w, z)$ is of the nth degree in z; the values of z are determined by the eliminant of the two equations which is of degree $n(n-1)$, and there are, therefore, $n(n-1)$ values of z which must be examined.

First, suppose that $\dfrac{\partial f(w, z)}{\partial z}$ does not vanish for a value of z, thus obtained, and the corresponding value of w; then we have the first case in the preceding investigation. And, on the hypothesis adopted in the present instance, $m=2$; so that *each such point z is a branch-point.*

Next, suppose that $\dfrac{\partial f(w, z)}{\partial z}$ vanishes for some of the $n(n-1)$ values of z; the value of m is still 2, owing to the hypothesis. The case will now be still further limited by assuming that $\dfrac{\partial^2 f(w, z)}{\partial z^2}$ does not vanish for the value of z and the corresponding value of w; and thus in the vicinity of $z=a$, $w=\alpha$ we have an equation

$$0 = Az'^2 + 2Bz'w' + Cw'^2 + \text{terms of the third degree} + \ldots\ldots,$$

where A, B, C are the values of $\dfrac{\partial^2 f}{\partial z^2}$, $\dfrac{\partial^2 f}{\partial z \partial w}$, $\dfrac{\partial^2 f}{\partial w^2}$ for $z=a$, $w=\alpha$.

If $B^2 \gtrless AC$, this equation leads to the solution

$$C'w + Bz' \propto \text{uniform function of } z'.$$

* Clebsch und Gordan, *Theorie der Abel'schen Functionen*, (Leipzig, Teubner, 1866).

The point $z=a$, $w=\alpha$ is *not* a branch-point; the values of w, equal at the point, are functionally distinct. Moreover, such a point z occurs doubly in the eliminant; so that, if there be δ such points, they account for 2δ in the eliminant of degree $n(n-1)$; and therefore, on their score, the number $n(n-1)$ must be diminished by 2δ. The case is, reverting to the generalisation of the geometry, that of a double point where the tangents are not coincident.

If, however, $B^2 = AC$, the equation leads to the solution

$$Cw' + Bz' = Lz'^{\frac{3}{2}} + Mz'^2 + Nz'^{\frac{5}{2}} + \ldots\ldots$$

The point $z=a$, $w=\alpha$ is a point where the two values of z interchange. Now such a point z occurs triply in the eliminant; so that, if there be κ such points, they account for 3κ of the degree of the equation. Each of them provides only one branch-point, and the aggregate therefore provides κ branch-points; hence, in counting the branch-points of this type as derived through the eliminant, its degree must be diminished by 2κ. The case is, reverting to the generalisation of the geometry, that of a double point (real or imaginary) where the tangents are coincident.

It is assumed that all the $n(n-1)$ points z are accounted for under the three classes considered. Hence *the number of branch-points of the equation is*

$$\Omega = n(n-1) - 2\delta - 2\kappa,$$

where n is the degree of the equation, δ is the number of double points (in the generalised geometrical sense) at which tangents to the curve do not coincide, and κ is the number of double points at which tangents to the curve do coincide.

And at each of these branch-points, Ω in number, two branches of the function are equal and, for a small circuit round it, interchange.

99. The following theorem is a combined converse of many of the theorems which have been proved:

A function w, which has n (and only n) values for each value of z, and which has a finite number of infinities and of branch-points in any part of the plane, is a root of an equation in w of degree n, the coefficients of which are uniform functions of z in that part of the plane.

We shall first prove that every integral symmetric function of the n values is a uniform function in the part of the plane under consideration.

Let S_k denote $\sum_{i=1}^{n} w_i^k$, where k is a positive integer. At an ordinary point of the plane, S_k is evidently a one-valued function and that value is finite; S_k is continuous; and therefore the function S_k is uniform in the immediate vicinity of an ordinary point of the plane.

For a point a, which is a branch-point of the function w, we know that the branches can be arranged in cyclical systems. Let $w_1, \dots, w_\mu$ be such a system. Then these branches interchange in cyclical order for a description of a small circuit round a; and, if $z - a = Z^\mu$, it is known (§ 93) that, in the vicinity of $Z = 0$, a branch w is a uniform function of Z, say

$$w = G\left(\frac{1}{Z}\right) + P(Z).$$

Therefore
$$w^k = G_k\left(\frac{1}{Z}\right) + P_k(Z)$$

in the vicinity of $Z = 0$; say

$$w_1^k = A_k + \sum_{m=1} B_{k,m} Z^{-m} + \sum_{m=1} C_{k,m} Z^m.$$

Now the other branches of the function which are equal at a are derivable from any one of them by taking the successive values which that one acquires as the variable describes successive circuits round a. A circuit of w round a changes the argument of $z - a$ by 2π, and therefore gives Z reproduced but multiplied by a factor which is a primitive μth root of unity, say by a factor α; a second circuit will reproduce Z with a factor α^2; and so on. Hence

$$w_2^k = A_k + \sum_{m=1} B_{k,m}\, \alpha^{-m} Z^{-m} + \sum_{m=1} C_{k,m}\, \alpha^m Z^m,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots \vdots \cdots\cdots\cdots\cdots\cdots\cdots$$

$$w_r^k = A_k + \sum_{m=1} B_{k,m}\, \alpha^{-rm} Z^{-m} + \sum_{m=1} C_{k,m}\, \alpha^{rm} Z^m,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots \vdots \cdots\cdots\cdots\cdots\cdots\cdots$$

and therefore

$$\sum_{r=1}^{\mu} w_r^k = \mu A_k + \sum_{m=1} B_{km} Z^{-m} (1 + \alpha^{-m} + \alpha^{-2m} + \dots + \alpha^{-m\mu+m})$$
$$+ \sum_{m=1} C_{km} Z^m (1 + \alpha^m + \alpha^{2m} + \dots + \alpha^{m\mu-m}).$$

Now, since α is a primitive μth root of unity,

$$1 + \alpha^s + \alpha^{2s} + \dots + \alpha^{s(\mu-1)}$$

is zero for all integral values of s which are not integral multiples of μ, and it is μ for those values of s which are integral values of μ; hence

$$\frac{1}{\mu}\sum_{r=1}^{\mu} w_r^k = A_k + B_{k,\mu} Z^{-\mu} + B_{k,2\mu} Z^{-2\mu} + B_{k,3\mu} Z^{-3\mu} + \dots$$
$$+ C_{k,\mu} Z^\mu + C_{k,2\mu} Z^{2\mu} + C_{k,3\mu} Z^{3\mu} + \dots$$
$$= A_k + B'_{k,1}(z-a)^{-1} + B'_{k,2}(z-a)^{-2} + B'_{k,3}(z-a)^{-3} + \dots$$
$$+ C'_{k,1}(z-a) + C'_{k,2}(z-a)^2 + C'_{k,3}(z-a)^3 + \dots.$$

Hence the point $z = a$ may be a singularity of $\sum_{r=1}^{\mu} w_r^k$ but it is not a branch-point of the function; and therefore in the immediate vicinity of $z=a$ the quantity $\sum_{r=1}^{\mu} w_r^k$ is a uniform function.

The point a is an essential singularity of this uniform function, if the order of the infinity of w at a be infinite: it is an accidental singularity, if that order be a finite integer.

This result is evidently valid for all the cyclical systems at a, as well as for the individual branches which may happen to be one-valued at a. Hence S_k, being the sum of sums of the form $\sum_{r=1}^{\mu} w_r^k$ each of which is a uniform function of z in the vicinity of a, is itself a uniform function of z in that vicinity. Also a is an essential singularity of S_k, if the order of the infinity at $z=a$ for any one of the branches of w be infinite; and it is an accidental singularity of S_k, if the order of the infinity at $z=a$ for all the branches of w be finite. Lastly, it is an ordinary point of S_k, if there be no branch of w for which it is an infinity. Similarly for each of the branch-points.

Again, let c be a regular singularity of any one (or more) of the branches of w; then c is a regular singularity of every power of each of those branches, the singularities being simultaneously accidental or simultaneously essential. Hence c is a singularity of S_k: and therefore in the vicinity of c, S_k is a uniform function, having c for an accidental singularity if it be so for each of the branches w affected by it, and having c for an essential singularity if it be so for any one of the branches w.

It thus appears that in the part of the plane under consideration the function S_k is one-valued; and it is continuous and finite, except at certain isolated points each of which is a singularity. It is therefore a uniform function in that part of the plane; and the singularity of the function at any point is essential, if the order of the infinity for any one of the branches w at that point be infinite, but it is accidental, if the order of the infinity for all the branches w there be finite. And the number of these singularities is finite, being not greater than the combined number of the infinities of the function w, whether regular singularities or branch-points.

Since the sums of the kth powers for all positive values of the integer k are uniform functions and since any integral symmetric function of the n values is a rational integral algebraical function of the sums of the powers, it follows that any integral symmetric function of the n values is a uniform function of z in the part of the plane under consideration; and every infinity of a branch w leads to a singularity of the symmetric function, which is essential or accidental according as the orders of infinity of the various branches are not all finite or are all finite.

Since w has n (and only n) values $w_1, \dots, w_n$ for each value of z, the equation which determines w is

$$(w - w_1)(w - w_2) \dots (w - w_n) = 0.$$

The coefficients of the various powers of w are symmetric functions of the branches $w_1, \dots, w_n$; and therefore they are uniform functions of z in the part of the plane under consideration. They possess a finite number of singularities, which are accidental or essential according to the character of the infinities of the branches at the same points.

COROLLARY. *If all the infinities of the branches in the finite part of the whole plane be of finite order, then the finite singularities of all the coefficients of the powers of w in the equation satisfied by w are all accidental; and the coefficients themselves then take the form of a quotient of an integral uniform function (which may be either transcendental or algebraical, in the sense of* § 47) *by another function of a similar character.*

If $z = \infty$ be an essential singularity for at least one of the coefficients, through being an infinity of unlimited order for a branch of w, then one or both of the functions in the quotient-form of one at least of the coefficients must be transcendental.

If $z = \infty$ be an accidental singularity or an ordinary point for all the coefficients, through being either an infinity of finite order or an ordinary point for the branches of w, then all the functions which occur in all the coefficients are rational, algebraical expressions. When the equation is multiplied throughout by the least common multiple of the denominators of the coefficients, it takes the form

$$w^n h_0(z) + w^{n-1} h_1(z) + \dots + w\, h_{n-1}(z) + h_n(z) = 0,$$

where the functions $h_0(z), h_1(z), \dots, h_n(z)$ are rational, integral, algebraical functions of z, in the sense of § 47.

A knowledge of the number of infinities of w gives an upper limit of the degree of the equation in z in the last form. Thus, let a_i be a regular singularity of the function; and let $\alpha_i, \beta_i, \gamma_i, \dots$ be the orders of the infinities of the branches at a_i; then

$$w_1 w_2 \dots w_n (z - a_i)^{\lambda_i},$$

where λ_i denotes $\alpha_i + \beta_i + \gamma_i + \dots$, is finite (but not zero) for $z = a_i$.

Let c_i be a branch-point, which is an infinity; and let μ branches w form a system for c_i, such that $w(z - c_i)^{\frac{\theta_i}{\mu}}$ is finite (but not zero) at the point; then

$$w_1 w_2 \dots w_\mu (z - c_i)^{\theta_i}$$

is finite (but not zero) at the point, and therefore also

$$w_1 \dots w_n (z - c_i)^{\theta_i + \phi_i + \psi_i + \dots}$$

is finite, where θ_i, ϕ_i, ψ_i, ... are numbers belonging to the various systems; or, if ϵ_i denote $\theta_i + \phi_i + \psi_i + \dots$, then

$$w_1 \dots w_n (z - c_i)^{\epsilon_i}$$

is finite for $z = c_i$. Similarly for other symmetric functions of w.

Hence, if a_1, a_2, ... be the regular singularities with numbers λ_1, λ_2, ... defined as above, and if c_1, c_2, ... be the branch-points, that are also infinities, with numbers ϵ_1, ϵ_2, ... defined as above, then the product

$$(w - w_1) \dots\dots (w - w_n) \prod_{i=1} (z - a_i)^{\lambda_i} \prod_{i=1} (z - c_i)^{\epsilon_i}$$

is finite at all the points a_i and at all the points c_i. The points a and the points c are the only points in the finite part of the plane that can make the product infinite: hence it is finite everywhere in the finite part of the plane, and it is therefore an integral function of z.

Lastly, let ρ be the number for $z = \infty$ corresponding to λ_i for a_i or to ϵ_i for c_i, so that for the coefficient of any power of w in $(w - w_1) \dots (w - w_n)$ the greatest difference in degree between the numerator and the denominator is ρ in favour of the excess of the former.

Then the preceding product is of order

$$\rho + \Sigma\lambda_i + \Sigma\epsilon_i,$$

which is therefore the order of the equation in z when it is expressed in a holomorphic form.

CHAPTER IX.

Periods of Definite Integrals, and Periodic Functions in general.

100. Instances have already occurred in which the value of a function of z is not dependent solely upon the value of z but depends also on the course of variation by which z obtains that value; for example, integrals of uniform functions, and multiform functions. And it may be expected that, *a fortiori*, the value of an integral connected with a multiform function will depend upon the course of variation of the variable z. Now as integrals which arise in this way through multiform functions and, generally, integrals connected with differential equations are a fruitful source of new functions, it is desirable that the effects on the value of an integral caused by variations of a z-path be assigned so that, within the limits of algebraic possibility, the expression of the integral may be made completely determinate.

There are two methods which, more easily than others, secure this result; one of them is substantially due to Cauchy, the other to Riemann.

The consideration of Riemann's method, both for multiform functions and for integrals of such functions, will be undertaken later, in Chapters XV., XVI. Cauchy's method has already been used in preceding sections relating to uniform functions, and it can be extended to multiform functions. Its characteristic feature is the isolation of critical points, whether regular singularities or branch-points, by means of small curves each containing one and only one critical point.

Over the rest of the plane the variable z ranges freely and, under certain conditions, any path of variation of z from one point to another can, as will be proved immediately, be deformed without causing any change in the value of the integral, provided that the path does not meet any of the small curves in the course of the deformation. Further, from a knowledge of the relation of any point thus isolated to the function, it is possible to calculate the change caused by a deformation of the z-path over such a point; and thus, for defined deformations, the value of the integral can be assigned precisely.

The properties proved in Chapter II. are useful in the consideration of the integrals of uniform functions; it is now necessary to establish the propositions which give the effects of deformation of path on the integrals of multiform function. The most important of these propositions is the following:—

If w be a multiform function, the value of $\int_a^b w\,dz$, *taken between two ordinary points, is unaltered for a deformation of the path, provided that the initial branch of w be the same and that no branch-point or infinity be crossed in the deformation.*

Consider two paths acb, adb, (fig. 16, p. 152), satisfying the conditions specified in the proposition. Then in the area between them the branch w has no infinity and no point of discontinuity; and there is no branch-point in that area. Hence, by § 90, Corollary VI., the branch w is a uniform monogenic function for that area; it is continuous and finite everywhere within it and, by the same Corollary, we may treat w as a uniform, monogenic, finite and continuous function. Hence, by § 17, we have

$$(c)\int_a^b w\,dz + (d)\int_b^a w\,dz = 0,$$

the first integral being taken along acb and the second along bda; and therefore

$$(c)\int_a^b w\,dz = -(d)\int_b^a w\,dz = (d)\int_a^b w\,dz,$$

shewing that the values of the integral along the two paths are the same under the specified conditions.

It is evident that, if some critical point be crossed in the deformation, the branch w cannot be declared uniform and finite in the area and the theorem of § 17 cannot then be applied.

COROLLARY I. *The integral round a closed curve containing no critical point is zero.*

COROLLARY II. *A curve round a branch-point, containing no other critical point of the function, can be deformed into a loop without altering the value of* $\int w\,dz$; for the deformation satisfies the condition of the proposition. Hence, when the value of the integral for the loop is known, the value of the integral is known for the curve.

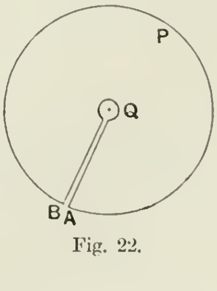

Fig. 22.

COROLLARY III. From the proposition it is possible to infer conditions, under which *the integral* $\int w\,dz$ *round the whole of any curve remains unchanged, when the whole curve is deformed,* without leaving an infinitesimal arc common as in Corollary II.

Let CDC', ABA' be the curves: join two consecutive points AA' to two consecutive points CC'. Then if the area $CABA'C'DC$ enclose no critical point of the function w, the value of $\int wdz$ along CDC' is by the proposition the same as its value along $CABA'C'$. The latter is made up of the value along CA, the value along ABA', and the value along $A'C'$, say

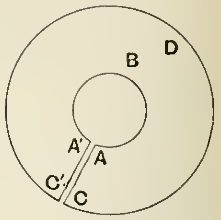

Fig. 23.

$$\int_C^A wdz + \int_B wdz + \int_{A'}^{C'} w'dz,$$

where w' is the changed value of w consequent on the description of a simple curve reducible to B (§ 90, Cor. II.).

Now since w is finite everywhere, the difference between the values of w at A and at A' consequent on the description of ABA' is finite: hence as $A'A$ is infinitesimal the value of $\int wdz$ necessary to complete the value for the whole curve B is infinitesimal and therefore the complete value can be taken as the foregoing integral $\int_B wdz$. Similarly for the complete value along the curve D: and therefore the difference of the integrals round B and round D is

$$\int_C^A wdz + \int_{A'}^{C'} w'dz,$$

say

$$\int_C^A (w - w')\, dz.$$

In general this integral is not zero, so that the values of the integral round B and round D are not equal to one another: and therefore the curve D cannot be deformed into the curve B without affecting the value of $\int wdz$ round the whole curve, even when the deformation does not cause the curve to pass over a critical point of the function.

But in special cases it may vanish. The most important and, as a matter of fact, the one of most frequent occurrence is that in which the description of the curve B restores at A' the initial value of w at A. It easily follows, by the use of § 90, Cor. II., that the description of D (assuming that the area between B and D includes no critical point) restores at C' the initial value of w at C. In such a case, $w = w'$ for corresponding points on AC and $A'C'$, and the integral, which expresses the difference, is zero: the value of the integral for the curve B is then the same as that for D. Hence we have the proposition:—

If a curve be such that the description of it by the independent variable restores the initial value of a multiform function w, then the value of $\int wdz$ taken round the curve is unaltered when the curve is deformed into any other curve, provided that no branch-point or point of discontinuity of w is crossed in the course of deformation.

This is the generalisation of the proposition of § 19 which has thus far been used only for uniform functions.

Note. Two particular cases, which are very simple, may be mentioned here: special examples will be given immediately.

The first is that in which the curve B, and therefore also D, encloses no branch-point or infinity; the initial value of w is restored after a description of either curve, and it is easy to see (by reducing B to a point, as may be done) that the value of the integral is zero.

The second is that in which the curve encloses more than one branch-point, the enclosed branch-points being such that a circuit of all the loops, into which (by Corollary V., § 90) the curve can be deformed, restores the initial branch of w. This case is of especial importance when w is two-valued: the curves then enclose an even number of branch-points.

101. It is important to know the value of the integral of a multiform function round a small curve enclosing a branch-point.

Let c be a point at which m branches of an algebraical function are equal and interchange in a single cycle; and let c, if an infinity, be of only finite order, say k/m. Then in the vicinity of c, any of the branches w can be expressed in the form

$$w = \sum_{s=-k}^{\infty} g_s (z-c)^{\frac{s}{m}},$$

where k is a finite integer.

The value of $\int w dz$ taken round a small curve enclosing c is the sum of the integrals

$$g_s \int (z-c)^{\frac{s}{m}} dz,$$

the value of which, taken once round the curve and beginning at a point z_1, is

$$\frac{m g_s}{m+s} (z_1 - c)^{\frac{s}{m}+1} [\alpha^s - 1],$$

where α is a primitive mth root of unity, provided $m+s$ is not zero. If then $s+m$ be positive, the value is zero in the limit when the curve is infinitesimal: if $m+s$ be negative, the value is ∞ in the limit.

But, if $m+s$ be zero, the value is $2\pi i g_s$.

Hence we have the proposition: *If, in the vicinity of a branch-point c, where m branches w are equal to one another and interchange cyclically, the expression of one of the branches be*

$$g_k (z-c)^{-\frac{k}{m}} + g_{k-1} (z-c)^{-\frac{k-1}{m}} + \dots\dots$$

then $\int w dz$, taken once round a small curve enclosing c, is zero, if $k < m$; is infinite, if $k > m$; and is $2\pi i g_k$, if $k = m$.

It is easy to see that, if the integral be taken m times round the small curve enclosing c, then the value of the integral is $2m\pi i g_m$ when k is greater than m, so that the integral vanishes unless there be a term involving $(z-c)^{-1}$ in the expansion of a branch w in the vicinity of the point. The reason that the integral, which can furnish an infinite value for a single circuit, ceases to do so for m circuits, is that the quantity $(z_1 - c)^{-\frac{\lambda}{m}}$, which becomes indefinitely great in the limit, is multiplied for a single circuit by $\alpha^\lambda - 1$, for a second circuit by $\alpha^{2\lambda} - \alpha^\lambda$, and so on, and for the mth circuit by $\alpha^{m\lambda} - \alpha^{(m-1)\lambda}$, the sum of all of which coefficients is zero.

Ex. The integral $\int \{(z-a)(z-b)\ldots(z-f)\}^{-\frac{1}{2}} dz$ taken round an indefinitely small curve enclosing a is zero, provided no one of the quantities $b, \ldots, f$ is equal to a.

102. Some illustrations have already been given in Chapter II., but they relate solely to definite, not to indefinite, integrals of uniform functions. The whole theory will not be considered at this stage; we shall merely give some additional illustrations, which will shew how the method can be applied to indefinite integrals of uniform functions and to integrals of multiform functions, and which will also form a simple and convenient introduction to the theory of periodic functions of a single variable.

We shall first consider indefinite integrals of uniform functions.

Ex. 1. Consider the integral $\int \frac{dz}{z}$, and denote* it by $f(z)$.

The function to be integrated is uniform, and it has an accidental singularity of the first order at the origin, which is its only singularity. The value of $\int z^{-1} dz$ taken positively along a small curve round the origin, say round a circle with the origin as centre, is $2\pi i$; but the value of the integral is zero when taken along any closed curve which does not include the origin.

Taking $z=1$ as the lower limit of the integral, and any point z as the upper limit, we consider the possible paths from 1 to z. Any path from 1 to z can be deformed, without crossing the origin, into a path which circumscribes the origin positively some number of times, say m_1, and negatively some number of times, say m_2, all in any order, and then leads in a straight line from 1 to z. For this path the value of the integral is equal to

$$(2\pi i)\, m_1 + (-2\pi i)\, m_2 + \int_1^z \frac{dz}{z},$$

that is, to

$$2m\pi i + \int_1^z \frac{dz}{z},$$

where m is an integer, and in the last integral the variation of z is along a straight line from 1 to z. Let the last integral be denoted by u; then

$$f(z) = u + 2m\pi i,$$

* See Chrystal, ii, pp. 266—272, for the elementary properties of the function and its inverse, when the variable is complex.

and therefore, inverting the function and denoting f^{-1} by ϕ, we have

$$z = \phi(u + 2m\pi i).$$

Hence the general integral is a function of z with an infinite number of values; and z is a periodic function of the integral, the period being $2\pi i$.

Ex. 2. Consider the function $\int \frac{dz}{1+z^2}$; and again denote it by $f(z)$.

The one-valued function to be integrated has two accidental singularities $\pm i$, each of the first order. The value of the integral taken positively along a small curve round i is π, and along a small curve round $-i$ is $-\pi$.

We take the origin O as the lower limit and any point z as the upper limit. Any path from O to z can be deformed, without crossing either of the singularities and therefore without changing the value of the integral, into

(i) any numbers of positive (m_1, m_2) and of negative (m_1', m_2') circuits round i and round $-i$, and

(ii) a straight line from O to z.

Then we have

$$\begin{aligned} f(z) &= m_1\pi + m_1'(-\pi) + m_2(-\pi) + m_2'\{-(-\pi)\} + \int_0^z \frac{dz}{1+z^2} \\ &= n\pi + \int_0^z \frac{dz}{1+z^2} \\ &= n\pi + u, \end{aligned}$$

where n is an integer and the integral on the right-hand side is taken along a straight line from O to z.

Inverting the function and denoting f^{-1} by ϕ, we have

$$z = \phi(u + n\pi).$$

The integral, as before, is a function of z with an infinite number of values; and z is a periodic function of the integral, the period being π.

103. Before passing to the integrals of multiform functions, it is convenient to consider the method in which Hermite* discusses the multiplicity in value of a definite integral of a uniform function.

Taking a simple case, let $\phi(z) = \int_0^z \frac{dZ}{1+Z}$

and introduce a new variable t such that $Z = zt$; then

$$\phi(z) = \int_0^1 \frac{zdt}{1+zt}.$$

When the path of t is assigned, the integral is definite, finite and unique in value for all points of the plane except for those for which $1 + zt = 0$; and, according to the path of variation of t from 0 to 1, there will be a z-curve which is a curve of discontinuity for the subject of integration. Suppose the path of t to be the straight line from 0 to 1; then the curve of discontinuity

* *Crelle*, t. xci, (1881), pp. 62—77; *Cours à la Faculté des Sciences*, 4ème éd. (1891), pp. 76—79, 154—164, and elsewhere.

is the axis of x between -1 and $-\infty$. In this curve let any point $-\xi$ be taken where $\xi > 1$; and consider a point $z_1 = -\xi + i\epsilon$ and a point $z_2 = -\xi - i\epsilon$, respectively on the positive and the negative sides of the axis of x, both being ultimately taken as infinitesimally near the point $-\xi$. Then

$$\phi(z_1) - \phi(z_2) = \int_0^1 \left(\frac{-\xi + i\epsilon}{1 - \xi t + i\epsilon t} + \frac{\xi + i\epsilon}{1 - \xi t - i\epsilon t}\right) dt$$

$$= \int_0^1 \frac{2i\epsilon}{(1-\xi t)^2 + \epsilon^2 t^2}\, dt = \int_1^\infty \frac{2i\epsilon}{(t-\xi)^2 + \epsilon^2}\, dt$$

$$= 2i\left[\tan^{-1}\frac{t-\xi}{\epsilon}\right]_1^\infty.$$

Let ϵ become infinitesimal; then, when t is infinite, we have

$$\tan^{-1}\left(\frac{t-\xi}{\epsilon}\right) = \tfrac{1}{2}\pi,$$

for ϵ is positive; and, when t is unity, we have

$$\tan^{-1}\frac{t-\xi}{\epsilon} = -\tfrac{1}{2}\pi,$$

for ξ is > 1. Hence $\qquad \phi(z_1) - \phi(z_2) = 2\pi i.$

The part of the axis of x from -1 to $-\infty$ is therefore a line of discontinuity in value of $\phi(z)$, such that there is a sudden change in passing from one edge of it to the other. If the plane be cut along this line so that it cannot be crossed by the variable which may not pass out of the plane, then the integral is everywhere finite and uniform in the modified surface. If the plane be not cut along the line, it is evident that a single passage across the line from one edge to the other makes a difference of $2\pi i$ in the value, and consequently any number of passages across will give rise to the multiplicity in value of the integral.

Such a line is called a *section** by Hermite, after Riemann who, in a different manner, introduces these lines of singularity into his method of representing the variable on surfaces†.

When we take the general integral of a uniform function of Z and make the substitution $Z = zt$, the integral that arises for consideration is of the form

$$\Phi(z) = \int_{t_0}^{t_1} \frac{F(t, z)}{G(t, z)}\, dt.$$

We shall suppose that the path of variation of t is the axis of real quantities: and the subject of integration will be taken to be a general function of t and z, without special regard to its derivation from a uniform function of Z.

* *Coupure;* see *Crelle*, t. xci, p. 62. † See Chapter XV.

It is easy, after the special example, to see that Φ is a continuous function of z in any space that does not include a z-point which, for values of t included within the range of integration, would satisfy the equation.

$$G(t, z) = 0.$$

But in the vicinity of a z-point, say ζ, corresponding to the value $t = \theta$ in the range of integration, there will be discontinuity in the subject of integration and also, as will now be proved, in the value of the integral.

Let Z be the point ζ and draw the curve through Z corresponding to $t =$ real constant; let N_1 be a point on the positive side and N_2 a point on the negative side of this curve positively described, both points being on the normal at Z; and let $ZN_1 = ZN_2 = \epsilon'$, supposed small. Then for N_1 we have

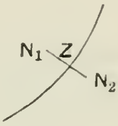

Fig. 24.

$$x_1 = \xi - \epsilon' \sin \psi, \qquad y_1 = \eta + \epsilon' \cos \psi,$$

so that
$$z_1 = \zeta + i\epsilon' (\cos \psi + i \sin \psi),$$

where ψ is the inclination of the tangent to the axis of real quantities. But, if $d\sigma$ be an arc of the curve at Z,

$$\frac{d\sigma}{dt} (\cos \psi + i \sin \psi) = \frac{d\xi}{dt} + i \frac{d\eta}{dt} = \frac{d\zeta}{dt}$$

for variations along the tangent at Z, that is,

$$\frac{d\sigma}{dt} (\cos \psi + i \sin \psi) = - \frac{\dfrac{\partial}{\partial \theta} G(\theta, \zeta)}{\dfrac{\partial}{\partial \zeta} G(\theta, \zeta)}.$$

Thus, since $\dfrac{d\sigma}{dt}$ may be taken as finite on the supposition that Z is an ordinary point of the curve, we have

$$z_1 = \zeta - i\epsilon \frac{P}{Q},$$

where
$$\epsilon = \epsilon' \frac{dt}{d\sigma}, \qquad P = \frac{\partial}{\partial \theta} G(\theta, \zeta), \qquad Q = \frac{\partial}{\partial \zeta} G(\theta, \zeta).$$

Similarly
$$z_2 = \zeta + i\epsilon \frac{P}{Q}.$$

Hence
$$\Phi(z_1) = \int_{t_0}^{t_1} \frac{F(t, z_1)}{G(t, z_1)} dt$$

$$= \int_{t_0}^{t_1} \frac{F(t, \zeta) - i\epsilon \left\{\dfrac{\partial}{\partial \zeta} F(t, \zeta)\right\} \dfrac{P}{Q}}{G(t, \zeta) - i\epsilon \left\{\dfrac{\partial}{\partial \zeta} G(t, \zeta)\right\} \dfrac{P}{Q}} dt,$$

with a similar expression for $\Phi(z_2)$; and therefore

$$\Phi(z_1) - \Phi(z_2) = 2i \int_{t_0}^{t_1} \epsilon \frac{F(t, \zeta) \dfrac{\partial}{\partial \zeta}\{G(t, \zeta)\} \dfrac{P}{Q} - G(t, \zeta) \dfrac{\partial}{\partial \zeta}\{F(t, \zeta)\} \dfrac{P}{Q}}{G^2(t, \zeta) + \epsilon^2 \dfrac{P^2}{Q^2} \left\{\dfrac{\partial}{\partial \zeta} G(t, \zeta)\right\}^2} dt.$$

The subject of integration is infinitesimal, except in the immediate vicinity of $t = \theta$; and there

$$G(t, \zeta) = (t - \theta) P, \qquad F(t, \zeta) = F(\theta, \zeta),$$

$$\frac{\partial}{\partial \zeta}\{G(t, \zeta)\} = Q, \qquad \frac{\partial}{\partial \zeta}\{F(t, \zeta)\} = \frac{\partial}{\partial \zeta}\{F(\theta, \zeta)\},$$

powers of small quantities other than those retained being negligible. Let the limiting values of t, that need be retained, be denoted by $\theta + \nu$ and $\theta - \mu$; then, after reduction, we have

$$\Phi(z_1) - \Phi(z_2) = 2i \int_{\theta-\mu}^{\theta+\nu} \frac{F(\theta, \zeta)}{P} \frac{\epsilon dt}{(t-\theta)^2 + \epsilon^2}$$

$$= 2\pi i \frac{F(\theta, \zeta)}{\dfrac{\partial}{\partial \theta}\{G(\theta, \zeta)\}},$$

in the limit when ϵ is made infinitesimal.

Hence a line of discontinuity of the subject of integration is a section for the integral; and the preceding expression is the magnitude, by numerical multiples of which the values of the integral differ*.

Ex. 1. Consider the integral

$$\Phi(z) = \int \frac{dZ}{1+Z^2}$$

$$= \int \frac{zdt}{1+z^2 t^2}.$$

We have

$$\frac{F(\theta, \zeta)}{\dfrac{\partial}{\partial \theta}\{G(\theta, \zeta)\}} = \frac{\zeta}{2\zeta^2 \theta} = \frac{1}{2\zeta\theta} = \frac{1}{2i},$$

so that π is the period for the above integral.

Ex. 2. Shew that the sections for the integral

$$\int_0^\infty \frac{t^a \sin z}{1 + 2t \cos z + t^2} dt,$$

* The memoir and the *Cours d'Analyse* of Hermite should be consulted for further developments; and, in reference to the integral treated above, Jordan, *Cours d'Analyse*, t. iii, pp. 610—614, may be consulted with advantage. See also, generally, for functions defined by definite integrals, Goursat, *Acta Math.*, t. ii, (1883), pp. 1—70, and ib., t. v, (1884), pp. 97—120; and Pochhammer, *Math. Ann.*, t. xxxv, (1890), pp. 470—494, 495—526. Goursat also discusses double integrals.

where a is positive and less than 1, are the straight lines $x=(2k+1)\pi$, where k assumes all integral values; and that the period of the integral at any section at a distance η from the axis of real quantities is $2\pi \cosh(a\eta)$. (Hermite.)

Ex. 3. Shew that the integral

$$\int_0^1 u^{\beta-1}(1-u)^{\gamma-\beta-1}(1-zu)^{-a}\,du,$$

where the real parts of β and $\gamma-\beta$ are positive, has the part of the axis of real quantities between 1 and $+\infty$ for a section.

Shew also that the integral

$$\phi(z)=\int_0^{\frac{1}{z}} u^{\beta-1}(1-u)^{\gamma-\beta-1}(1-zu)^{-a}\,du,$$

where the real parts of β and $1-a$ are positive, has the part of the axis of real quantities between 0 and 1 for a section: but that, in order to render $\phi(z)$ a uniform function of z, it is necessary to prevent the variable from crossing, not merely the section, but also the part of the axis of real quantities between 1 and $+\infty$. (Goursat.)

(The latter line is called a section of the *second* kind.)

Ex. 4. Discuss generally the effect of changing the path of t on a section of the integral; and, in particular, obtain the section for $\int_0^z \frac{dZ}{1+Z}$ when, after the substitution $Z=zt$, the path of t is made a semi-circle on the line joining 0 and 1 as diameter.

Note. It is manifestly impossible to discuss all the important bearings of theorems and principles, which arise from time to time in our subject; we can do no more than mention the subject of those definite integrals involving complex variables, which first occur as solutions of the better-known linear differential equations of the second order.

Thus for the definite integral connected with the hypergeometric series, memoirs by Jacobi* and Goursat† should be consulted; for the definite integral connected with Bessel's functions, memoirs by Hankel‡ and Weber§ should be consulted; and Heine's *Handbuch der Kugelfunctionen* for the definite integrals connected with Legendre's functions.

104. We shall now consider integrals of multiform functions.

Ex. 1. To find the integral of a multiform function round one loop; and round a number of loops.

Let the function be

$$w=\{(z-a_1)(z-a_2)\ldots(z-a_n)\}^{\frac{1}{m}},$$

where m may be a negative or positive integer, and the quantities a are unequal to one another; and let the loop be from the origin round the point a_1. Then, if I be the value of the integral with an assigned initial branch w, we have

$$I=\int_0^{a_1} wdz+\int_c wdz+\int_{a_1}^0 awdz,$$

where a is $e^{\frac{2\pi i}{m}}$ and the middle integral is taken round the circle at a_1 of infinitesimal radius.

* *Crelle*, t. lvi, (1859), pp. 149—165; the memoir was not published until after his death.

† *Sur l'équation différentielle linéaire qui admet pour intégrale la série hypergéométrique*, (Thèse, Gauthier-Villars, Paris, 1881).

‡ *Math. Ann.*, t. i, (1869), pp. 467—501.

§ *Math. Ann.*, t. xxxvii, (1890), pp. 404—416.

But, since the limit of $(z-a_1)w$ when $z=a_1$ is zero, the middle integral vanishes by § 101; and therefore

$$I_{a_1}=(1-a)\int_0^{a_1} wdz,$$

where the integral may, if convenient, be considered as taken along the straight line from O to a_1.

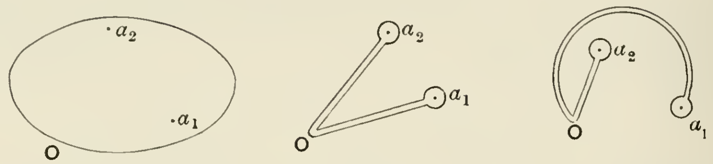

Fig. 25.

Next, consider a circuit for an integral of w which (fig. 25) encloses two branch-points, say a_1 and a_2, but no others; the circuit in (1) can be deformed into that in (2) or into that in (3) as well as into other forms. Hence the integral round all the three circuits must be the same. Beginning with the same branch as in the first case, we have

$$(1-a)\int_0^{a_1} wdz,$$

as the integral after the first loop in (2). And the branch with which the second loop begins is aw, so that the integral described as in the second loop is

$$(1-a)\int_0^{a_2} awdz;$$

and therefore, for the circuit as in (2), the integral is

$$I=(1-a)\int_0^{a_1} wdz+a(1-a)\int_0^{a_2} wdz.$$

Proceeding similarly with the integral for the circuit in (3), we find that its expression is

$$I=(1-a)\int_0^{a_2} wdz+a(1-a)\int_0^{a_1} wdz,$$

and these two values must be equal.

But the integrals denoted by the same symbols are not the same in the two cases; the function $\int_0^{a_1} wdz$ is different in the second value of I from that in the first, for the deformation of path necessary to change from the one to the other passes over the branch-point a_2. In fact, the equality of the two values of I really determines the value of the integral for the loop Oa_1 in (3).

And, in general, equations thus obtained by varied deformations do not give relations among loop-integrals but define the values of those loop-integrals for the deformed paths.

We therefore take that deformation of the circuit into loops which gives the simplest path. Usually *the path is changed into a group of loops round the branch-points as they occur, taken in order in a trigonometrically positive direction.*

The value of the integral round a circuit, equivalent to any number of loops, is obvious.

Ex. 2. *To find the value of* $\int wdz$, *taken round a simple curve which includes all the branch-points of* w *and all the infinities.*

If $z=\infty$ be a branch-point or an infinity, then all the branch-points and all the infinities of w lie on what is usually regarded as the exterior of the curve, or the curve may in one sense be said to exclude all these points. The integral round the curve is then the integral of a function round a curve, such that over the area included by it the function is uniform, finite and continuous ; hence the integral is zero.

If $z=\infty$ be neither a branch-point nor an infinity, the curve can be deformed until it is a circle, centre the origin and of very great radius. If then the limit of zw, when $|z|$ is infinitely great, be zero, the value of the integral again is zero, by II., § 24.

Another method of considering the integral, is to use Neumann's sphere for the representation of the variable. Any simple closed curve divides the area of the sphere into two parts ; when the curve is defined as above, one of those parts is such that the function is uniform, finite and continuous throughout and therefore its integral round the curve, regarded as the boundary of that part, is zero. (See Corollary III., § 90.)

Ex. 3. To find the general value of $\int(1-z^2)^{-\frac{1}{2}}dz$. The function to be integrated is two-valued : the two values interchange round each of the branch-points ± 1, which are the only branch-points of the function.

Let I be the value of the integral for a loop from the origin round $+1$, beginning with the branch which has the value $+1$ at the origin ; and let I' be the corresponding value for the loop from the origin round -1, beginning with the same branch. Then, by Ex. 1,

$$I=2\int_0^1(1-z^2)^{-\frac{1}{2}}dz, \qquad I'=2\int_0^{-1}(1-z^2)^{-\frac{1}{2}}dz$$
$$=-I,$$

the last equality being easily obtained by changing variables.

Now consider the integral when taken round a circle, centre the origin and of indefinitely great radius R ; then by § 24, II., if the limit of zw for $z=\infty$ be k, the value of $\int wdz$ round this circle is $2\pi ik$. In the present case $w=(1-z^2)^{-\frac{1}{2}}$ so that the limit of zw is $+\frac{1}{i}$; hence

$$\int(1-z^2)^{-\frac{1}{2}}dz=2\pi,$$

the integral being taken round the circle. But since a description of the circle restores the initial value, it can be deformed into the two loops from O to A and from O to A'. The value round the first is I ; and the branch with which the second begins to be described has the value -1 at the origin, so that the consequent value round the second is $-I'$; hence

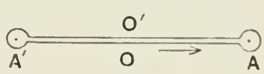

Fig. 26.

$$I-I'=2\pi^*,$$

and therefore

$$I=-I'=\pi,$$

verifying the ordinary result that

$$\int_0^1(1-z^2)^{-\frac{1}{2}}dz=\tfrac{1}{2}\pi,$$

when the integral is taken along a straight line.

To find the general value of u for any path of variation between O and z, we proceed as follows. Let Ω be any circuit which restores the initial branch of $(1-z^2)^{-\frac{1}{2}}$. Then by § 100, Corollary II., Ω may be composed of

(i) a set of double circuits round $+1$, say m',

(ii) a set of double circuits round -1, say m'',

and (iii) a set of circuits round $+1$ and -1 ;

* It is interesting to obtain this equation when O' is taken as the initial point, instead of O.

and these may come in any order and each may be described in either direction. Now for a double circuit positively described, the value of the integral for the first description is I and for the second description, which begins with the branch $-(1-z^2)^{-\frac{1}{2}}$, it is $-I$; hence for the double circuit it is zero when positively described, and therefore it is zero also when negatively described. Hence each of the m' double circuits yields zero as its nett contribution to the integral.

Similarly, each of the m'' double circuits round -1 yields zero as its nett contribution to the integral.

For a circuit round $+1$ and -1 described positively, the value of the integral has just been proved to be $I-I'$, and therefore when described negatively it is $I'-I$. Hence, if there be n_1 positive descriptions and n_2 negative descriptions, the nett contribution of all these circuits to the value of the integral is $(n_1-n_2)(I-I')$, that is, $2n\pi$ where n is an integer.

Hence the complete value for the circuit Ω is $2n\pi$.

Now any path from O to z can be resolved into a circuit Ω, which restores the initial branch of $(1-z^2)^{-\frac{1}{2}}$, chosen to have the value $+1$ at the origin, and either (i) a straight line Oz;

or (ii) the path $OACz$, viz., a loop round $+1$ and the line Oz;

or (iii) the path $OA'Cz$, viz., a loop round -1 and the line Oz.

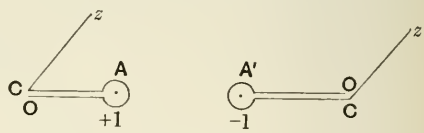

Fig. 27.

Let u denote the value for the line Oz, so that

$$u=\int_0^z (1-z^2)^{-\frac{1}{2}}\,dz.$$

Hence, for case (i), the general value of the integral is

$$2n\pi+u.$$

For the path $OACz$, the value is I for the loop OAC, and is $(-u)$ for the line Cz, the negative sign occurring because, after the loop, the branch of the function for integration along the line is $-(1-z^2)^{-\frac{1}{2}}$; this value is $I-u$, that is, it is $\pi-u$. Hence, for case (ii), the value of the integral is

$$2n\pi+\pi-u.$$

For the path $OA'Cz$, the value is similarly found to be $-\pi-u$; and therefore, for case (iii), the value of the integral is

$$2n\pi-\pi-u.$$

If $f(z)$ denote the general value of the integral, we have either

$$f(z)=2n\pi+u,$$

or

$$f(z)=(2m+1)\pi-u,$$

where n and m are any integers, so that $f(z)$ is a function with two infinite series of values.

Lastly, if $z=\phi(\theta)$ be the inverse of $f(z)=\theta$, then the relation between u and z given by

$$u=\int_0^z (1-z^2)^{-\frac{1}{2}}\,dz$$

can be represented in the form

$$\left.\begin{aligned}\phi(u)=z&=\phi(2n\pi+u)\\ \text{and}\quad \phi(u)=z&=\phi(2m\pi+\pi-u)\end{aligned}\right\},$$

both equations being necessary for the full representation. Evidently z is a simply-periodic function of u, the period being 2π; and from the definition it is easily seen to be an odd function.

Let $y=(1-z^2)^{\frac{1}{2}}=\chi(u)$, so that y is an even function of u; from the consideration of the various paths from O to z, it is easy to prove that

$$\left.\begin{aligned}\chi(u)&=\chi(2n\pi+u)\\&=-\chi(2m\pi+\pi-u)\end{aligned}\right\}.$$

Ex. 4. To find the general value of $\int\{(1-z^2)(1-k^2z^2)\}^{-\frac{1}{2}}dz$. It will be convenient (following Jordan*) to regard this integral as a special case of

$$Z=\int\{(z-a)(z-b)(z-c)(z-d)\}^{-\frac{1}{2}}dz=\int wdz.$$

The two-valued function to be integrated has a, b, c, d (but not ∞) as the complete system of branch-points; and the two values interchange at each of them. We proceed as in the last example, omitting mere re-statements of reasons there given that are applicable also in the present example.

Any circuit Ω, which restores an initial branch of w, can be made up of

(i) sets of double circuits round each of the branch-points,

and (ii) sets of circuits round any two of the branch-points.

The value of $\int wdz$ for a loop from the origin to a branch-point k (where $k=a$, b, c, or d) is

$$2\int_0^k wdz;$$

and this may be denoted by K, where $K=A$, B, C, or D.

The value of the integral for a double circuit round a branch-point is zero. Hence the amount contributed to the value of the integral by all the sets in (i) as this part of Ω is zero.

The value of the integral for a circuit round a and b taken positively is $A-B$; for one round b and c is $B-C$; for one round c and d is $C-D$; for one round a and c is $A-C$, which is the sum of $A-B$ and $B-C$; and similarly for circuits round a and d and round b and d. There are therefore three distinct values, say $A-B$, $B-C$, $C-D$, the values for circuits round a and b, b and c, c and d respectively; the values for circuits round any other pair can be expressed linearly in terms of these values. Suppose then that the part of Ω represented by (ii), when thus resolved, is the nett equivalent of the description of m' circuits round a and b, of n' circuits round b and c, and of l' circuits round c and d. Then the value of the integral contributed by this part of Ω is

$$m'(A-B)+n'(B-C)+l'(C-D),$$

which is therefore the whole value of the integral for Ω.

But the values of A, B, C, D are not independent†. Let a circle with centre the origin and very great radius be drawn; then since the limit of zw for $|z|=\infty$ is zero and since $z=\infty$ is not a branch-point, the value of $\int wdz$ round this circle is zero (Ex. 2). The circle can be deformed into four loops round a, b, c, d respectively in order; and therefore the value of the integral is $A-B+C-D$, that is,

$$A-B+C-D=0.$$

Hence the value of the integral for the circuit Ω is

$$m(A-B)+n(B-C),$$

where m and n denote $m'-l'$ and $n'-l'$ respectively.

* *Cours d'Analyse*, t. ii, p. 343.

† For a purely analytical proof of the following relation, see Greenhill's *Elliptic Functions*, Chapter II.

Now any path from the origin to z can be resolved into Ω, together with either

(i) a straight line from O to z,

or (ii) a loop round a and then a straight line to z.

It might appear that another resolution would be given by a combination of Ω with, say, a loop round b and then a straight line to z; but it is resoluble into the second of the above combinations. For at C, after the description of the loop B, introduce a double description of the loop A, which adds nothing to the value of the integral and does not in the end affect the branch of w at C; then the new path can be regarded as made up of (α) the circuit constituted by the loop round b and the first loop round a, (β) the second loop round a, which begins with the initial branch of w, followed by a straight path to z. Of these (α) can be absorbed into Ω, and (β) is the same as (ii); hence the path is not essentially new. Similarly for the other points.

Let u denote the value of the integral with a straight path from O to z; then the whole value of the integral for the combination of Ω with (i) is of the form

$$m(A-B)+n(B-C)+u.$$

For the combination of Ω with (ii), the value of the integral for the part (ii) of the path is A, for the loop round a, $+(-u)$, for the straight path which, owing to the description of the loop round a, begins with $-w$; hence the whole value of the integral is of the form

$$m(A-B)+n(B-C)+A-u^*.$$

Hence, if $f(z)$ denote the general value of the integral, it has two systems of values, each containing a doubly-infinite number of terms; and, if $z=\phi(u)$ denote the inverse of $u=f(z)$, we have

$$\begin{aligned}\phi(u)&=\phi\{m(A-B)+n(B-C)+u\}\\&=\phi\{m(A-B)+n(B-C)+A-u\},\end{aligned}$$

where m and n are any integers. Evidently z is a doubly-periodic function of u, with periods $A-B$ and $B-C$.

Ex. 5. The case of the foregoing integral which most frequently occurs is the elliptic integral in the form used by Legendre and Jacobi, viz.:

$$u=\int\{(1-z^2)(1-k^2z^2)\}^{-\frac{1}{2}}dz=\int w\,dz,$$

where k is real. The branch-points of the function to be integrated are 1, -1, $\frac{1}{k}$ and $-\frac{1}{k}$, and the values of the integral for the corresponding loops from the origin are

$$2\int_0^1 w\,dz,$$

$$2\int_0^{-1} w\,dz=-2\int_0^1 w\,dz,$$

$$2\int_0^{\frac{1}{k}} w\,dz,$$

and

$$2\int_0^{-\frac{1}{k}} w\,dz=-2\int_0^{\frac{1}{k}} w\,dz.$$

Now the values for the loops are connected by the equation

$$A-B+C-D=0,$$

* The value for a loop round b and then a straight line to z, just considered, is $B-u$

$$=-(A-B)+A-u,$$

being the value in the text with m changed to $m-1$.

and so it will be convenient that, as all the points lie on the axis of real variables, we arrange the order of the loops so that this relation is identically satisfied. Otherwise, the relation will, after Ex. 1, be a definition of the paths of integration chosen for the loops.

Among the methods of arrangement, which secure the identical satisfaction of the

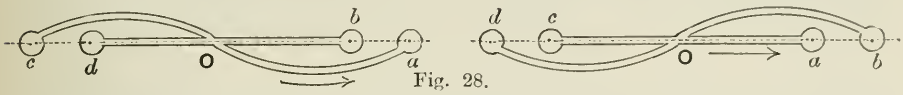

Fig. 28.

relation, the two in the figure* are the simplest, the curved lines being taken straight in the limit; for, by the first arrangement when $k < 1$, we have

$$\left\{2\int_0^{\frac{1}{k}} - 2\int_0^1 + 2\int_0^{-\frac{1}{k}} - 2\int_0^{-1}\right\} wdz = 0,$$

and, by the second when $k > 1$, we have

$$\left\{2\int_0^1 - 2\int_0^{\frac{1}{k}} + 2\int_0^{-1} - 2\int_0^{-\frac{1}{k}}\right\} wdz = 0,$$

both of which are identically satisfied. We may therefore take either of them; let the former be adopted.

The periods are $A-B$, $B-C$, (and $C-D$, which is equal to $B-A$), and any linear combination of these is a period: we shall take $A-B$, and $B-D$. The latter, $B-D$, is equal to

$$2\int_0^1 wdz - 2\int_0^{-1} wdz,$$

which, being denoted by $4K$, gives

$$4K = 4\int_0^1 \frac{dz}{\{(1-z^2)(1-k^2z^2)\}^{\frac{1}{2}}}$$

as one period. The former, $A-B$, is equal to

$$2\int_0^{\frac{1}{k}} wdz - 2\int_0^1 wdz,$$

which is

$$2\int_1^{\frac{1}{k}} wdz;$$

this, being denoted by $2iK'$, gives

$$2iK' = 2\int_1^{\frac{1}{k}} \frac{dz}{\{(1-z^2)(1-k^2z^2)\}^{\frac{1}{2}}}$$

$$= 2i\int_0^1 \frac{dz'}{\{(1-z'^2)(1-k'^2z'^2)\}^{\frac{1}{2}}},$$

where $k'^2 + k^2 = 1$ and the relation between the variables of the integrals is $k^2z^2 + k'^2z'^2 = 1$.

Hence the periods of the integral are $4K$ and $2iK'$. Moreover, A is $2\int_0^{\frac{1}{k}} wdz$, which is

$$2\int_0^1 wdz + 2\int_1^{\frac{1}{k}} wdz = 2K + 2iK'.$$

Hence the general value of $\int_0^z \{(1-z^2)(1-k^2z^2)\}^{\frac{1}{2}} dz$ is

$$u + 4mK + 2niK',$$

* Jordan, *Cours d'Analyse*, t. ii, p. 356.

or $$2K+2iK'-u+4mK+2niK',$$
that is, $$2K-u+4mK+2niK',$$
where u is the integral taken from O to z along an assigned path, often taken to be a straight line; so that there are two systems of values for the integral, each containing a doubly-infinite number of terms.

If z be denoted by $\phi(u)$—evidently, from the integral definition, an odd function of u—, then
$$\phi(u)=\phi(u+4mK+2niK')$$
$$=\phi(2K-u+4mK+2niK'),$$
so that z is a doubly-periodic function of u, the periods being $4K$ and $2iK'$.

Now consider the function $z_1=(1-z^2)^{\frac{1}{2}}$. A z-path round $\frac{1}{k}$ does not affect z_1 by way of change, provided the curve does not include the point 1; hence, if $z_1=\chi(u)$, we have
$$\chi(u)=\chi(u+2K+2iK').$$
But a z-path round the point 1 does change z_1 into $-z_1$; so that
$$\chi(u)=-\chi(u+2K).$$
Hence $\chi(u)$, which is an even function, has two periods, viz., $4K$ and $2K+2iK'$, whence
$$\chi(u)=\chi(u+4mK+2nK+2niK').$$
Similarly, taking $z_2=(1-k^2z^2)^{\frac{1}{2}}=\psi(u)$, it is easy to see that
$$\psi(u)=\psi(u+2K),$$
$$-\psi(u)=\psi(u+2K+2iK')=\psi(u+2iK'),$$
so that $\psi(u)$, which is an even function, has two periods, viz., $2K$ and $4iK'$; whence
$$\psi(u)=\psi(u+2mK+4niK').$$

The functions $\phi(u)$, $\chi(u)$, $\psi(u)$ are of course sn u, cn u, dn u respectively.

Ex. 6. To find the general value of the integral
$$\int_z^\infty \{4(z-e_1)(z-e_2)(z-e_3)\}^{-\frac{1}{2}}dz=w^*.$$

The function to be integrated has e_1, e_2, e_3, and ∞ for its branch-points; and for paths round each of them the two branches interchange.

A circuit Ω which restores the initial branch of the function to be integrated can be resolved into:—

(i) Sets of double circuits round each of the branch-points alone: as before, the value of the integral for each of these double circuits is zero.

(ii) Sets of circuits, each enclosing two of the branch-points: it is convenient to retain circuits including ∞ and e_1, ∞ and e_2, ∞ and e_3, the other three combinations being reducible to these.

The values of the integral for these three retained are respectively
$$E_1=2\int_{e_1}^\infty \{4(z-e_1)(z-e_2)(z-e_3)\}^{-\frac{1}{2}}dz=2\omega_1,$$
$$E_2=2\int_{e_2}^\infty \{4(z-e_1)(z-e_2)(z-e_3)\}^{-\frac{1}{2}}dz=2\omega_2,$$
$$E_3=2\int_{e_3}^\infty \{4(z-e_1)(z-e_2)(z-e_3)\}^{-\frac{1}{2}}dz=2\omega_3,$$

* The choice of ∞ for the upper limit is made on a ground which will subsequently be considered, viz., that, when the integral is zero, z is infinite.

and therefore the value of the integral for the circuit Ω is of the form

$$m'E_1 + n'E_2 + l'E_3.$$

But E_1, E_2, E_3 are not linearly independent. The integral of the function round any curve in the finite part of the plane, which does not include e_1, e_2 or e_3 within its boundary, is zero, by Ex. 2; and this curve can be deformed to the shape in the figure, until it becomes infinitely large, without changing the value of the integral.

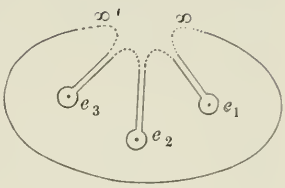

Fig. 29.

Since the limit of zw for $|z| = \infty$ is zero, the value of the integral from ∞' to ∞ is zero, by § 24, II.; and if the description begin with a branch w, the branch at ∞ is $-w$. The rest of the integral consists of the sum of the values round the loops, which is

$$-E_1 + E_2 - E_3,$$

because a path round a loop changes the branch of w and the last branch after describing the loop round e_3 is $+w$ at ∞', the proper value (§ 90, III). Hence, as the whole integral is zero, we have

$$-E_1 + E_2 - E_3 = 0,$$

or say

$$E_2 = E_1 + E_3.$$

Thus the value of the integral for any circuit Ω, which restores the initial branch of w, can be expressed in any of the equivalent forms $mE_1 + nE_3$, $m'E_1 + n'E_2$, $m''E_2 + n''E_3$, where the m's and n's are integers.

Now any path from ∞ to z can be resolved into a circuit Ω, which restores at ∞ the initial branch of w, combined with either

(i) a straight path from ∞ to z,

or (ii) a loop between ∞ and e_1, together with a straight path from ∞ to z.

(The apparently distinct alternatives, of a loop between ∞ and e_2 together with a straight path from ∞ to z and of a similar path round e_3, are inclusible in the second alternative above; the reasons are similar to those in Ex. 5.)

If u denote $\int_z^\infty \{4(z-e_1)(z-e_2)(z-e_3)\}^{-\frac{1}{2}} dz$ when the integral is taken in a straight line, then the value of the integral for part (i) of a path is u; and the value of the integral for part (ii) of a path is $E_1 - u$, the initial branch in each case for these parts being the initial branch of w for the whole path. Hence the most general value of the integral for any path is

$$2m\omega_1 + 2n\omega_3 + u,$$

or

$$2m\omega_1 + 2n\omega_3 + 2\omega_1 - u,$$

the two being evidently included in the form

$$2m\omega_1 + 2n\omega_3 \pm u.$$

If, then, we denote by $z = \wp(u)$ the relation which is inverse to

$$u = \int_z^\infty \{4(z-e_1)(z-e_2)(z-e_3)\}^{-\frac{1}{2}} dz,$$

we have

$$\wp(u) = \wp(2m\omega_1 + 2n\omega_3 \pm u).$$

In the same way as in the preceding example, it follows that

$$\wp'(u) = \wp'(2m\omega_1 + 2n\omega_3 + u) = -\wp'(2m\omega_1 + 2n\omega_3 - u),$$

where $\wp'(u)$ is $-\{4(z-e_1)(z-e_2)(z-e_3)\}^{\frac{1}{2}}$.

The foregoing simple examples are sufficient illustrations of the multiplicity of value of an integral of a uniform function or of a multiform function, when branch-points or discontinuities occur in the part of the plane in which the path of integration lies. They also shew one of the modes in which singly-periodic and doubly-periodic functions arise, the periodicity consisting in the addition of arithmetical multiples of constant quantities to the argument. And it is to be noted that, as only a single value of z is used in the integration, so only a single value of z occurs in the inversion; that is, the functions just obtained are uniform functions of their variables. To the properties of such periodic functions we shall return in the succeeding chapters.

105. We proceed to the theory of uniform periodic functions, some special examples of which have just been considered; and limitation will be made here to periodicity of the linear additive type, which is only a very special form of periodicity.

A function $f(z)$ is said to be periodic when there is a quantity ω such that the equation

$$f(z+\omega)=f(z)$$

is an identity for all values of z. Then $f(z+n\omega)=f(z)$, where n is any integer positive or negative; and it is assumed that ω is the smallest quantity for which the equation holds, that is, that no submultiple of ω will satisfy the equation. The quantity ω is called a *period* of the function.

A function is said to be *simply-periodic* when there is only a single period: to be *doubly-periodic* when there are two periods; and so on, the periodicity being for the present limited to additive modification of the argument.

It is convenient to have a graphical representation of the periodicity of a function.

(i) For simply-periodic functions, we take a series of points O, A_1, $A_2, \ldots$, A_{-1}, $A_{-2}, \ldots$ representing 0, ω, $2\omega, \ldots$, $-\omega$, $-2\omega, \ldots$; and through these points we draw a series of parallel lines, dividing the plane into bands. Let P be any point z in the band between the lines through O and through A_1; through P draw a line parallel to OA_1 and measure off $PP_1=P_1P_2=\ldots=PP_{-1}=P_{-1}P_{-2}=\ldots$, each equal to OA_1; then all the points P_1, $P_2, \ldots$, P_{-1}, $P_{-2}, \ldots$ are represented by $z+n\omega$ for positive and negative integral values of n. But $f(z+n\omega)=f(z)$; and therefore the value of the function at a point P_n in any of the bands is

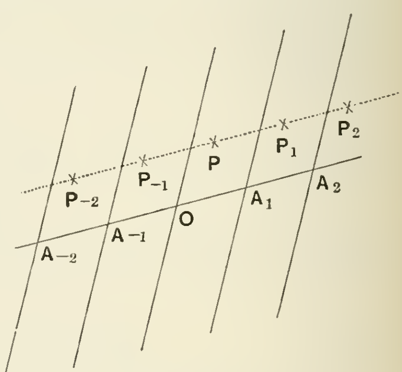

Fig. 30.

the same as the value at P. Moreover to a point in any of the bands there corresponds a point in any other of the bands; and therefore, owing to the periodic resumption of the value at the points corresponding to each point P, *it is sufficient to consider the variation of the function for points within one band,* say the band between the lines through O and through A_1. A point P within the band is sometimes called *irreducible,* the corresponding points P in the other bands *reducible.*

If it were convenient, the boundary lines of the bands could be taken through points other than $A_1, A_2, \ldots$; for example, through points $(m+\frac{1}{2})\,\omega$ for positive and negative integral values of m. Moreover, they need not be *straight* lines. The essential feature of the graphic representation is the division of the plane into bands.

(ii) For doubly-periodic functions a similar method is adopted. Let ω and ω' be the two periods of such a function $f(z)$, so that

$$f(z+\omega)=f(z)=f(z+\omega');$$

then $$f(z+n\omega+n'\omega')=f(z),$$

where n and n' are any integers positive or negative.

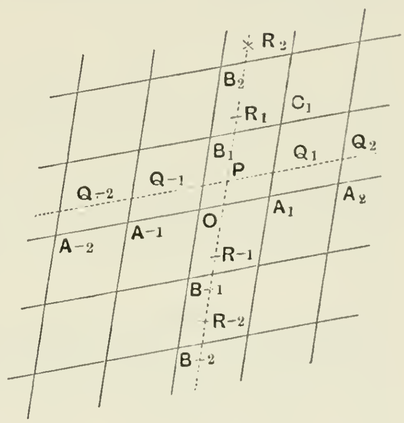

Fig. 31.

For graphic purposes, we take points $O, A_1, A_2, \ldots, A_{-1}, A_{-2}, \ldots$ representing $0, \omega, 2\omega, \ldots, -\omega, -2\omega, \ldots$; and we take another series $O, B_1, B_2, \ldots, B_{-1}, B_{-2}, \ldots$ representing $0, \omega', 2\omega', \ldots, -\omega', -2\omega', \ldots$; through the points A we draw lines parallel to the line of points B, and through the points B we draw lines parallel to the line of points A. The intersection of the lines through A_n and $B_{n'}$ is evidently the point $n\omega+n'\omega'$, that is, the angular points of the parallelograms into which the plane is divided represent the points $n\omega+n'\omega'$ for the values of n and n'.

Let P be any point z in the parallelogram $OA_1C_1B_1$; on lines through P, parallel to the sides of the parallelogram, take points $Q_1, Q_2, \ldots, Q_{-1}, Q_{-2}, \ldots$ such that $PQ_1=Q_1Q_2=\ldots=\omega$ and points $R_1, R_2, \ldots, R_{-1}, R_{-2}, \ldots$ such that $PR_1=R_1R_2=\ldots=\omega'$; and through these new points draw lines parallel to the sides of the parallelogram. Then the variables of the points in which these lines intersect are all represented by $z+m\omega+m'\omega'$ for positive and negative integral values of m and m'; and the point represented by $z+m_1\omega+m'\omega'$ is situated in the parallelogram, the angular points of which are $m\omega+m'\omega'$, $(m+1)\,\omega+m'\omega'$, $m\omega+(m'+1)\,\omega'$, and $(m+1)\,\omega+(m'+1)\,\omega'$, exactly as P is situated in $OA_1C_1B_1$. But

$$f(z+m_1\omega+m_1'\omega')=f(z),$$

and therefore the value of the function at such a point is the same as the value at P. Since the parallelograms are all equal and similarly situated, to any point in any of them there corresponds a point in $OA_1C_1B_1$; and the value of the function at the two points is the same. Hence *it is sufficient to consider the variation of the function for points within one parallelogram*, say, that which has 0, ω, $\omega+\omega'$, ω' for its angular points. A point P within this parallelogram is sometimes called *irreducible*, the corresponding points within the other parallelograms *reducible* to P; the whole aggregate of the points thus reducible to any one are called *homologous* points. And the parallelogram to which the reduction is made is called the parallelogram of periods.

As in the case of simply-periodic functions, it may prove convenient to choose the position of the *fundamental parallelogram* so that the origin is not on its boundary; thus it might be the parallelogram the middle points of whose sides are $\pm\frac{1}{2}\omega$, $\pm\frac{1}{2}\omega'$.

106. In the preceding representation it has been assumed that the line of points A is different in direction from the line of points B. If $\omega = u + iv$ and $\omega' = u' + iv'$, this assumption implies that v'/u' is unequal to v/u, and therefore that the real part of $\omega'/i\omega$ does not vanish. The justification of this assumption is established by the proposition, due to Jacobi*:—

The ratio of the periods of a uniform doubly-periodic function cannot be real.

Let $f(z)$ be a function, having ω and ω' as its periods. If the ratio ω'/ω be real, it must be either commensurable or incommensurable.

If it be commensurable, let it be equal to n'/n, where n and n' are integers, neither of which is unity owing to the definition of the periods ω and ω_1.

Let n'/n be developed as a continued fraction, and let m'/m be the last convergent before n'/n, where m and m' are integers. Then

$$\frac{n'}{n} \sim \frac{m'}{m} = \frac{1}{mn},$$

that is,

$$mn' \sim m'n = 1,$$

so that

$$m'\omega \sim m\omega' = \frac{\omega}{n}(m'n \sim mn') = \frac{\omega}{n}.$$

Therefore

$$f(z) = f(z + m'\omega \sim m\omega'),$$

since m and m' are integers; so that

$$f(z) = f\left(z + \frac{\omega}{n}\right),$$

contravening the definition of ω as a period, viz., that no submultiple of ω is a period. Hence the ratio of the periods is not a commensurable real quantity.

* *Ges. Werke*, t. ii, pp. 25, 26.

If it be incommensurable, we express ω'/ω as a continued fraction. Let p/q and p'/q' be two consecutive convergents: their values are separated by the value of ω'/ω, so that we may write

$$\frac{\omega'}{\omega} = \frac{p}{q} + h\left(\frac{p'}{q'} - \frac{p}{q}\right),$$

where $$1 > h > 0.$$

Now $pq' \sim p'q = 1$, so that

$$\frac{\omega'}{\omega} = \frac{p}{q} + \frac{\epsilon}{qq'},$$

where ϵ is real and $|\epsilon| < 1$; hence

$$q\omega' - p\omega = \frac{\epsilon}{q'}\,\omega.$$

Therefore $$f(z) = f(z + q\omega' - p\omega),$$

since p and q are integers; so that

$$f(z) = f\left(z + \frac{\epsilon}{q'}\,\omega\right).$$

Now since ω'/ω is incommensurable, the continued fraction is unending. We therefore take an advanced convergent, so that q' is very large. Then $\frac{\epsilon}{q'}\omega$ is a very small quantity and $z + \frac{\epsilon}{q'}\omega$ is a point infinitesimally near to z, that is, the function $f(z)$, under the present hypothesis, resumes its value at a point infinitesimally near to z. Passing along the line joining these two points infinitesimally near another, we should have $f(z)$ constant along a line and therefore (§ 37) constant everywhere; it would thus cease to be a varying function.

The ratio of the periods is thus not an incommensurable real quantity.

We therefore infer Jacobi's theorem that the ratio of the periods cannot be real. In general, the ratio is a complex quantity; it may, however, be a pure imaginary*.

Corollary. If a uniform function have two periods ω_1 and ω_2 such that a relation

$$m_1\omega_1 + m_2\omega_2 = 0$$

exists for integral values of m_1 and m_2, the function is only simply-periodic. And such a relation cannot exist between two periods of a simply-periodic function, if m_1 and m_2 be real and incommensurable; for then the function would be constant.

* It was proved, in Ex. 5 and Ex. 6 of § 104, that certain uniform functions are doubly-periodic. A direct proof, that the ratio of the distinct periods of the functions there obtained is not a real quantity, is given by Falk, *Acta Math.*, t. vii, (1885), pp. 197—200, and by Pringsheim, *Math. Ann.*, t. xxvii, (1886), pp. 151—157.

Similarly, if a uniform function have three periods ω_1, ω_2, ω_3, connected by two relations

$$m_1\omega_1 + m_2\omega_2 + m_3\omega_3 = 0,$$

$$n_1\omega_1 + n_2\omega_2 + n_3\omega_3 = 0,$$

where the coefficients m and n are integers, then the function is only simply-periodic.

107. The two following propositions, also due to Jacobi*, are important in the theory of uniform periodic functions of a single variable:—

If a uniform function have three periods ω_1, ω_2, ω_3 such that a relation

$$m_1\omega_1 + m_2\omega_2 + m_3\omega_3 = 0$$

is satisfied for integral values of m_1, m_2, m_3, then the function is only a doubly-periodic function.

What has to be proved, in order to establish this proposition, is that two periods exist of which ω_1, ω_2, ω_3 are integral multiple combinations.

Evidently we may assume that m_1, m_2, m_3 have no common factor: let f be the common factor (if any) of m_2 and m_3, which is prime to m_1. Then since

$$\frac{m_1}{f}\omega_1 = -\frac{m_2}{f}\omega_2 - \frac{m_3}{f}\omega_3$$

and the right-hand side is an integral combination of periods, it follows that $\frac{m_1}{f}\omega_1$ is a period.

Now $\frac{m_1}{f}$ is a fraction in its lowest terms. Change it into a continued fraction and let $\frac{p}{q}$ be the last convergent before the proper value; then

$$\frac{m_1}{f} - \frac{p}{q} = \pm\frac{1}{fq}$$

so that

$$q\frac{m_1}{f} - p = \pm\frac{1}{f}.$$

But ω_1 is a period and $\frac{m_1}{f}\omega_1$ is a period; therefore $q\frac{m_1}{f}\omega_1 - p\omega_1$ is a period, or ω_1/f is a period, $= \omega_1'$ say.

Let $m_2/f = m_2'$, $m_3/f = m_3'$, so that $m_1\omega_1' + m_2'\omega_2 + m_3'\omega_3 = 0$. Change m_2'/m_3' into a continued fraction, taking $\frac{r}{s}$ to be the last convergent before the proper value, so that

$$\frac{m_2'}{m_3'} - \frac{r}{s} = \pm\frac{1}{sm_3'}.$$

* *Ges. Werke*, t. ii, pp. 27—32.

Then $r\omega_2 + s\omega_3$, being an integral combination of periods, is a period. But

$$\pm\,\omega_2 = \omega_2(sm_2' - rm_3')$$
$$= -r\omega_2 m_3' - s(m_1\omega_1' + m_3'\omega_3)$$
$$= -m_1 s\omega_1' - m_3'(r\omega_2 + s\omega_3);$$

also
$$\pm\,\omega_3 = \omega_3(sm_2' - rm_3')$$
$$= sm_2'\omega_3 + r(m_1\omega_1' + m_2'\omega_2)$$
$$= m_1 r\omega_1' + m_2'(r\omega_2 + s\omega_3);$$

and
$$\omega_1 = f\omega_1'.$$

Hence two periods ω_1' and $r\omega_2 + s\omega_3$ exist of which ω_1, ω_2, ω_3 are integral multiple combinations; and therefore all the periods are equivalent to ω_1' and $r\omega_2 + s\omega_3$, that is, the function is only doubly-periodic.

Corollary. If a function have four periods ω_1, ω_2, ω_3, ω_4 connected by two relations

$$m_1\omega_1 + m_2\omega_2 + m_3\omega_3 + m_4\omega_4 = 0,$$
$$n_1\omega_1 + n_2\omega_2 + n_3\omega_3 + n_4\omega_4 = 0,$$

where the coefficients m and n are integers, the function is only doubly-periodic.

108. *If a uniform function of one variable have three periods ω_1, ω_2, ω_3, then a relation of the form*

$$m_1\omega_1 + m_2\omega_2 + m_3\omega_3 = 0$$

must be satisfied for some integral values of m_1, m_2, m_3.

Let $\omega_r = \alpha_r + i\beta_r$, for $r = 1, 2, 3$; in consequence of § 106, we shall assume that no one of the ratios of ω_1, ω_2, ω_3 in pairs is real, for, otherwise, either the three periods reduce to two immediately, or the function is a constant. Then, determining two quantities λ and μ by the equations

$$\alpha_3 = \lambda\alpha_1 + \mu\alpha_2,\quad \beta_3 = \lambda\beta_1 + \mu\beta_2,$$

so that λ and μ are real quantities and neither zero nor infinity, we have

$$\omega_3 = \lambda\omega_1 + \mu\omega_2,$$

for real values of λ and μ.

Then, first, if either λ or μ be commensurable, the other is also commensurable. Let $\lambda = a/b$, where a and b are integers; then

$$b\mu\omega_2 = b\omega_3 - b\lambda\omega_1$$
$$= b\omega_3 - a\omega_1,$$

so that $b\mu\omega_2$ is a period. Now, if $b\mu$ be not commensurable, change it into a continued fraction, and let p/q, p'/q' be two consecutive convergents, so that, as in § 106,

$$b\mu = \frac{p}{q} + \frac{x}{qq'},$$

where $1 > x > -1$. Then $\frac{p}{q}\omega_2 + \frac{x\omega_2}{qq'}$ is a period, and so is ω_2; hence

$$q\left(\frac{p}{q}\omega_2 + \frac{x\omega_2}{qq'}\right) - p\omega_2$$

is a period, that is, $\frac{x}{q'}\omega_2$ is a period. We may take q' indefinitely large, and then the function has an infinitesimal quantity for a period, that is, it would be a constant under the hypothesis. Hence $b\mu$ (and therefore μ) cannot be incommensurable, if λ be commensurable; and thus λ and μ are simultaneously commensurable or simultaneously incommensurable.

If λ and μ be simultaneously commensurable, let $\lambda = \frac{a}{b}$, $\mu = \frac{c}{d}$, so that

$$\omega_3 = \frac{a}{b}\omega_1 + \frac{c}{d}\omega_2,$$

and therefore $$bd\omega_3 = ad\omega_1 + bc\omega_2,$$

a relation of the kind required.

If λ and μ be simultaneously incommensurable, express λ as a continued fraction; then by taking any convergent r/s, we have

$$\lambda - \frac{r}{s} = \frac{x}{s^2},$$

where $1 > x > -1$, so that $$s\lambda - r = \frac{x}{s};$$

by taking the convergent sufficiently advanced the right-hand side can be made infinitesimal.

Let r_1 be the nearest integer to the value of $s\mu$, so that, if

$$s\mu - r_1 = \Delta,$$

we have Δ numerically less than $\frac{1}{2}$. Then

$$s\omega_3 - r\omega_1 - r_1\omega_2 = \frac{x}{s}\omega_1 + \Delta\omega_2,$$

and the quantity $\frac{x}{s}\omega_1$ can be made so small as to be negligible. Hence integers r, r_1, s can be chosen so as to give a new period $\omega_2' (= \Delta\omega_2)$, such that $|\omega_2'| < \frac{1}{2}|\omega_2|$.

We now take $\omega_1, \omega_2', \omega_3$: they will be connected by a relation of the form

$$\omega_3 = \lambda'\omega_1 + \mu'\omega_2',$$

and λ' and μ' must be incommensurable: for otherwise the substitution for ω_2' of its value just obtained would lead to a relation among $\omega_1, \omega_2, \omega_3$ that would imply commensurability of λ and of μ.

Proceeding just as before, we may similarly obtain a new period ω_2'' such that $|\omega_2''| < \frac{1}{2}|\omega_2'|$; and so on in succession. Hence we shall obtain, after n

such processes, a period $\omega_2^{(n)}$ such that $|\omega_2^{(n)}| < \frac{1}{2^n}|\omega_2|$, so that by making n sufficiently large we shall ultimately obtain a period less than any assigned quantity. Let such period be ω; then

$$f(z+\omega) = f(z),$$

and so for points along the ω-line we have an infinite number close together at which the function is unaltered in value. The function, being uniform, must in that case be constant.

It thus appears that, if λ and μ be simultaneously incommensurable, the function is a constant. Hence the only tenable result is that λ and μ are simultaneously commensurable, and then there is a period-equation of the form

$$m_1\omega_1 + m_2\omega_2 + m_3\omega_3 = 0,$$

where m_1, m_2, m_3 are integers.

The foregoing proof is substantially due to Jacobi (l.c.). The result can be obtained from geometrical considerations by shewing that the infinite number of points, at which the function resumes its value, along a line through z parallel to the ω_3-line will, unless the condition be satisfied, reduce to an infinite number of points in the ω_1, ω_2 parallelogram which will form either a continuous line or a continuous area, in either of which cases the function would be a constant. But, if the condition be satisfied, then the points along the line through z reduce to only a finite number of points.

COROLLARY I. Uniform functions of a single variable cannot have three independent periods; in other words, *triply-periodic uniform functions of a single variable do not exist*; and, a fortiori, uniform functions of a single variable with a number of independent periods greater than two do not exist.*

But functions involving more than one variable can have more than two periods, e.g., Abelian transcendents; and a function of one variable, having more than two periods, is not uniform.

COROLLARY II. *All the periods of a uniform periodic function of a single variable reduce either to integral multiples of one period or to linear combinations of integral multiples of two periods whose ratio is not a real quantity.*

109. It is desirable to have the parallelogram, in which a doubly-periodic function is considered, as small as possible. If in the parallelogram (supposed, for convenience, to have the origin for an angular point) there be a point ω'' such that

$$f(z+\omega'') = f(z)$$

for all values of z, then the parallelogram can be replaced by another.

* This theorem is also due to Jacobi, (l.c., p. 202, note).

It is evident that ω'' is a period of the function; hence (§ 108) we must have

$$\omega''=\lambda\omega+\mu\omega';$$

and both λ and μ, which are commensurable quantities, are less than unity since the point is within the parallelogram. Moreover, $\omega+\omega'-\omega''$, which is equal to $(1-\lambda)\,\omega+(1-\mu)\,\omega'$, is another point within the parallelogram; and

$$f(z+\omega+\omega'-\omega'')=f(z),$$

since ω, ω', ω'' are periods. Thus there cannot be a single such point, unless

$$\lambda=\tfrac{1}{2}=\mu.$$

But the number of such points within the parallelogram must be finite; if there were an infinite number, they would form a continuous line or a continuous area where the uniform function had an unvarying value, and consequently (§ 37) the function would have a constant value everywhere.

To construct a new parallelogram when all the points are known, we first choose the series of points parallel to the ω-line through the origin O, and of that series we choose the point nearest O, say A_1. We similarly choose the point, nearest the origin, of the series of points parallel to the ω-line and nearest to it after the series that includes A_1, say B_1: we take OA_1, OB_1 as adjacent sides of the parallelogram, and these lines as the vectorial representations of the periods. No point lies within this parallelogram where the function has the same value as at O; hence the angular points of the original parallelograms coincide with angular points of the new parallelograms.

When a parallelogram has thus been obtained, containing no internal point Ω such that the function can satisfy the equation

$$f(z+\Omega)=f(z)$$

for all values of z, it is called a *fundamental*, or a *primitive, parallelogram*: and the parallelogram of reference in subsequent investigations will be assumed to be of a fundamental character.

But *a fundamental parallelogram is not unique.*

Let ω and ω' be the periods for a given fundamental parallelogram, so that every other period ω'' is of the form $\lambda\omega+\mu\omega'$, where λ and μ are integers. Take any four integers a, b, c, d such that $ad-bc=\pm 1$, as may be done in an infinite variety of ways; and adopt two new periods ω_1 and ω_2, such that

$$\omega_1=a\omega+b\omega',\qquad \omega_2=c\omega+d\omega'.$$

Then the parallelogram with ω_1 and ω_2 for adjacent sides is fundamental. For we have

$$\pm\,\omega=d\omega_1-b\omega_2,\qquad \pm\,\omega'=-\,c\omega_1+a\omega_2,$$

and therefore any period ω''

$$=\lambda\omega+\mu\omega'$$
$$=(\lambda d-\mu c)\,\omega_1+(-\,\lambda b+\mu a)\,\omega_2,\text{ save as to signs of }\lambda\text{ and }\mu.$$

The coefficients of ω_1 and ω_2 are integers, that is, the point ω'' lies outside the new parallelogram of reference; there is therefore no point in it such that

$$f(z+\omega'')=f(z),$$

and hence the parallelogram is fundamental.

COROLLARY. *The aggregate of the angular points in one division of the plane into fundamental parallelograms coincides with their aggregate in any other division into fundamental parallelograms; and all fundamental parallelograms for a given function are of the same area.*

The method suggested above for the construction of a fundamental parallelogram is geometrical, and it assumes a knowledge of all the points ω'' within a given parallelogram for which the equation $f(z+\omega'')=f(z)$ is satisfied.

Such a point ω_3 within the ω_1, ω_2 parallelogram is given by

$$\omega_3=\frac{m_1}{m_3}\omega_1+\frac{m_2}{m_3}\omega_2,$$

where m_1, m_2, m_3 are integers. We may assume that no two of these three integers have a common factor; were it otherwise, say for m_1 and m_2, then, as in § 107, a submultiple of ω_3 would be a period—a result which may be considered as excluded. Evidently all the points in the parallelogram are the reduced points homologous with ω_3, $2\omega_3$, ..., $(m_3-1)\omega_3$; when these are obtained, the geometrical construction is possible.

The following is a simple and practicable analytical method for the construction.

Change m_1/m_3 and m_2/m_3 into continued fractions; and let p/q and r/s be the last convergents before the respective proper values, so that

$$\frac{m_1}{m_3}-\frac{p}{q}=\frac{\epsilon}{qm_3},\qquad \frac{m_2}{m_3}-\frac{r}{s}=\frac{\epsilon'}{sm_3},$$

where ϵ and ϵ' are each of them ± 1. Let

$$q\frac{m_2}{m_3}=\theta+\frac{\mu}{m_3},\qquad s\frac{m_1}{m_3}=\phi+\frac{\lambda}{m_3},$$

where λ and μ are taken to be less than m_3, but they do not vanish because q and s are less than m_3. Then

$$q\omega_3-p\omega_1-\theta\omega_2=\frac{1}{m_3}(\mu\omega_2+\epsilon\omega_1),\qquad s\omega_3-r\omega_2-\phi\omega_1=\frac{1}{m_3}(\lambda\omega_1+\epsilon'\omega_2);$$

the left-hand sides are periods, say Ω_1 and Ω_2 respectively, and since $\mu+\epsilon$ is not $>m_3$ and $\lambda+\epsilon'$ is not $>m_3$, the points Ω_1 and Ω_2 determine a parallelogram smaller than the initial parallelogram.

Thus
$$\epsilon\omega_1+\mu\omega_2=m_3\Omega_1,\qquad \lambda\omega_1+\epsilon'\omega_2=m_3\Omega_2,$$
are equations defining new periods Ω_1, Ω_2. Moreover

$$\phi+\frac{\lambda}{m_3}=s\frac{m_1}{m_3}=s\frac{p}{q}+\frac{\epsilon s}{qm_3},\qquad \theta+\frac{\mu}{m_3}=q\frac{m_2}{m_3}=q\frac{r}{s}+\frac{\epsilon' q}{sm_3}:$$

so that, multiplying the right-hand sides together and likewise the left-hand sides, we at once see that $\lambda\mu-\epsilon\epsilon'$ is divisible by m_3 if it be not zero: let

$$\lambda\mu-\epsilon\epsilon'=m_3\Delta.$$

Then, as λ and μ are less than m_3, they are greater than Δ; and they are prime to it, because $\epsilon\epsilon'$ is ± 1.

Hence we have $\Delta\omega_1 = \mu\Omega_2 - \epsilon'\Omega_1, \quad \Delta\omega_2 = \lambda\Omega_1 - \epsilon\Omega_2.$

Since λ and μ are both greater than Δ, let

$$\lambda = \lambda_1\Delta + \lambda', \qquad \mu = \mu_1\Delta + \mu',$$

where λ' and μ' are $<\Delta$. Then $\lambda'\mu' - \epsilon\epsilon'$ is divisible by Δ if it be not zero, say

$$\lambda'\mu' - \epsilon\epsilon' = \Delta\Delta';$$

then λ' and μ' are $>\Delta'$ and are prime to it. And now

$$\Delta(\omega_1 - \mu_1\Omega_2) = \mu'\Omega_2 - \epsilon'\Omega_1, \qquad \Delta(\omega_2 - \lambda_1\Omega_1) = \lambda'\Omega_1 - \epsilon\Omega_2;$$

and therefore, if $\omega_1 - \mu_1\Omega_2 = \Omega_3$, $\omega_2 - \lambda_1\Omega_1 = \Omega_4$, which are periods, we have

$$\Delta\Omega_3 = \mu'\Omega_2 - \epsilon'\Omega_1, \qquad \Delta\Omega_4 = \lambda'\Omega_1 - \epsilon\Omega_2.$$

With Ω_3 and Ω_4 we can construct a parallelogram smaller than that constructed with Ω_1 and Ω_2.

We now have $\Delta'\Omega_1 = \epsilon\Omega_3 + \mu'\Omega_4, \quad \Delta'\Omega_2 = \lambda'\Omega_3 + \epsilon'\Omega_4,$

that is, equations of the same form as before. We proceed thus in successive stages: each quantity Δ thus obtained is distinctly less than the preceding Δ, and so finally we shall reach a stage when the succeeding Δ would be unity, that is, the solution of the pair of equations then leads to periods that determine a fundamental parallelogram. It is not difficult to prove that ω_1, ω_2, ω_3 are combinations of integral multiples of these periods.

If one of the quantities, such as $\lambda'\mu' - \epsilon\epsilon'$, be zero, then $\lambda' = \mu' = 1$, $\epsilon = \epsilon' = \pm 1$; and then Ω_3 and Ω_4 are identical. If $\epsilon = \epsilon' = +1$, then $\Delta\Omega_3 = \Omega_2 - \Omega_1$, and the fundamental parallelogram is determined by

$$\Omega_3' = \Omega_1 + \frac{1}{\Delta}(\Omega_2 - \Omega_1), \qquad \Omega_4' = \Omega_2 - \frac{1}{\Delta}(\Omega_2 - \Omega_1).$$

If $\epsilon = \epsilon' = -1$, then $\Delta\Omega_3 = \Omega_2 + \Omega_1$, so that, as Δ is not unity in this case, the fundamental parallelogram is determined by Ω_2 and Ω_3.

Ex. If a function be periodic in ω_1, ω_2, and also in ω_3 where

$$29\omega_3 = 17\omega_1 + 11\omega_2,$$

periods for a fundamental parallelogram are

$$\Omega_1' = 5\omega_1 + 3\omega_2 - 8\omega_3, \qquad \Omega_2' = 3\omega_1 + 2\omega_2 - 5\omega_3,$$

and the values of ω_1, ω_2, ω_3 in terms of Ω_1' and Ω_2' are

$$\omega_1 = \Omega_2' + 3\Omega_1', \qquad \omega_2 = 9\Omega_2' - 2\Omega_1', \qquad \omega_3 = 4\Omega_2' + \Omega_1'.$$

Further discussion relating to the transformation of periods and of fundamental parallelograms will be found in Briot and Bouquet's *Théorie des fonctions elliptiques*, pp. 234, 235, 268—272.

110. It has been proved that uniform periodic functions of a single variable cannot have more than two periods, independent in the sense that their ratio is not a real quantity. If then a function exist, which has two periods with a real incommensurable ratio or has more than two independent periods, either it is not uniform or it is a function (whether uniform or multiform) of more variables than one.

When restriction is made to uniform functions, the only alternative is that the function should depend on more than one variable.

In the case when three periods $\omega_1, \omega_2, \omega_3$ (each of the form $\alpha + i\beta$) were assigned, it was proved that the necessary condition for the existence of a uniform function of a single variable is that finite integers m_1, m_2, m_3 can be found such that

$$m_1\alpha_1 + m_2\alpha_2 + m_3\alpha_3 = 0,$$

$$m_1\beta_1 + m_2\beta_2 + m_3\beta_3 = 0\,;$$

and that, if these conditions be not satisfied, then finite integers m_1, m_2, m_3 can be found such that both $\Sigma m\alpha$ and $\Sigma m\beta$ become infinitesimally small.

This theorem is purely algebraical, and is only a special case of a more general theorem as follows:

Let $\alpha_{11}, \alpha_{12},\ldots, \alpha_{1, r+1}\,; \alpha_{21}, \alpha_{22},\ldots, \alpha_{2, r+1}\,; \ldots\,; \alpha_{r1}, \alpha_{r2},\ldots, \alpha_{r, r+1}$ *be r sets of real quantities such that a relation of the form*

$$n_1\alpha_{s1} + n_2\alpha_{s2} + \ldots + n_{r+1}\,\alpha_{sr+1} = 0$$

is not satisfied among any one set. Then finite integers $m_1,\ldots, m_{r+1}$ can be determined such that each of the sums

$$m_1\alpha_{s1} + m_2\alpha_{s2} + \ldots + m_{r+1}\,\alpha_{s,\, r+1}$$

(*for* $s = 1, 2,\ldots, r$) *is an infinitesimally small quantity.* And, *a fortiori*, if fewer than r sets, each containing $r+1$ quantities be given, the $r+1$ integers can be determined so as to lead to the result enunciated; all that is necessary for the purpose being an arbitrary assignment of sets of real quantities necessary to make the number of sets equal to r. But the result is not true if more than r sets be given.

We shall not give a proof of this general theorem*; it would follow the lines of the proof in the limited case, as given in § 108. But the theorem will be used to indicate how the value of an integral with more than two periods is affected by the periodicity.

Let I be the value of the integral taken along some assigned path from an initial point z_0 to a final point z; and let the periods be $\omega_1, \omega_2,\ldots, \omega_r$, (where $r > 2$), so that the general value is

$$I + m_1\omega_1 + m_2\omega_2 + \ldots + m_r\omega_r,$$

where $m_1, m_2,\ldots, m_r$ are integers. Now if $\omega_s = \alpha_s + i\beta_s$, for $s = 1, 2,\ldots, r$, when it is divided into its real and its imaginary parts, then finite integers $n_1, n_2,\ldots, n_r$ can be determined such that

$$n_1\alpha_1 + n_2\alpha_2 + \ldots + n_r\alpha_r$$

and

$$n_1\beta_1 + n_2\beta_2 + \ldots + n_r\beta_r$$

are both infinitesimal; and then $\left|\sum_{s=1}^{r} n_s\omega_s\right|$ is infinitesimal. But the addition of $\sum_{s=1}^{r} n_s\omega_s$ still gives a value of the integral; hence the value can be modified

* A proof will be found in Clebsch and Gordan's *Theorie der Abel'schen Functionen*, § 38.

by infinitesimal quantities, and the modification can be repeated indefinitely. The modifications of the value correspond to modifications of the path from z_0 to z; and hence the integral, regarded as depending on a single variable, can be made, by modifications of the path of the variable, to assume any value. The integral, in fact, has not a definite value dependent solely upon the final value of the variable; to make the value definite, the path by which the variable passes from the lower to the upper limit must be specified.

It will subsequently (§ 239) be shewn how this limitation is avoided by making the integral, regarded as a function, depend upon a proper number of independent variables—the number being greater than unity.

Ex. 1. If V_0 be the value of $\int_0^z \frac{dz}{(1-z^n)^{\frac{1}{2}}}$, ($n$ integral), taken along an assigned path, and if

$$P = 2\int_0^1 \frac{dx}{(1-x^n)^{\frac{1}{2}}} \quad (x \text{ real}),$$

then the general value of the integral is

$$(-1)^q V_0 + P\left[\tfrac{1}{2}\{1-(-1)^q\} + \sum_{p=1}^{n} m_p e^{\frac{2p\pi i}{n}}\right],$$

where q is any integer and m_p any positive or negative integer such that $\sum_{p=1}^{n} m_p = 0$.

(Math. Trip. Part II, 1889.)

Ex. 2. Prove that $v = \int_0^z u\,dz$, where

$$u^3 - 3zu + z^3 = 0,$$

is an algebraical function satisfying the equation

$$8(v+\tfrac{3}{2})^3 - 12(v+\tfrac{3}{2})^2 - 12z^3(v+\tfrac{3}{2}) + z^6 + 16z^3 = 0;$$

and obtain the conditions necessary and sufficient to ensure that

$$v = \int u\,dz$$

should be an algebraical function, when u is an algebraical function satisfying an equation

$$f(z, u) = 0.$$

(Liouville, Briot and Bouquet.)

CHAPTER X.

Simply-Periodic and Doubly-Periodic Functions.

111. Only a few of the properties of simply-periodic functions will be given, partly because some of them are connected with Fourier's series the detailed discussion of which lies beyond our limits, and partly because, as will shortly be explained, many of them can at once be changed into properties of uniform non-periodic functions which have already been considered.

When we use the graphical method of § 105, it is evident that we need consider the variation of the function within only a single band. Within that band any function must have at least one infinity, for, if it had not, it would not have an infinity anywhere in the plane and so would be a constant; and it must have at least one zero, for, if it had not, its reciprocal, also a simply-periodic function, would not have an infinity in the band. The infinities may, of course, be accidental or essential: their character is reproduced at the homologous points in all the bands.

For purposes of analytical representation, it is convenient to use a relation

$$Z = e^{\frac{2\pi i}{\omega} z},$$

so that, if the point Z in its plane have R and Θ for polar coordinates,

$$z = \frac{\omega}{2\pi i} \log R + \frac{\Theta}{2\pi} \omega.$$

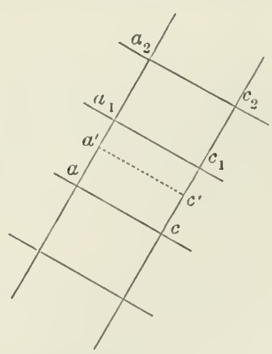

Fig. 32.

If we take any point A in the Z-plane and a corresponding point a in the z-plane, then, as Z describes a complete circle through A with the origin as centre, z moves along a line aa_1, where a_1 is $a + \omega$. A second description of the circle makes z move from a_1 to a_2, where $a_2 = a_1 + \omega$; and so on in succession.

For various descriptions, positive and negative, the point a describes a line, the inclination of which to the axis of real quantities is the argument of ω.

Instead of making Z describe a circle through A, let us make it describe a part of the straight line from the origin through A, say from A, where $OA = R$, to C, where $OC = R'$. Then z describes a line through a perpendicular to aa_1, and it moves to c where

$$c - a = \frac{\omega}{2\pi i}(\log R' - \log R).$$

Similarly, if any point A' on the former circumference move radially to a point C at a distance R' from the Z-origin, the corresponding z point a' moves through a distance $a'c'$, parallel and equal to ac: and all the points c' lie on a line parallel to aa_1. Repeated description of a Z-circumference with the origin as centre makes z describe the whole line cc_1c_2.

If then a function be simply-periodic in ω, we may conveniently take any point a, and another point $a_1 = a + \omega$, through a and a_1 draw straight lines perpendicular to aa_1, and then consider the function within this band. The aggregate of points within this band is obtained by taking

(i) all points along a straight line, perpendicular to a boundary of the band, as aa_1;

(ii) the points along all straight lines, which are drawn through the points of (i) parallel to a boundary of the band.

In (i), the value of z varies from 0 to ω in an expression $a + z$, that is, in the Z-plane for a given value of R, the angle Θ varies from 0 to 2π.

In (ii), the value of $\log R$ varies from $-\infty$ to $+\infty$ in an expression $\frac{\omega}{2\pi i}\log R + \frac{\Theta}{2\pi}\omega$, that is, the radius R must vary from 0 to ∞.

Hence the band in the z-plane and the whole of the Z-plane are made equivalent to one another by the transformation

$$Z = e^{\frac{2\pi i}{\omega} z}.$$

Now let z_0 be any special point in the finite part of the band for a given simply-periodic function, and let Z_0 be the corresponding point in the Z-plane. Then for points z in the immediate vicinity of z_0 and for points Z which are consequently in the immediate vicinity of Z_0, we have

$$\begin{aligned} Z - Z_0 &= e^{\frac{2\pi i}{\omega} z} - e^{\frac{2\pi i}{\omega} z_0} \\ &= e^{\frac{2\pi i}{\omega} z_0}\left\{e^{\frac{2\pi i}{\omega}(z - z_0)} - 1\right\} \\ &= \lambda \frac{2\pi i}{\omega} e^{\frac{2\pi i z_0}{\omega}} (z - z_0), \end{aligned}$$

where $|\lambda|$ differs from unity only by an infinitesimal quantity.

If then w, a function of z, be changed into W a function of Z, the following relations subsist:—

When a point z_0 is a zero of w, the corresponding point Z_0 is a zero of W.

When a point z_0 is an accidental singularity of w, the corresponding point Z_0 is an accidental singularity of W.

When a point z_0 is an essential singularity of w, the corresponding point Z_0 is an essential singularity of W.

When a point z_0 is a branch-point of any order for a function w, the corresponding point Z_0 is a branch-point of the same order for W.

And the converses of these relations also hold.

Since the character of any finite critical point for w is thus unchanged by the transformation, it is often convenient to change the variable to Z so as to let the variable range over the whole plane, in which case the theorems already proved in the preceding chapters are applicable. But special account must be taken of the point $z = \infty$.

112. We can now apply Laurent's theorem to deduce what is practically Fourier's series, as follows.

Let $f(z)$ be a simply-periodic function having ω as its period, and suppose that in a portion of the z-plane bounded by any two parallel lines, the inclination of which to the axis of real quantities is equal to the argument of ω, the function is uniform and has no singularities; then, at points within that portion of the plane, the function can be expressed in the form of a converging series of positive and of negative integral powers of $e^{\frac{2\pi z i}{\omega}}$.

In figure 32, let $aa_1a_2\ldots$ and $cc_1c_2\ldots$ be the two lines which bound the portion of the plane: the variations of the function will all take place within that part of the portion of the plane which lies within one of the representative bands, say within the band bounded by $\ldots ac\ldots$ and $\ldots a_1c_1\ldots$: that is, we may consider the function within the rectangle acc_1a_1a, where it has no singularities and is uniform.

Now the rectangle acc_1a_1a in the z-plane corresponds to a portion of the Z-plane which, after the preceding explanation, is bounded by two circles with the origin for common centre and of radii $\left|e^{\frac{2\pi i}{\omega}z_a}\right|$ and $\left|e^{\frac{2\pi i}{\omega}z_c}\right|$; and the variations of the function within the rectangle are given by the variations of a transformed function within the circular ring. The characteristics of the one function at points in the rectangle are the same as the characteristics of the other at points in the circular ring: and therefore, from the character of the assigned function, the transformed function has no singularities and it is uniform within the circular ring. Hence, by Laurent's Theorem (§ 28), the transformed function is expressible in the form

$$F(Z) = \sum_{n=-\infty}^{n=+\infty} a_n Z^n,$$

a series which converges within the ring: and the value of the coefficient a_n is given by

$$\frac{1}{2\pi i}\int \frac{F(Z)}{Z^{n+1}}\, dZ,$$

taken along any circle in the ring concentric with the boundaries.

Retransforming to the variable z, the expression for the original function is

$$f(z) = \sum_{n=-\infty}^{n=+\infty} a_n e^{\frac{2n\pi i z}{\omega}}.$$

The series converges for points within the rectangle and therefore, as it is periodic, it converges within the portion of the plane assigned. And the value of a_n is

$$a_n = \frac{1}{\omega}\int f(z)\, e^{-\frac{2n\pi i z}{\omega}}\, dz,$$

taken along a path which is the equivalent of any circle in the ring concentric with the boundaries, that is, along any line $a'c'$ perpendicular to the lines which bound the assigned portion of the plane.

The expression of the function can evidently be changed into the form

$$f(z) = \frac{1}{\omega}\int \sum_{n=-\infty}^{n=\infty} e^{\frac{2n\pi i}{\omega}(z-\zeta)} f(\zeta)\, d\zeta,$$

where the integral is taken along the piece of a line, perpendicular to the boundaries and intercepted between them.

If one of the boundaries of the portion of the plane be at infinity, (so that the periodic function has no singularities within one part of the plane), then the corresponding portion of the Z-plane is either the part within or the part without a circle, centre the origin, according as the one or the other of the boundaries is at ∞. In the former case, the terms with negative indices n are absent; in the latter, the terms with positive indices are absent.

113. On account of the consequences of the relation subsisting between the variables z and Z, many of the propositions relating to general uniform functions, as well as of those relating to multiform functions, can be changed, merely by the transformation of the variables, into propositions relating to simply-periodic functions. One such proposition occurs in the preceding section; the following are a few others, the full development being unnecessary here, in consequence of the foregoing remark. The band of reference for the simply-periodic functions considered will be supposed to include the origin: and, when any point is spoken of, it is that one of the series of homologous points in the plane, which lies in the band.

We know that, if a uniform function of Z have no essential singularity, then it is a rational algebraical function, which is integral if $z=\infty$ be the only accidental singularity and is meromorphic if there be accidental singularities in the finite part of the plane; and every such function has as many zeros as it has accidental singularities.

Hence *a uniform simply-periodic function with $z=\infty$ as its sole essential singularity has as many zeros as it has infinities in each band of the plane; the number of points at which it assumes a given value is equal to the number of its zeros; and, if the period be ω, the function is a rational algebraical function of $e^{\frac{2\pi iz}{\omega}}$, which is integral if all the singularities be at an infinite distance and is meromorphic if some (or all) of them be in a finite part of the plane.* But any number of the zeros and any number of the infinities may be absorbed in the essential singularity at $z=\infty$.

The simplest function of Z, thus restricted to have the same number of zeros as of infinities, is one which has a single zero and a single infinity in the finite part of the plane; the possession of a single zero and a single infinity will therefore characterise the most elementary simply-periodic function. Now, bearing in mind the relation

$$Z=e^{\frac{2\pi iz}{\omega}},$$

the simplest z-point to choose for a zero is the origin, so that $Z=1$; and then the simplest z-point to choose for an infinity at a finite distance is $\frac{1}{2}\omega$, (being half the period), so that $Z=-1$. The expression of the function in the Z-plane with 1 for a zero and -1 for an accidental singularity is

$$A\frac{Z-1}{Z+1},$$

and therefore assuming as the most elementary simply-periodic function that which in the plane has a series of zeros and a series of accidental singularities all of the first order, the points of the one being midway between those of the other, its expression is

$$A\frac{e^{\frac{2\pi iz}{\omega}}-1}{e^{\frac{2\pi iz}{\omega}}+1},$$

which is a constant multiple of $\tan\frac{\pi z}{\omega}$. Since $e^{\frac{2\pi iz}{\omega}}$ is a rational fractional function of $\tan\frac{\pi z}{\omega}$, part of the foregoing theorem can be re-stated as follows:—

If the period of the function be ω, the function is a rational algebraical function of $\tan\frac{\pi z}{\omega}$.

Moreover, in the general theory of uniform functions, it was found convenient to have a simple element for the construction of products, there (§ 53) called a primary factor: it was of the type

$$\left(\frac{Z-a}{Z-c}\right) e^{G\left(\frac{1}{Z-c}\right)},$$

where the function $G\left(\frac{1}{Z-c}\right)$ could be a constant; and it had only one infinity and one zero.

Hence for simply-periodic functions we may regard $\tan\frac{\pi z}{\omega}$ as a typical primary factor when the number of irreducible zeros and the (equal) number of irreducible accidental singularities are finite. If these numbers should tend to an infinite limit, then an exponential factor might have to be associated with $\tan\frac{\pi z}{\omega}$; and the function in that case might have essential singularities elsewhere than at $z=\infty$.

114. We can now prove that *every uniform function, which has no essential singularities in the finite part of the plane and is such that all its accidental singularities and its zeros are arranged in groups equal and finite in number at equal distances along directions parallel to a given direction, is a simply-periodic function.*

Let ω be the common period of the groups of zeros and of singularities: and let the plane be divided into bands by parallel lines, perpendicular to any line representing ω. Let $a, b, \ldots$ be the zeros, $\alpha, \beta, \ldots$ the singularities in any one band.

Take a uniform function $\phi(z)$, simply-periodic in ω and having a single zero and a single singularity in the band: we might take $\tan\frac{\pi z}{\omega}$ as a value of $\phi(z)$. Then

$$\frac{\phi(z)-\phi(a)}{\phi(z)-\phi(\alpha)}$$

is a simply-periodic function having only a single zero, viz., $z=a$ and a single singularity, viz., $z=\alpha$; for as $\phi(z)$ has only a single zero, there is only a single point for which $\phi(z)=\phi(a)$, and a single point for which $\phi(z)=\phi(\alpha)$. Hence

$$\frac{\{\phi(z)-\phi(a)\}\{\phi(z)-\phi(b)\}\ldots}{\{\phi(z)-\phi(\alpha)\}\{\phi(z)-\phi(\beta)\}\ldots}$$

is a simply-periodic function with all the zeros and with all the infinities of the given function within the band. But on account of its periodicity it has all the zeros and all the infinities of the given function over the whole plane; hence its quotient by the given function has no zero and no singularity over the whole plane and therefore it is a constant; that is, the given function, save as to a constant factor, can be expressed in the foregoing form. It is thus a simply-periodic function.

This method can evidently be used to construct simply-periodic functions, having assigned zeros and assigned singularities. Thus if a function have $a+m\omega$ as its zeros and $c+m'\omega$ as its singularities, where m and m' have all integral values from $-\infty$ to $+\infty$, the simplest form is obtained by taking a constant multiple of

$$\frac{\tan\frac{\pi z}{\omega}-\tan\frac{\pi a}{\omega}}{\tan\frac{\pi z}{\omega}-\tan\frac{\pi c}{\omega}}.$$

Ex. Construct a function, simply-periodic in ω, having zeros given by $(m+\frac{1}{2})\omega$ and $(m+\frac{3}{4})\omega$ and singularities by $(m+\frac{1}{3})\omega$ and $(m+\frac{2}{3})\omega$.

The irreducible zeros are $\frac{1}{2}\omega$ and $\frac{3}{4}\omega$; the irreducible singularities are $\frac{1}{3}\omega$ and $\frac{2}{3}\omega$. Now

$$A'\frac{\left(\tan\frac{\pi z}{\omega}-\tan\frac{1}{2}\pi\right)\left(\tan\frac{\pi z}{\omega}-\tan\frac{3}{4}\pi\right)}{\left(\tan\frac{\pi z}{\omega}-\tan\frac{1}{3}\pi\right)\left(\tan\frac{\pi z}{\omega}-\tan\frac{2}{3}\pi\right)}$$

is evidently a function, initially satisfying the required conditions. But, as $\tan\frac{1}{2}\pi$ is infinite, we divide out by it and absorb it into A' as a factor; the function then takes the form

$$A\frac{1+\tan\frac{\pi z}{\omega}}{3-\tan^2\frac{\pi z}{\omega}}.$$

We shall not consider simply-periodic functions, which have essential singularities elsewhere than at $z=\infty$; adequate investigation will be found in the second part of Guichard's memoir, (l.c., p. 147). But before leaving the consideration of the present class of functions, one remark may be made. It was proved, in our earlier investigations, that uniform functions can be expressed as infinite series of functions of the variable and also as infinite products of functions of the variable. This general result is true when the functions in the series and in the products are simply-periodic in the same period. But the function, so represented, though periodic in that common period, may also have another period: and, in fact, many doubly-periodic functions of different kinds (§ 136) are often conveniently expressed as infinite converging series or infinite converging products of simply-periodic functions.

Any detailed illustration of this remark belongs to the theory of elliptic functions: one simple example must suffice.

Let the real part of $\frac{i\pi\omega'}{\omega}$ be negative, and let q denote $e^{\frac{i\pi\omega'}{\omega}}$; then the function

$$\theta(z)=\sum_{n=-\infty}^{n=\infty}(-1)^n q^{n^2} e^{\frac{2ni\pi z}{\omega}},$$

being an infinite converging series of powers of the simply-periodic function $e^{\frac{2i\pi z}{\omega}}$, is finite everywhere in the plane. Evidently $\theta(z)$ is periodic in ω, so that

$$\theta(z+\omega)=\theta(z).$$

Again,
$$\theta(z+\omega')=\sum_{n=-\infty}^{n=\infty}(-1)^n q^{n^2} e^{\frac{2n i\pi(z+\omega')}{\omega}}$$
$$=\sum_{n=-\infty}^{n=\infty}(-1)^n q^{n^2} e^{\frac{2ni\pi z}{\omega}} q^{2n}$$
$$=-\frac{1}{q}e^{-\frac{2i\pi z}{\omega}}\sum_{n=-\infty}^{n=\infty}\left\{(-1)^{n+1} q^{(n+1)^2} e^{\frac{2(n+1)i\pi z}{\omega}}\right\}$$
$$=-\frac{1}{q}e^{-\frac{2i\pi z}{\omega}}\theta(z),$$

the change in the summation so as to give $\theta(z)$ being permissible because the extreme terms for the infinite values of n can be neglected on account of the assumption with regard to q. There is thus a pseudo-periodicity for $\theta(z)$ in a period ω'.

Similarly, if
$$\theta_3(z)=\sum_{n=-\infty}^{n=\infty} q^{n^2} e^{\frac{2ni\pi z}{\omega}},$$
we can prove that
$$\theta_3(z+\omega)=\theta(z),$$
$$\theta_3(z+\omega')=\frac{1}{q}e^{-\frac{2i\pi z}{\omega}}\theta(z).$$

Then $\theta_3(z)\div\theta(z)$ is doubly-periodic in ω and $2\omega'$, though constructed only from functions simply-periodic in ω: it is a function with an infinite number of irreducible accidental singularities in a band.

115. We now pass to doubly-periodic functions of a single variable, the periodicity being additive. The properties, characteristic of this important class of functions, will be given in the form either of new theorems or appropriate modifications of theorems, already established; and the development adopted will follow, in a general manner, the theory given by Liouville*. It will be assumed that the functions are uniform, unless multiformity be explicitly stated, and that all the singularities in the finite part of the plane are accidental†.

The geometrical representation of double-periodicity, explained in § 105, will be used concurrently with the analysis; and the parallelogram of periods, to which the variable argument of the function is referred, is a fundamental parallelogram (§ 109) with periods‡ 2ω and $2\omega'$. An angular point z_0 for the parallelogram of reference can be chosen so that neither a zero nor a pole of the function lies on the perimeter; for the number of zeros and the number of poles in any finite area must be finite, as otherwise they would form a continuous line or a continuous area, or they would be in the vicinity of an essential singularity. This choice will, in

* In his lectures of 1847, edited by Borchardt and published in *Crelle*, t. lxxxviii, (1880), pp. 277—310. They are the basis of the researches of Briot and Bouquet, the most complete exposition of which will be found in their *Théorie des fonctions elliptiques*, (2nd ed.), pp. 239—280.

† For doubly-periodic functions, which have essential singularities, reference should be made to Guichard's memoir, (the introductory remarks and the third part), already quoted on p. 147, *note*.

‡ The factor 2 is introduced merely for the sake of convenience.

general, be made; but, in particular cases, it is convenient to have the origin as an angular point of the parallelogram and then it not infrequently occurs that a zero or a pole lies on a side or at a corner. If such a point lie on a side, the homologous point on the opposite side is assigned to the parallelogram which has that opposite side as homologous; and if it be at an angular point, the remaining angular points are assigned to the parallelograms which have them as homologous corners.

The parallelogram of reference will therefore, in general, have z_0, $z_0 + 2\omega$, $z_0 + 2\omega'$, $z_0 + 2\omega + 2\omega'$ for its angular points; but occasionally it is desirable to take an equivalent parallelogram having $z_0 \pm \omega \pm \omega'$ as its angular points.

When the function is denoted by $\phi(z)$, the equations indicating the periodicity are

$$\phi(z + 2\omega) = \phi(z) = \phi(z + 2\omega').$$

116. We now proceed to the fundamental propositions relating to doubly-periodic functions.

I. *Every doubly-periodic function must have zeros and infinities within the fundamental parallelogram.*

For the function, not being a constant, has zeros somewhere in the plane and it has infinities somewhere in the plane; and, being doubly-periodic, it experiences within the parallelogram all the variations that it can have over the plane.

COROLLARY. *The function cannot be a rational integral function of z.*

For within a parallelogram of finite dimensions an integral function has no infinities and therefore cannot represent a doubly-periodic function.

An analytical form for $\phi(z)$ can be obtained which will put its singularities in evidence. Let a be such a pole, of multiplicity n; then we know that, as the function is uniform, coefficients A can be determined so that the function

$$\phi(z) - \frac{A_n}{(z-a)^n} - \frac{A_{n-1}}{(z-a)^{n-1}} - \dots - \frac{A_2}{(z-a)^2} - \frac{A_1}{z-a}$$

is finite in the vicinity of a; but the remaining poles of $\phi(z)$ are singularities of this modified function. Proceeding similarly with the other singularities $b, c, \dots$, which are finite in number and each of which is finite in degree, we have coefficients $A, B, C, \dots$ determined so that

$$\phi(z) - \sum_{\kappa=a,\, b,\dots} \left\{ \sum_{r=1}^{n_\kappa} \frac{K_r}{(z-k)^r} \right\}$$

is finite in the vicinity of every pole of $\phi(z)$ within the parallelogram and therefore is finite everywhere within the parallelogram. Let its value be

$\chi(z)$; then for points lying within the parallelogram, the function $\phi(z)$ is expressed in the form

$$\chi(z)+\frac{A_1}{z-a}+\frac{A_2}{(z-a)^2}+\dots+\frac{A_n}{(z-a)^n}$$
$$+\frac{B_1}{z-b}+\frac{B_2}{(z-b)^2}+\dots+\frac{B_m}{(z-b)^m}$$
$$+\dots\dots\dots\dots\dots\dots$$
$$+\frac{H_1}{z-h}+\frac{H_2}{(z-h)^2}+\dots+\frac{H_l}{(z-h)^l}.$$

But though $\phi(z)$ is periodic, $\chi(z)$ is not periodic. It has the property of being finite everywhere within the parallelogram; if it were periodic, it would be finite everywhere, and therefore could have only a constant value; and then $\phi(z)$ would be an algebraical meromorphic function, which is not periodic. The sum of the fractions in $\phi(z)$ may be called the fractional part of the function: owing to the meromorphic character of the function, it cannot be evanescent.

The analytical expression can be put in the form

$$(z-a)^{-n}(z-b)^{-m}\dots(z-h)^{-l}F(z),$$

where $F(z)$ is finite everywhere within the parallelogram. If $\alpha, \beta, \dots, \eta$ be all the zeros, of degrees $\nu, \mu, \dots, \lambda$, within the parallelogram, then

$$F(z)=(z-\alpha)^\nu(z-\beta)^\mu\dots(z-\eta)^\lambda G(z),$$

where $G(z)$ has no zero within the parallelogram; and so the function can be expressed in the form

$$\frac{(z-\alpha)^\nu(z-\beta)^\mu\dots(z-\eta)^\lambda}{(z-a)^n(z-b)^m\dots(z-h)^l}G(z),$$

where $G(z)$ has no zero and no infinity within the parallelogram or on its boundary; and $G(z)$ is not periodic.

The *order* of a doubly-periodic function is the sum of the multiplicities of all the poles which the function has within a fundamental parallelogram; and, the sum being n, the function is said to be of the nth order. All these singularities are, as already remarked, accidental; it is convenient to speak of any particular singularity as simple, double, ... according to its multiplicity.

If two doubly-periodic functions u and v be such that an equation

$$Au+Bv+C=0$$

is satisfied for constant values of A, B, C, the functions are said to be *equivalent* to one another. Equivalent functions evidently have the same accidental singularities in the same multiplicity.

II. *The integral of a doubly-periodic function round the boundary of a fundamental parallelogram is zero.*

Let $ABCD$ be a fundamental parallelogram, the boundary of it being taken so as to pass through no pole of the function. Let A be z_0, B be $z_0 + 2\omega$, and* D be $z_0 + 2\omega'$; then any point in AB is

$$z_0 + 2\omega t,$$

where t is a real quantity lying between 0 and 1; and therefore the integral along AB is

$$\int_0^1 \phi(z_0 + 2\omega t)\, 2\omega dt.$$

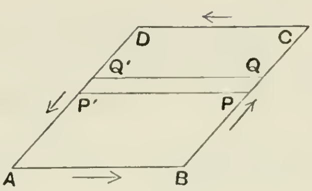

Fig. 33.

Any point in BC is $z_0 + 2\omega + 2\omega' t$, where t is a real quantity lying between 0 and 1; therefore the integral along BC is

$$\int_0^1 \phi(z_0 + 2\omega + 2\omega' t)\, 2\omega' dt$$

$$= \int_0^1 \phi(z_0 + 2\omega' t)\, 2\omega' dt,$$

since ϕ is periodic in 2ω.

Any point in DC is $z + 2\omega' + 2\omega t$, where t is a real quantity lying between 0 and 1; therefore the integral along CD is

$$\int_1^0 \phi(z_0 + 2\omega' + 2\omega t)\, 2\omega dt$$

$$= \int_1^0 \phi(z_0 + 2\omega t)\, 2\omega dt$$

$$= -\int_0^1 \phi(z_0 + 2\omega t)\, 2\omega dt.$$

Similarly, the integral along DA is

$$= -\int_0^1 \phi(z_0 + 2\omega' t)\, 2\omega' dt.$$

Hence the complete value of the integral, taken round the parallelogram, is

$$= \int_0^1 \phi(z_0 + 2\omega t)\, 2\omega dt + \int_0^1 \phi(z_0 + 2\omega' t)\, 2\omega' dt$$

$$- \int_0^1 \phi(z_0 + 2\omega t)\, 2\omega dt - \int_0^1 \phi(z_0 + 2\omega' t)\, 2\omega' dt,$$

which is manifestly zero, since each of the integrals is the integral of a continuous function.

Corollary. Let $\psi(z)$ be any uniform function of z, not necessarily doubly-periodic, but without singularities on the boundary. Then the

* The figure implies that the argument of ω' is greater than the argument of ω, a hypothesis which, though unimportant for the present proposition, must be taken account of hereafter (e.g., § 129).

integral $\int\psi(z)\,dz$ taken round the parallelogram of periods is easily seen to be

$$\int_0^1 \psi(z_0 + 2\omega t)\, 2\omega dt + \int_0^1 \psi(z_0 + 2\omega + 2\omega' t)\, 2\omega' dt$$
$$- \int_0^1 \psi(z_0 + 2\omega' + 2\omega t)\, 2\omega dt - \int_0^1 \psi(z_0 + 2\omega' t)\, 2\omega' dt;$$

or, if we write
$$\psi(\zeta + 2\omega) - \psi(\zeta) = \psi_1(\zeta),$$
$$\psi(\zeta + 2\omega') - \psi(\zeta) = \psi_2(\zeta),$$

then
$$\int\psi(z)\,dz = \int_0^1 \psi_1(z_0 + 2\omega' t)\, 2\omega' dt - \int_0^1 \psi_2(z_0 + 2\omega t)\, 2\omega dt,$$

where on the left-hand side the integral is taken positively round the boundary of the parallelogram and on the right-hand side the variable t in the integrals is real.

The result may also be written in the form

$$\int\psi(z)\,dz = \int_A^D \psi_1(z)\,dz - \int_A^B \psi_2(z)\,dz,$$

the integrals on the right-hand side being taken along the straight lines AD and AB respectively.

Evidently the foregoing main proposition is established, when $\psi_1(\zeta)$ and $\psi_2(\zeta)$ vanish for all values of ζ.

III. *If a doubly-periodic function $\phi(z)$ have infinities $a_1, a_2, \ldots$ within the parallelogram, and if $A_1, A_2, \ldots$ be the coefficients of $(z-a_1)^{-1}, (z-a_2)^{-1}, \ldots$ respectively in the fractional part of $\phi(z)$ when it is expanded in the parallelogram, then*

$$A_1 + A_2 + \ldots = 0.$$

As the function $\phi(z)$ is uniform, the integral $\int\phi(z)\,dz$ is, by (§ 19, II.), the sum of the integrals round a number of curves each including one and only one of the infinities within that parallelogram.

Taking the expression for $\phi(z)$ on p. 220, the integral $A_m\int(z-a)^{-m}\,dz$ round the curve enclosing a is 0, if m be not unity, and is $2\pi i A_1$, if m be unity; the integral $K_m\int(z-k)^{-m}\,dz$ round that curve is 0 for all values of m and for all points k other than a; and the integral $\int\chi(z)\,dz$ round the curve is zero, since $\chi(z)$ is uniform and finite everywhere in the vicinity of a. Hence the integral of $\phi(z)$ round a curve enclosing a_1 alone of all the infinities is $2\pi i A_1$.

Similarly the integral round a curve enclosing a_2 alone is $2\pi i A_2$; and so on, for each of the curves in succession.

Hence the value of the integral round the parallelogram is

$$2\pi i \Sigma A.$$

But by the preceding proposition, the value of $\int \phi(z)\,dz$ round the parallelogram is zero; and therefore

$$A_1 + A_2 + \ldots = 0.$$

This result can be expressed in the form that *the sum of the residues* of a doubly-periodic function relative to a fundamental parallelogram of periods is zero.*

COROLLARY 1. *A doubly-periodic function of the first order does not exist.*

Let such a function have a for its single simple infinity. Then an expression for the function within the parallelogram is

$$\frac{A}{z-a} + \chi(z),$$

where $\chi(z)$ is everywhere finite in the parallelogram. By the above proposition, A vanishes; and so the function has no infinity in the parallelogram. It therefore has no infinity anywhere in the plane, and so is merely a constant: that is, quà function of a variable, it does not exist.

COROLLARY 2. *Doubly-periodic functions of the second order are of two classes.*

As the function is of the second order, the sum of the degrees of the infinities is two. There may thus be either a single infinity of the second degree or two simple infinities.

In the former case, the analytical expression of the function is

$$\phi(z) = \frac{A_1}{z-a} + \frac{A_2}{(z-a)^2} + \chi(z),$$

where a is the infinity of the second degree and $\chi(z)$ is holomorphic within the parallelogram. But, by the preceding proposition, $A_1 = 0$; hence the analytical expression for a doubly-periodic function with a single irreducible infinity a of the second degree is

$$\frac{A_2}{(z-a)^2} + \chi(z)$$

within the parallelogram. Such functions of the second order, which have only a single irreducible infinity, may be called the first class.

In the latter case, the analytical expression of the function is

$$\phi(z) = \frac{C_1}{z-c_1} + \frac{C_2}{z-c_2} + \chi(z),$$

where c_1 and c_2 are the two simple infinities and $\chi(z)$ is finite within the parallelogram. Then

$$C_1 + C_2 = 0;$$

* See p. 42.

so that, if $C_1 = - C_2 = C$, the analytical expression for a doubly-periodic function with two simple irreducible infinities a_1 and a_2 is

$$C\left(\frac{1}{z-a_1} - \frac{1}{z-a_2}\right) + \chi(z)$$

within the parallelogram. Such functions of the second order, which have two irreducible infinities, may be called the second class.

COROLLARY 3. *If within any parallelogram of periods a function is only of the second order, the parallelogram is fundamental.*

COROLLARY 4. *A similar division of doubly-periodic functions of any order into classes can be effected according to the variety in the constitution of the order, the number of classes being the number of partitions of the order.*

The simplest class of functions of the nth order is that in which the functions have only a single irreducible infinity of the nth degree. Evidently the analytical expression of the function within the parallelogram is

$$\frac{G_2}{(z-a)^2} + \frac{G_3}{(z-a)^3} + \dots + \frac{G_n}{(z-a)^n} + \chi(z),$$

where $\chi(z)$ is holomorphic within the parallelogram. Some of the coefficients G may vanish; but all may not vanish, for the function would then be finite everywhere in the parallelogram.

It will however be seen, from the next succeeding propositions, that the division into classes is of most importance for functions of the second order.

IV. *Two functions, which are doubly-periodic in the same periods*, and which have the same zeros and the same infinities each in the same degrees respectively, are in a constant ratio.*

Let ϕ and ψ be the functions, having the same periods; and let α of degree ν, β of degree μ, ... be all the irreducible zeros of ϕ and ψ; and a of degree n, b of degree m, ... be all the irreducible infinities of ϕ and of ψ. Then a function $G(z)$, without zeros or infinities within the parallelogram, exists such that

$$\phi(z) = \frac{(z-\alpha)^\nu (z-\beta)^\mu \dots}{(z-a)^n (z-b)^m \dots} G(z);$$

and another function $H(z)$, without zeros or infinities within the parallelogram, exists such that

$$\psi(z) = \frac{(z-\alpha)^\nu (z-\beta)^\mu \dots}{(z-a)^n (z-b)^m \dots} H(z).$$

Hence $$\frac{\phi(z)}{\psi(z)} = \frac{G(z)}{H(z)}.$$

Now the function on the right-hand side has no zeros in the parallelogram, for G has no zeros and H has no infinities; and it has no infinities in the

* Such functions will be called *homoperiodic.*

parallelogram, for G has no infinities and H has no zeros: hence it has neither zeros nor infinities in the parallelogram. Since it is equal to the function on the left-hand side, which is a doubly-periodic function, it has no zeros and no infinities in the whole plane; it is therefore a constant, say A. Thus*

$$\phi(z) = A\psi(z).$$

V. *Two functions of the second order, doubly-periodic in the same periods and having the same infinities, are equivalent to one another.*

If one of the functions be of the first class in the second order, it has one irreducible double infinity, say at a; so that we have

$$\phi(z) = \frac{G}{(z-a)^2} + \chi(z),$$

where $\chi(z)$ is finite everywhere within the parallelogram. Then the other function also has $z = a$ for its sole irreducible infinity and that infinity is of the second degree; therefore we have

$$\psi(z) = \frac{H}{(z-a)^2} + \chi_1(z),$$

where $\chi_1(z)$ is finite everywhere within the parallelogram. Hence

$$H\phi(z) - G\psi(z) = H\chi(z) - G\chi_1(z).$$

Now χ and χ_1 are finite everywhere within the parallelogram, and therefore so is $H\chi - G\chi_1$. But $H\chi - G\chi_1$, being equal to the doubly-periodic function $H\phi - G\psi$, is therefore doubly-periodic; as it has no infinities within the parallelogram, it consequently can have none over the plane and therefore it is a constant, say I. Thus

$$H\phi(z) - G\psi(z) = I,$$

proving that the functions ϕ and ψ are equivalent.

If on the other hand one of the functions be of the second class in the second order, it has two irreducible simple infinities, say at b and c, so that we have

$$\phi(z) = C\left(\frac{1}{z-b} - \frac{1}{z-c}\right) + \theta(z),$$

where $\theta(z)$ is finite everywhere within the parallelogram. Then the other function also has $z = b$ and $z = c$ for its irreducible infinities, each of them being simple; therefore we have

$$\psi(z) = D\left(\frac{1}{z-b} - \frac{1}{z-c}\right) + \theta_1(z),$$

where $\theta_1(z)$ is finite everywhere within the parallelogram. Hence

$$D\phi(z) - C\psi(z) = D\theta(z) - C\theta_1(z).$$

* This proposition is the modified form of the proposition of § 52, when the generalising exponential factor has been determined so as to admit of the periodicity.

The right-hand side, being finite everywhere in the parallelogram, and equal to the left-hand side which is a doubly-periodic function, is finite everywhere in the plane; it is therefore a constant, say B, so that

$$D\phi(z) - C\psi(z) = B,$$

proving that ϕ and ψ are equivalent to one another.

It thus appears that in considering doubly-periodic functions of the second order, homoperiodic functions of the same class are equivalent to one another if they have the same infinities; so that, practically, it is by their infinities that homoperiodic functions of the second order and the same class are discriminated.

COROLLARY 1. *If two equivalent functions of the second order have one zero the same, all their zeros are the same.*

For in the one class the constant I, and in the other class the constant B, is seen to vanish on substituting for z the common zero; and then the two functions always vanish together.

COROLLARY 2. *If two functions, doubly-periodic in the same periods but not necessarily of the second order, have the same infinities occurring in such a way that the fractional parts of the two functions are the same except as to a constant factor, the functions are equivalent to one another. And if, in addition, they have one zero common, then all their zeros are common, so that the functions are then in a constant ratio.*

COROLLARY 3. *If two functions of the second order, doubly-periodic in the same periods, have their zeros the same, and one infinity common, they are in a constant ratio.*

VI. *Every doubly-periodic function has as many irreducible zeros as it has irreducible infinities.*

Let $\phi(z)$ be such a function. Then

$$\frac{\phi(z+h) - \phi(z)}{z+h-z}$$

is a doubly-periodic function for any value of h, for the numerator is doubly-periodic and the denominator does not involve z; so that, in the limit when $h = 0$, the function is doubly-periodic, that is, $\phi'(z)$ is doubly-periodic.

Now suppose $\phi(z)$ has irreducible zeros of degree m_1 at a_1, m_2 at a_2, ..., and has irreducible infinities of degree μ_1 at α_1, μ_2 at α_2, ...; so that the number of irreducible zeros is $m_1 + m_2 + \dots$, and the number of irreducible infinities is $\mu_1 + \mu_2 + \dots$, both of these numbers being finite. It has been shewn that $\phi(z)$ can be expressed in the form

$$\frac{(z-a_1)^{m_1}(z-a_2)^{m_2}\dots}{(z-\alpha_1)^{\mu_1}(z-\alpha_2)^{\mu_2}\dots} F(z),$$

where $F(z)$ has neither a zero nor an infinity within, or on the boundary of, the parallelogram of reference.

Since $F(z)$ has a value, which is finite, continuous and different from zero everywhere within the parallelogram or on its boundary, the function $\frac{F'(z)}{F(z)}$ is not infinite within the same limits. Hence we have

$$\frac{\phi'(z)}{\phi(z)} = g(z) + \frac{m_1}{z - a_1} + \frac{m_2}{z - a_2} + \dots$$
$$+ \frac{-\mu_1}{z - \alpha_1} + \frac{-\mu_2}{z - \alpha_2} + \dots,$$

where $g(z)$ has no infinities within, or on the boundary of, the parallelogram of reference. But, because $\phi'(z)$ and $\phi(z)$ are doubly-periodic, their quotient is also doubly-periodic; and therefore, applying Prop. II., we have

$$m_1 + m_2 + \dots - \mu_1 - \mu_2 - \dots = 0,$$

that is,

$$m_1 + m_2 + \dots = \mu_1 + \mu_2 + \dots,$$

or the number of irreducible zeros is equal to the number of irreducible infinities.

Corollary I. *The number of irreducible points for which a doubly-periodic function assumes a given value is equal to the number of irreducible zeros.*

For if the value be A, every infinity of $\phi(z)$ is an infinity of the doubly-periodic function $\phi(z) - A$; hence the number of the irreducible zeros of the latter is equal to the number of its irreducible infinities, which is the same as the number for $\phi(z)$ and therefore the same as the number of irreducible zeros of $\phi(z)$. And every irreducible zero of $\phi(z) - A$ is an irreducible point, for which $\phi(z)$ assumes the value A.

Corollary II. *A doubly-periodic function with only a single zero does not exist; a doubly-periodic function of the second order has two zeros; and, generally, the order of a function can be measured by its number of irreducible zeros.*

Note. It may here be remarked that the doubly-periodic functions (§ 115), that have only accidental singularities in the finite part of the plane, have $z = \infty$ for an essential singularity. It is evident that for infinite values of z, the finite magnitude of the parallelogram of periods is not recognisable; and thus for $z = \infty$ the function can have any value, shewing that $z = \infty$ is an essential singularity.

VII. *Let $a_1, a_2, \dots$ be the irreducible zeros of a function of degrees $m_1, m_2, \dots$ respectively; $\alpha_1, \alpha_2, \dots$ its irreducible infinities of degrees $\mu_1, \mu_2, \dots$ respectively; and $z_1, z_2, \dots$ the irreducible points where it assumes a value c, which is neither zero nor infinity, their degrees being $M_1, M_2, \dots$ respectively.*

Then, except possibly as to additive multiples of the periods, the quantities $\sum\limits_{r=1} m_r a_r$, $\sum\limits_{r=1} \mu_r \alpha_r$ *and* $\sum\limits_{r=1} M_r z_r$ *are equal to one another, so that*

$$\sum_{r=1} m_r a_r \equiv \sum_{r=1} M_r z_r \equiv \sum_{r=1} \mu_r \alpha_r \ (\text{mod. } 2\omega,\ 2\omega').$$

Let $\phi(z)$ be the function. Then the quantities which occur are the sums of the zeros, the assigned values, and the infinities, the degree of each being taken account of when there is multiple occurrence; and by the last proposition these degrees satisfy the relations

$$\Sigma m_r = \Sigma M_r = \Sigma \mu_r.$$

The function $\phi(z) - c$ is doubly-periodic in 2ω and $2\omega'$; its zeros are $z_1, z_2, \ldots$ of degrees $M_1, M_2, \ldots$ respectively; and its infinities are $\alpha_1, \alpha_2, \ldots$ of degrees $\mu_1, \mu_2, \ldots$, being the same as those of $\phi(z)$. Hence there exists a function $G(z)$, without either a zero or an infinity lying in the parallelogram or on its boundary, such that $\phi(z) - c$ can be expressed in the form

$$\frac{(z-z_1)^{M_1}(z-z_2)^{M_2}\ldots}{(z-\alpha_1)^{\mu_1}(z-\alpha_2)^{\mu_2}\ldots}\, G(z)$$

for all points not outside the parallelogram; and therefore, for points in that region

$$\frac{\phi'(z)}{\phi(z)-c} = \sum_{r=1} \frac{M_r}{z-z_r} - \sum \frac{\mu_r}{z-\alpha_r} + \frac{G'(z)}{G(z)}.$$

Hence

$$\begin{aligned}\frac{z\phi'(z)}{\phi(z)-c} &= \sum_{r=1} \frac{M_r z}{z-z_r} - \sum \frac{\mu_r z}{z-\alpha_r} + \frac{zG'(z)}{G(z)} \\ &= \sum_{r=1} M_r + \sum_{r=1} \frac{M_r z_r}{z-z_r} - \sum_{r=1} \mu_r - \sum \frac{\mu_r \alpha_r}{z-\alpha_r} + \frac{zG'(z)}{G(z)} \\ &= \sum_{r=1} \frac{M_r z_r}{z-z_r} - \sum \frac{\mu_r \alpha_r}{z-\alpha_r} + \frac{zG'(z)}{G(z)},\end{aligned}$$

because

$$\sum_{r=1} M_r = \sum_{r=1} \mu_r.$$

Integrate both sides round the boundary of the fundamental parallelogram. Because $G(z)$ has no zero and no infinity in the included region and does not vanish along the curve, the integral

$$\int \frac{zG'(z)}{G(z)}\, dz$$

vanishes. But the points z_i and α_i are enclosed in the area; and therefore the value of the right-hand side is

$$2\pi i \,\Sigma\, M_r z_r - 2\pi i \,\Sigma\, \mu_r \alpha_r,$$

so that

$$2\pi i\,(\Sigma M_r z_r - \Sigma \mu_r \alpha_r) = \int \frac{z\phi'(z)}{\phi(z)-c}\, dz,$$

the integral being extended round the parallelogram.

Denoting the subject of integration $\dfrac{z\phi'(z)}{\phi(z)-c}$ by $f(z)$, we have

$$f(z+2\omega)-f(z)=2\omega\,\frac{\phi'(z)}{\phi(z)-c},$$

$$f(z+2\omega')-f(z)=2\omega'\,\frac{\phi'(z)}{\phi(z)-c};$$

and therefore, by the Corollary to Prop. II., the value of the foregoing integral is

$$2\omega\int_A^D \frac{\phi'(z)}{\phi(z)-c}\,dz-2\omega'\int_A^B \frac{\phi'(z)}{\phi(z)-c}\,dz,$$

the integrals being taken along the straight lines AD and AB respectively (fig. 33, p. 221).

Let $w=\phi(z)-c$; then, as z describes a path, w will also describe a single path as it is a uniform function of z. When z moves from A to D, w moves from $\phi(A)-c$ by some path to $\phi(D)-c$, that is, it returns to its initial position since $\phi(D)=\phi(A)$; hence, as z describes AD, w describes a simple closed path, the area included by which may or may not contain zeros and infinities of w. Now

$$dw=\phi'(z)\,dz,$$

and therefore the integral $\displaystyle\int_A^D \frac{\phi'(z)}{\phi(z)-c}\,dz$ is equal to

$$\int\frac{dw}{w},$$

taken in some direction round the corresponding closed path for w. This integral vanishes, if no w-zero or w-infinity be included within the area bounded by the path; it is $\pm 2m'\pi i$, if m' be the excess of the number of included zeros over the number of included infinities, the $+$ or $-$ sign being taken with a positive or a negative description; hence we have

$$\int_A^D \frac{\phi'(z)}{\phi(z)-c}\,dz=2m\pi i,$$

where m is some positive or negative integer and may be zero. Similarly

$$\int_A^B \frac{\phi'(z)}{\phi(z)-c}\,dz=2n\pi i,$$

where n is some positive or negative integer and may be zero.

Thus $\qquad 2\pi i\,(\Sigma M_r z_r-\Sigma\mu_r\alpha_r)=2\omega\,.\,2m\pi i-2\omega'\,.\,2n\pi i,$

and therefore $\qquad \Sigma M_r z_r-\Sigma\mu_r\alpha_r=2m\omega-2n\omega'$

$$\equiv 0\ (\text{mod. } 2\omega,\ 2\omega').$$

Finally, since $\Sigma M_r z_r\equiv\Sigma\mu_r\alpha_r$ whatever be the value of c, for the right-hand side is independent of c, we may assign to c any value we please. Let the value zero be assigned; then $\Sigma M_r z_r$ becomes $\Sigma m_r a_r$, so that

$$\Sigma m_r a_r \equiv \Sigma \mu_r \alpha_r \text{ (mod. } 2\omega, 2\omega').$$

The combination of these results leads to the required theorem*, expressed by the congruences

$$\sum_{r=1} m_r a_r \equiv \sum_{r=1} M_r z_r \equiv \sum_{r=1} \mu_r \alpha_r \text{ (mod. } 2\omega, 2\omega').$$

Note. Any point within the parallelogram can be represented in the form $z_0 + a2\omega + b2\omega'$, where a and b are real positive quantities less than unity. Hence

$$\Sigma M_r z_r = A_z 2\omega + B_z 2\omega' + z_0 \Sigma M_r,$$

where A and B are real positive quantities each less than ΣM_r, that is, less than the order of the function.

In particular, for functions of the second order, we have

$$z_1 + z_2 = A_z 2\omega + B_z 2\omega' + 2z_0,$$

where A_z and B_z are positive quantities each less than 2. Similarly, if a and b be the zeros,

$$a + b = A_a 2\omega + B_a 2\omega' + 2z_0,$$

where A_a and B_a are each less than 2; hence, if

$$z_1 + z_2 - a - b = m2\omega + m'2\omega',$$

then m may have any one of the three values $-1, 0, 1$ and so may m', the simultaneous values not being necessarily the same.

Let α and β be the infinities of a function of the second class; then

$$\alpha + \beta - a - b = n2\omega + n'2\omega',$$

where n and n' may each have any one of the three values $-1, 0, 1$. By changing the origin of the fundamental parallelogram, so as to obtain a different set of irreducible points, we can secure that n and n' are zero, and then

$$\alpha + \beta = a + b.$$

Thus, if n be 1 with an initial parallelogram, so that

$$\alpha + \beta = a + b + 2\omega,$$

we should take either $\beta - 2\omega = \beta'$, or $\alpha - 2\omega = \alpha'$, according to the position of α and β, and then have a new parallelogram such that

$$\alpha + \beta' = a + b, \text{ or } \alpha' + \beta = a + b.$$

The case of exception is when the function is of the first class and has a repeated zero.

* The foregoing proof is suggested by Königsberger, *Theorie der elliptischen Functionen*, t. i, p. 342; other proofs are given by Briot and Bouquet and by Liouville, to whom the adopted form of the theorem is due. The theorem is substantially contained in one of Abel's general theorems in the comparison of transcendents.

VIII. *Let $\phi(z)$ be a doubly-periodic function of the second order. If γ be the one double infinity when the function is of the first class, and if α and β be the two simple infinities when the function is of the second class, then in the former case*

$$\phi(z) = \phi(2\gamma - z),$$

and in the latter case
$$\phi(z) = \phi(\alpha + \beta - z).$$

Since the function is of the second order, so that it has two irreducible infinities, there are two (and only two) irreducible points in a fundamental parallelogram at which the function can assume any the same value: let them be z and z'.

Then, for the first class of functions, we have

$$\begin{aligned} z + z' &\equiv 2\gamma \\ &= 2\gamma + 2m\omega + 2n\omega', \end{aligned}$$

where m and n are integers; and then, since $\phi(z) = \phi(z')$ by definition of z and z', we have

$$\begin{aligned} \phi(z) &= \phi(2\gamma - z + 2m\omega + 2n\omega') \\ &= \phi(2\gamma - z). \end{aligned}$$

For the second class of functions, we have

$$\begin{aligned} z + z' &\equiv \alpha + \beta \\ &= \alpha + \beta + 2m\omega + 2n\omega'; \end{aligned}$$

so that, as before,

$$\begin{aligned} \phi(z) &= \phi(\alpha + \beta - z + 2m\omega + 2n\omega') \\ &= \phi(\alpha + \beta - z). \end{aligned}$$

117. Among the functions which have the same periodicity as a given function $\phi(z)$, the one which is most closely related to it is its derivative $\phi'(z)$. We proceed to find *the zeros and the infinities of the derivative of a function, in particular, of a function of the second order.*

Since $\phi(z)$ is uniform, an irreducible infinity of degree n for $\phi(z)$ is an irreducible infinity of degree $n+1$ for $\phi'(z)$. Moreover $\phi'(z)$, being uniform, has no infinity which is not an infinity of $\phi(z)$; thus the order of $\phi'(z)$ is $\Sigma(n+1)$ or its order is greater than that of $\phi(z)$ by an integer which represents the number of distinct irreducible infinities of $\phi(z)$, no account being taken of their degree. If, then, a function be of order m, the order of its derivative is not less than $m+1$ and is not greater than $2m$.

Functions of the second order either possess one double infinity so that within the parallelogram they take the form

$$\phi(z) = \frac{A}{(z-\gamma)^2} + \chi(z),$$

and then

$$\phi'(z) = \frac{-2A}{(z-\gamma)^3} + \chi'(z),$$

that is, the infinity of $\phi(z)$ is the single infinity of $\phi'(z)$ and it is of the third degree, so that $\phi'(z)$ is of the third order; or they possess two simple infinities, so that within the parallelogram they take the form

$$\phi(z) = C\left(\frac{1}{z-\alpha_1} - \frac{1}{z-\alpha_2}\right) + \chi(z),$$

and then
$$\phi'(z) = -C\left\{\frac{1}{(z-\alpha_1)^2} - \frac{1}{(z-\alpha_2)^2}\right\} + \chi'(z),$$

that is, each of the simple infinities of $\phi(z)$ is an infinity for $\phi'(z)$ of the second degree, so that $\phi'(z)$ is of the fourth order.

It is of importance (as will be seen presently) to know the zeros of the derivative of a function of the second order.

For a function of the first class, let γ be the irreducible infinity of the second degree; then we have

$$\phi(z) = \phi(2\gamma - z),$$

and therefore
$$\phi'(z) = -\phi'(2\gamma - z).$$

Now $\phi'(z)$ is of the third order, having γ for its irreducible infinity in the third degree: hence it has three irreducible zeros.

In the foregoing equation, take $z = \gamma$: then

$$\phi'(\gamma) = -\phi'(\gamma),$$

shewing that γ is either a zero or an infinity. It is known to be the only infinity of $\phi'(z)$.

Next, take $z = \gamma + \omega$; then

$$\begin{aligned}\phi'(\gamma+\omega) &= -\phi'(\gamma-\omega)\\ &= -\phi'(\gamma-\omega+2\omega)\\ &= -\phi'(\gamma+\omega),\end{aligned}$$

shewing that $\gamma+\omega$ is either a zero or an infinity. It is known not to be an infinity; hence it is a zero.

Similarly $\gamma+\omega'$ and $\gamma+\omega+\omega'$ are zeros. Thus three zeros are obtained, distinct from one another; and only three zeros are required; if they be not within the parallelogram, we take the irreducible points homologous with them. Hence:

IX. *The three zeros of the derivative of a function, doubly-periodic in* 2ω *and* $2\omega'$ *and having* γ *for its double (and only) irreducible infinity, are*

$$\gamma+\omega, \quad \gamma+\omega', \quad \gamma+\omega+\omega'.$$

For a function of the second class, let α and β be the two simple irreducible infinities; then we have

$$\phi(z) = \phi(\alpha+\beta-z),$$

and therefore
$$\phi'(z) = -\phi'(\alpha+\beta-z).$$

Now $\phi'(z)$ is of the fourth order, having α and β as its irreducible infinities each in the second degree; hence it must have four irreducible zeros.

In the foregoing equation, take $z = \frac{1}{2}(\alpha + \beta)$; then

$$\phi'\{\tfrac{1}{2}(\alpha+\beta)\} = -\phi'\{\tfrac{1}{2}(\alpha+\beta)\},$$

shewing that $\frac{1}{2}(\alpha+\beta)$ is either a zero or an infinity. It is known not to be an infinity; hence it is a zero.

Next, take $z = \frac{1}{2}(\alpha+\beta)+\omega$; then

$$\begin{aligned}\phi'\{\tfrac{1}{2}(\alpha+\beta)+\omega\} &= -\phi'\{\tfrac{1}{2}(\alpha+\beta)-\omega\}\\ &= -\phi'\{\tfrac{1}{2}(\alpha+\beta)-\omega+2\omega\}\\ &= -\phi'\{\tfrac{1}{2}(\alpha+\beta)+\omega\},\end{aligned}$$

shewing that $\frac{1}{2}(\alpha+\beta)+\omega$ is either a zero or an infinity. As before, it is a zero.

Similarly $\frac{1}{2}(\alpha+\beta)+\omega'$ and $\frac{1}{2}(\alpha+\beta)+\omega+\omega'$ are zeros. Four zeros are thus obtained, distinct from one another; and only four zeros are required. Hence:

X. *The four zeros of the derivative of a function, doubly-periodic in* 2ω *and* $2\omega'$ *and having* α *and* β *for its simple (and only) irreducible infinities, are*

$$\tfrac{1}{2}(\alpha+\beta),\quad \tfrac{1}{2}(\alpha+\beta)+\omega,\quad \tfrac{1}{2}(\alpha+\beta)+\omega',\quad \tfrac{1}{2}(\alpha+\beta)+\omega+\omega'.$$

The verification in each of these two cases of Prop. VII., that the sum of the zeros of the doubly-periodic function $\phi'(z)$ is congruent with the sum of its infinities, is immediate.

Lastly, it may be noted that, *if* z_1 *and* z_2 *be the two irreducible points for which a doubly-periodic function of the second order assumes a given value, then the values of its derivative for* z_1 *and for* z_2 *are equal and opposite.* For

$$\phi(z) = \phi(\alpha+\beta-z) = \phi(z_1+z_2-z),$$

since $z_1 + z_2 \equiv \alpha+\beta$; and therefore

$$\phi'(z) = -\phi'(z_1+z_2-z),$$

that is,

$$\phi'(z_1) = -\phi'(z_2),$$

which proves the statement.

118. We now come to a different class of theorems.

XI. *Any doubly-periodic function of the second order can be expressed algebraically in terms of an assigned doubly-periodic function of the second order, if the periods be the same.*

The theorem will be sufficiently illustrated and the line of proof sufficiently indicated, if we express a function $\phi(z)$ of the second class, with irreducible infinities α, β and irreducible zeros a, b such that $\alpha+\beta = a+b$, in terms of a function Φ of the first class with γ as its irreducible double infinity.

Consider a function $$\frac{\Phi(z+h)-\Phi(h')}{\Phi(z+h)-\Phi(h'')}.$$

A zero of $\Phi(z+h)$ is neither a zero nor an infinity of this function; nor is an infinity of $\Phi(z+h)$ a zero or an infinity of the function. It will have a and b for its irreducible zeros, if

$$a+h=h',$$
$$b+h+h'=2\gamma;$$

and these will be the only zeros, for Φ is of the second order. It will have α and β for its irreducible infinities, if

$$\alpha+h=h'',$$
$$\beta+h+h''=2\gamma;$$

and these will be the only infinities, for Φ is of the second order. These equations are satisfied by

$$h''=\tfrac{1}{2}(2\gamma-\beta+\alpha),$$
$$h'=\tfrac{1}{2}(2\gamma-b+a),$$
$$h=\tfrac{1}{2}(2\gamma-\alpha-\beta)=\tfrac{1}{2}(2\gamma-a-b).$$

Hence the assigned function, with these values of h, has the same zeros and the same infinities as $\phi(z)$; and it is doubly-periodic in the same periods. The ratio of the two functions is therefore a constant, by Prop. IV., so that

$$\phi(z)=A\frac{\Phi(z+h)-\Phi(h')}{\Phi(z+h)-\Phi(h'')}.$$

If the expression be required in terms of $\Phi(z)$ alone and constants, then $\Phi(z+h)$ must be expressed in terms of $\Phi(z)$ and constants which are values of $\Phi(z)$ for special values of z. This will be effected later.

The preceding proposition is a special case of a more general theorem which will be considered later; the following is another special case of that theorem: viz.:

XII. *A doubly-periodic function with any number of simple infinities can be expressed either as a sum or as a product, of functions of the second order and the second class which are doubly-periodic in the same periods.*

Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the irreducible infinities of the function Φ, and suppose that the fractional part of $\Phi(z)$ is

$$\frac{A_1}{z-\alpha_1}+\frac{A_2}{z-\alpha_2}+\ldots\ldots+\frac{A_n}{z-\alpha_n},$$

with the condition $A_1+A_2+\ldots\ldots+A_n=0$. Let $\phi_{ij}(z)$ be a function, doubly-periodic in the same periods, with α_i, α_j as its only irreducible infinities, supposed simple; where i and j have the values $1, \ldots\ldots, n$. Then the fractional parts of the functions $\phi_{12}(z)$, $\phi_{23}(z), \ldots$ are

$$G_1\left(\frac{1}{z-\alpha_1}-\frac{1}{z-\alpha_2}\right),$$

$$G_2\left(\frac{1}{z-\alpha_2}-\frac{1}{z-\alpha_3}\right),$$

$$\vdots$$

respectively; and therefore the fractional part of

$$\frac{A_1}{G_1}\phi_{12}(z)+\frac{A_1+A_2}{G_2}\phi_{23}(z)+\ldots+\frac{A_1+A_2+\ldots+A_{n-1}}{G_{n-1}}\phi_{n-1,n}(z)$$

is
$$\frac{A_1}{z-\alpha_1}+\frac{A_2}{z-\alpha_2}+\ldots+\frac{A_{n-1}}{z-\alpha_{n-1}}-\frac{A_1+A_2+\ldots+A_{n-1}}{z-\alpha_n}$$

$$=\frac{A_1}{z-\alpha_1}+\ldots+\frac{A_{n-1}}{z-\alpha_{n-1}}+\frac{A_n}{z-\alpha_n},$$

since $\sum_{i=1}^{n} A_i = 0$. This is the same as the fractional part of $\Phi(z)$; and therefore

$$\Phi(z)-\frac{A_1}{G_1}\phi_{12}(z)-\frac{A_1+A_2}{G_2}\phi_{23}(z)-\ldots-\frac{A_1+\ldots+A_{n-1}}{G_{n-1}}\phi_{n-1,n}(z)$$

has no fractional part. It thus has no infinity within the parallelogram; it is a doubly-periodic function and therefore has no infinity anywhere in the plane; and it is therefore merely a constant, say B. Hence, changing the constants, we have

$$\Phi(z)-B_1\phi_{12}(z)-B_2\phi_{23}(z)-\ldots-B_{n-1}\phi_{n-1,n}(z)=B,$$

giving an expression for $\Phi(z)$ as a linear combination of functions of the second order and the second class. But as the assignment of the infinities is arbitrary, the expression is not unique.

For the expression in the form of a product, we may denote the n irreducible zeros, supposed simple, by $a_1, \ldots, a_n$. We determine $n-2$ new irreducible quantities c, such that

$$\begin{aligned}
c_1 &\equiv \alpha_1+\alpha_2-a_1,\\
c_2 &\equiv \alpha_3+c_1-a_2,\\
c_3 &\equiv \alpha_4+c_2-a_3,\\
&\ldots\ldots\ldots\ldots \vdots \ldots\ldots\ldots\ldots\\
c_{n-2} &\equiv \alpha_{n-1}+c_{n-3}-a_{n-2},\\
a_n &\equiv \alpha_n+c_{n-2}-a_{n-1},
\end{aligned}$$

this being possible because $\sum_{r=1}^{n}\alpha_r \equiv \sum_{r=1}^{n} a_r$; and we denote by $\phi(z;\, \alpha, \beta;\, e, f)$ a function of z, which is doubly-periodic in the periods of the given function, has α and β for simple irreducible infinities and has e and f for simple irreducible zeros. Then the function

$$\phi(z;\ \alpha_1, \alpha_2;\ a_1, c_1)\,\phi(z;\ \alpha_3, c_1;\ a_2, c_2) \ldots \phi(z;\ \alpha_n, c_{n-2};\ a_{n-1}, a_n)$$

has neither a zero nor an infinity at c_1, at $c_2, \ldots$, and at c_{n-2}; it has simple infinities at $\alpha_1, \alpha_2, \ldots, \alpha_n$, and simple zeros at $a_1, a_2, \ldots, a_{n-1}, a_n$. Hence it has the same irreducible infinities and the same irreducible zeros in the same degree as the given function $\Phi(z)$; and therefore, by Prop. IV., $\Phi(z)$ is a mere constant multiple of the foregoing product.

The theorem is thus completely proved.

Other developments for functions, the infinities of which are not simple, are possible; but they are relatively unimportant in view of a theorem, Prop. XV., about to be proved, which expresses any periodic function in terms of a single function of the second order and its derivative.

XIII. *If two doubly-periodic functions have the same periods, they are connected by an algebraical equation.*

Let u be one of the functions, having n irreducible infinities, and v be the other, having m irreducible infinities.

By Prop. VI., Corollary I., there are n irreducible values of z for a value of u; and to each irreducible value of z there is a doubly-infinite series of values of z over the plane. The function v has the same value for all the points in any one series, so that a single value of v can be associated uniquely with each of the irreducible values of z, that is, there are n values of v for each value of u. Hence, (§ 99), v is a root of an algebraical equation of the nth degree, the coefficients of which are functions of u.

Similarly u is a root of an algebraical equation of the mth degree, the coefficients of which are functions of v.

Hence, combining these results, we have an algebraical equation between u and v of the nth degree in v and the mth in u, where m and n are the respective orders of v and u.

Corollary I. *If both the functions be even functions of z, then n and m are even integers; and the algebraical relation between u and v is of degree $\tfrac{1}{2}n$ in v and of degree $\tfrac{1}{2}m$ in u.*

Corollary II. *If a function u be doubly-periodic in ω and ω', and a function v be doubly-periodic in Ω and Ω', where*

$$\Omega = m\omega + n\omega',\ \ \Omega' = m'\omega + n'\omega',$$

m, n, m', n' being integers, then there is an algebraic relation between u and v.

119. It has been proved that, if a doubly-periodic function u be of order m, then its derivative du/dz is doubly-periodic in the same periods and is of an order n, which is not less than $m+1$ and not greater than $2m$. Hence, by

Prop. XIII., there subsists between u and u' an algebraical equation of order m in u' and of order n in u; let it be arranged in powers of u' so that it takes the form

$$U_0 u'^m + U_1 u'^{m-1} + \dots + U_{m-2} u'^2 + U_{m-1} u' + U_m = 0,$$

where U_0, U_1, ..., U_m are rational integral algebraical functions of u one at least of which must be of degree n.

Because the only distinct infinities of u' are infinities of u, it is impossible that u' should become infinite for finite values of u: hence $U_0 = 0$ can have no finite roots for u, that is, it is a constant and so it may be taken as unity.

And because the m values of z, for which u assumes a given value, have their sum constant save as to integral multiples of the periods, we have

$$\delta z_1 + \delta z_2 + \dots + \delta z_m = 0$$

corresponding to a variation δu; or

$$\frac{dz_1}{du} + \frac{dz_2}{du} + \dots + \frac{dz_m}{du} = 0.$$

Now $\dfrac{du}{dz_1}$ is one of the values of u' corresponding to the value of u, and so for the others; hence

$$\sum_{r=1}^{m} \frac{1}{u_r'} = 0,$$

that is, by the foregoing equation,

$$\frac{U_{m-1}}{U_m} = 0,$$

and therefore U_{m-1} vanishes. Hence:

XIV. *There is a relation, between a doubly-periodic function u of order m and its derivative, of the form*

$$u'^m + U_1 u'^{m-1} + \dots + U_{m-2} u'^2 + U_m = 0,$$

where U_1, ..., U_{m-2}, U_m are rational integral algebraical functions of u, at least one of which must be of degree n, the order of the derivative, and n is not less than $m+1$ and not greater than $2m$.

Further, by taking $v = \dfrac{1}{u}$, which is a function of order m because it has the m irreducible zeros of u for its infinities, and substituting, we have

$$v'^m - v^2 U_1 v'^{m-1} + v^4 U_2 v'^{m-2} - \dots \pm v^{2m-4} U_{m-2} v'^2 \mp v^{2m} U_m = 0.$$

The coefficients of this equation must be integral functions of v; hence the *degree of U_r in u cannot be greater than $2r$.*

COROLLARY. The foregoing equation becomes very simple in the case of doubly-periodic functions of the second order.

Then $$m = 2.$$

If the function have one infinity of the second degree, its derivative has that infinity in the third degree, and is of the third order, so that $n=3$; and the equation is

$$\left(\frac{du}{dz}\right)^2 = \lambda u^3 + 3\mu u^2 + 3\nu u + \rho,$$

where λ, μ, ν, ρ are constants. If θ be the infinity, so that

$$u = \phi(z) = \frac{A}{(z-\theta)^2} + \chi(z),$$

where $\chi(z)$ is everywhere finite in the parallelogram, then $\frac{1}{\lambda} = \frac{1}{4}A$; and the zeros of $\frac{du}{dz}$ are $\theta+\omega$, $\theta+\omega'$, $\theta+\omega+\omega'$; so that

$$\tfrac{1}{4}A\left(\frac{d\phi}{dz}\right)^2 = \{\phi(z)-\phi(\theta+\omega)\}\{\phi(z)-\phi(\theta+\omega')\}\{\phi(z)-\phi(\theta+\omega+\omega')\}.$$

This is *the general differential equation of Weierstrass's elliptic functions.*

If the function have two simple infinities α and β, its derivative has each of them as an infinity of the second degree, and is of the fourth order, so that $n=4$; and the equation is

$$\left(\frac{du}{dz}\right)^2 = c_0u^4 + 4c_1u^3 + 6c_2u^2 + 4c_3u + c_4,$$

where c_0, c_1, c_2, c_3, c_4 are constants. Moreover

$$u = \phi(z) = G\left(\frac{1}{z-\alpha} - \frac{1}{z-\beta}\right) + \chi(z),$$

where $\chi(z)$ is finite everywhere in the parallelogram. Then $c_0 = G^{-2}$; and the zeros of $\frac{du}{dz}$ are $\frac{1}{2}(\alpha+\beta)$, $\frac{1}{2}(\alpha+\beta)+\omega$, $\frac{1}{2}(\alpha+\beta)+\omega'$, $\frac{1}{2}(\alpha+\beta)+\omega+\omega'$, so that the equation is

$$G^2\left(\frac{d\phi}{dz}\right)^2 = [\phi(z) - \phi\{\tfrac{1}{2}(\alpha+\beta)\}]\,[\phi(z)-\phi\{\tfrac{1}{2}(\alpha+\beta)+\omega\}]$$
$$\times[\phi(z)-\phi\{\tfrac{1}{2}(\alpha+\beta)+\omega'\}]\,[\phi(z)-\phi\{\tfrac{1}{2}(\alpha+\beta)+\omega+\omega'\}].$$

This is *the general differential equation of Jacobi's elliptic functions.*

The canonical forms of both of these equations will be obtained in Chapter XI., where some properties of the functions are investigated as special illustrations of the general theorems.

Note. All the derivatives of a doubly-periodic function are doubly-periodic in the same periods, and have the same infinities as the function but in different degrees. In the case of a function of the second order, which must satisfy one or other of the two foregoing equations, it is easy to see that a derivative of even rank is a rational, integral, algebraical function of u, and that a derivative of odd rank is the product of a rational, integral, algebraical function of u by the first derivative of u.

It may be remarked that the form of these equations confirms the result at the end of § 117, by giving two values of u' for one value of u, the two values being equal and opposite.

Ex. If u be a doubly-periodic function having a single irreducible infinity of the third degree so as to be expressible in the form

$$-\frac{2}{z^3}+\frac{\theta}{z^2}+\text{integral function of } z$$

within the parallelogram of periods, then the differential equation of the first order which determines u is

$$u'^3+(a+3\theta u)\,u'^2=U_4,$$

where U_4 is a quartic function of u and where a is a constant which does not vanish with θ.

(Math. Trip., Part II, 1889.)

XV. *Every doubly-periodic function can be expressed rationally in terms of a function of the second order, doubly-periodic in the same periods, and its derivative.*

Let u be a function of the second order and the second class, having the same two periods as v, a function of the mth order; then, by Prop. XIII., there is an algebraical relation between u and v which, being of the second degree in v and the mth degree in u, may be taken in the form

$$Lv^2-2Mv+P=0,$$

where the quantities L, M, P are rational, integral, algebraical functions of u and at least one of them is of degree m. Taking

$$Lv-M=w,$$

we have

$$w^2=M^2-LP,$$

a rational, integral, algebraical function of u of degree not higher than $2m$.

Thus w cannot be infinite for any finite value of u: an infinite value of u makes w infinite, of finite multiplicity. To each value of u there correspond two values of w equal to one another but opposite in sign.

Moreover w, being equal to $Lv-M$, is a uniform function of z, say $F(z)$, while it is a two-valued function of u. A value of u gives two distinct values of z, say z_1 and z_2; hence the values of w, which arise from an assigned value of u, are values of w arising as uniform functions of the two distinct values of z. Hence as the two values of w are equal in magnitude and opposite in sign, we have

$$F(z_1)+F(z_2)=0,$$

that is, since $z_1+z_2\equiv\alpha+\beta$ where α and β are the irreducible infinities of u,

$$F(z_1)+F(\alpha+\beta-z_1)=0,$$

so that $\tfrac{1}{2}(\alpha+\beta)$, $\tfrac{1}{2}(\alpha+\beta)+\omega$, $\tfrac{1}{2}(\alpha+\beta)+\omega'$, and $\tfrac{1}{2}(\alpha+\beta)+\omega+\omega'$ are either zeros or infinities of w. They are known not to be infinities of u, and w is infinite only for infinite values of u; hence the four points are zeros of w.

But these are all the irreducible zeros of u'; hence the zeros of u' are included among the zeros of w.

Now consider the function w/u'. The numerator has two values equal and opposite for an assigned value of u; so also has the denominator. Hence w/u' is a uniform function of u.

This uniform function of u may become infinite for

(i) infinities of the numerator,

(ii) zeros of the denominator.

But, so far as concerns (ii), we know that the four irreducible zeros of the denominator are all simple zeros of u' and each of them is a zero of w; hence w/u' does not become infinite for any of the points in (ii). And, so far as concerns (i), we know that all of them are infinities of u. Hence w/u', a uniform function of u, can become infinite only for an infinite value of u, and its multiplicity for such a value is finite; hence it is a rational, integral, algebraical function of u, say N, so that

$$w = Nu'.$$

Moreover, because w^2 is of degree in u not higher than $2m$, and u'^2 is of the fourth degree in u, it follows that *N is of degree not higher than $m-2$.*

We thus have
$$Lv - M = Nu',$$
or
$$v = \frac{M + Nu'}{L} = \frac{M}{L} + \frac{N}{L}u',$$

where L, M, N are rational, integral, algebraical functions of u; the degrees of L and M are not higher than m, and that of N is not higher than $m-2$.

Note 1. The function u, which has been considered in the preceding proof, is of the second order and the second class. If a function u of the second order and the first class, having a double irreducible infinity, be chosen, the course of proof is similar; the function w has the three irreducible zeros of u' among its zeros and the result, as before, is

$$w = Nu'.$$

But, now, w^2 is of degree in u not higher than $2m$ and u'^2 is of the third degree in u; hence N is of degree not higher than $m-2$ and the degree of w^2 in u cannot be higher than $2m-1$.

Hence, if L, M, P be all of degree m, the terms of degree $2m$ in $LP - M^2$ disappear. If all of them be not of degree m, the degree of M must not be higher than $m-1$; the degree of either L or P must be m, but the degree of the other must not be greater than $m-1$, for otherwise the algebraical equation between u and v would not be of degree m in u.

We thus have

$$Lv^2 - 2Mv + P = 0, \quad Lv - M = Nu',$$

where the degree of N in u is not higher than $m-2$. If the degree of L be less than m, the degree of M is not higher than $m-1$ and the degree of P is m. If the degree of L be m, the degree of M may also be m provided that the degree of P be m and that the highest terms be such that the coefficient of u^{2m} in $LP-M^2$ vanishes.

Note 2. The theorem expresses a function v rationally in terms of u and u': but u' is an irrational function of u, so that v is not expressed rationally in terms of u alone.

But, in Propositions XI. and XII., it was indicated that a function such as v could be rationally expressed in terms of a doubly-periodic function, such as u. The apparent contradiction is explained by the fact that, in the earlier propositions, the arguments of the function u in the rational expression and of the function v are not the same; whereas, in the later proposition whereby v is expressed in general irrationally in terms of u, the arguments are the same. The transition from the first (which is the less useful form) to the second is made by expressing the functions of those different arguments in terms of functions of the same argument when (as will appear subsequently, in § 121, in proving the so-called addition-theorem) the irrational function of u, represented by the derivative u', is introduced.

COROLLARY I. Let Ω denote the sum of the irreducible infinities or of the irreducible zeros of the function u of the second order, so that $\Omega \equiv 2\gamma$ for functions of the first class, and $\Omega \equiv \alpha+\beta$ for functions of the second class. Let u be represented by $\phi(z)$ and v by $\psi(z)$, when the argument must be put in evidence. Then

$$\phi(\Omega-z)=\phi(z),$$

$$-\phi'(\Omega-z)=\phi'(z),$$

so that

$$\psi(\Omega-z)=\frac{M+N\phi'(\Omega-z)}{L}=\frac{M}{L}-\frac{N}{L}\phi'(z).$$

Hence

$$\psi(z)+\psi(\Omega-z)=2\frac{M}{L}=2R,$$

$$\psi(z)-\psi(\Omega-z)=2\frac{N}{L}\phi'(z)=2S\phi'(z).$$

First, if $\psi(z)=\psi(\Omega-z)$, then $S=0$ and $\psi(z)=R$: that is, *a function* $\psi(z)$, *which satisfies the equation*

$$\psi(z)=\psi(\Omega-z),$$

can be expressed as a rational algebraical meromorphic function of $\phi(z)$ *of the second order, doubly-periodic in the same periods and having the sum of its irreducible infinities congruent with* Ω.

Second, if $\psi(z)=-\psi(\Omega-z)$, then $R=0$ and $\psi(z)=S\phi'(z)$; that is, *a function* $\psi(z)$, *which satisfies the equation*

$$\psi(z)=-\psi(\Omega-z),$$

can be expressed as a rational algebraical meromorphic function of $\phi(z)$, *multiplied by* $\phi'(z)$, *where* $\phi(z)$ *is doubly-periodic in the same periods, is of the second order, and has the sum of its irreducible infinities congruent with* Ω.

Third, if $\psi(z)$ have no infinities except those of u, it cannot become infinite for finite values of u; hence $L=0$ has no roots, that is, L is a constant which may be taken to be unity. Then $\psi(z)$ a function of order m can be expressed in the form

$$M+N\phi'(z),$$

where, if the function $\phi(z)$ be of the second class, the degree of M is not higher than m; but, if it be of the first class, the degree of M is not higher than $m-1$; and in each case the degree of N is not higher than $m-2$.

It will be found in practice, with functions of the first class, that these upper limits for degrees can be considerably reduced by counting the degrees of the infinities in

$$M+N\phi'(z).$$

Thus, if the degree of M in u be μ and of N be λ, the highest degree of an infinity is either 2μ or $2\lambda+3$; so that, if the order of $\psi(z)$ be m, we should have

$$m=2\mu \text{ or } m=2\lambda+3,$$

according as m is even or odd.

When functions of the second class are used to represent a function $\psi(z)$, which has two infinities α and β each of degree n, then it is easy to see that M is of degree n and N of degree $n-2$; and so for other cases.

COROLLARY II. *Any doubly-periodic function can be expressed rationally in terms of any other function* u *of any order* n, *doubly-periodic in the same periods, and of its derivative; and this rational expression can always be taken in the form*

$$U_0+U_1u'+U_2u'^2+\ldots+U_{n-1}u'^{n-1},$$

where $U_0, \ldots, U_{n-1}$ *are algebraical, rational, meromorphic functions of* u.

COROLLARY III. *If* ϕ *be a doubly-periodic function, then* $\phi(u+v)$ *can be expressed in the form*

$$\frac{A+B\psi'(u)+C\psi'(v)+D\psi'(u)\psi'(v)}{E},$$

where ψ *is a doubly-periodic function in the same periods and of the second order: each of the functions* A, D, E *is a symmetric function of* $\psi(u)$ *and* $\psi(v)$, *and* B *is the same function of* $\psi(v)$ *and* $\psi(u)$ *as* C *is of* $\psi(u)$ *and* $\psi(v)$.

The degrees of A and E are not greater than m in $\psi(u)$ and than m in $\psi(v)$, where m is the order of ϕ; the degree of D is not greater than $m-2$ in $\psi(u)$ and than $m-2$ in $\psi(v)$; the degree of B is not greater than $m-2$ in $\psi(u)$ and than m in $\psi(v)$, and the degree of C is not greater than $m-2$ in $\psi(v)$ and than m in $\psi(u)$.

CHAPTER XI.

Doubly-Periodic Functions of the Second Order.

The present chapter will be devoted, in illustration of the preceding theorems, to the establishment of some of the fundamental formulæ relating to doubly-periodic functions of the second order which, as has already (in § 119, Cor. to Prop. XIV.) been indicated, are substantially elliptic functions: but for any development of their properties, recourse must be had to treatises on elliptic functions.

It may be remarked that, in dealing with doubly-periodic functions, we may restrict ourselves to a discussion of even functions and of odd functions. For, if $\phi(z)$ be any function, then $\frac{1}{2}\{\phi(z)+\phi(-z)\}$ is an even function, and $\frac{1}{2}\{\phi(z)-\phi(-z)\}$ is an odd function, both of them being doubly-periodic in the periods of $\phi(z)$; and the new functions would, in general, be of order double that of $\phi(z)$. We shall practically limit the discussion to even functions and odd functions of the second order.

120. Consider a function $\phi(z)$, doubly-periodic in 2ω and $2\omega'$; and let it be an odd function of the second class, with α and β as its irreducible infinities, and a and b as its irreducible zeros*.

Then we have
$$\phi(z)=\phi(\alpha+\beta-z)$$
which always holds, and
$$\phi(-z)=-\phi(z)$$
which holds because $\phi(z)$ is an odd function. Hence
$$\begin{aligned}\phi(\alpha+\beta+z)&=\phi(-z)\\&=-\phi(z)\end{aligned}$$
so that $\alpha+\beta$ is not a period; and
$$\begin{aligned}\phi(\alpha+\beta+\alpha+\beta+z)&=-\phi(\alpha+\beta+z)\\&=\phi(z),\end{aligned}$$

* To fix the ideas, it will be convenient to compare it with $\operatorname{sn} z$, for which $2\omega=4K$, $2\omega'=2iK'$, $\alpha=iK'$, $\beta=iK'+2K$, $a=0$, and $b=2K$.

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whence $2(\alpha+\beta)$ is a period. Since $\alpha+\beta$ is not a period, we take $\alpha+\beta \equiv \omega$, or $\equiv \omega'$, or $\equiv \omega+\omega'$; the first two alternatives merely interchange ω and ω', so that we have either

$$\alpha+\beta \equiv \omega,$$

or

$$\alpha+\beta \equiv \omega+\omega'.$$

And we know that, in general,

$$a+b \equiv \alpha+\beta.$$

First, for the zeros: we have

$$\phi(0) = -\phi(-0) = -\phi(0),$$

so that $\phi(0)$ is either zero or infinite. The choice is at our disposal; for $\dfrac{1}{\phi(z)}$ satisfies all the equations which have been satisfied by $\phi(z)$ and an infinity of either is a zero of the other. We therefore take

$$\phi(0) = 0,$$

so that we have

$$a = 0,$$

$$b = \omega \quad \text{or} \quad \omega+\omega'.$$

Next, for the infinities: we have

$$\phi(z) = -\phi(-z)$$

and therefore

$$\phi(-\alpha) = -\phi(\alpha) = \infty.$$

The only infinities of ϕ are α and β, so that either

$$-\alpha \equiv \alpha,$$

or

$$-\alpha \equiv \beta.$$

The latter cannot hold, because it would give $\alpha+\beta \equiv 0$ whereas $\alpha+\beta \equiv \omega$ or $\equiv \omega+\omega'$; hence

$$2\alpha \equiv 0,$$

which must be associated with $\alpha+\beta \equiv \omega$ or with $\alpha+\beta \equiv \omega+\omega'$.

Hence α, being a point inside the fundamental parallelogram, is either 0, ω, ω', or $\omega+\omega'$.

It cannot be 0 in any case, for that is a zero.

If $\alpha+\beta \equiv \omega$, then α cannot be ω, because that value would give $\beta = 0$, which is a zero, not an infinity. Hence either $\alpha = \omega'$, and then $\beta = \omega'+\omega$; or $\alpha = \omega'+\omega$, and then $\beta = \omega'$. These are effectively one solution; so that, if $\alpha+\beta \equiv \omega$, we have

$$\left.\begin{aligned} \alpha, \beta &= \omega', \omega'+\omega \\ \text{and} \qquad a, b &= 0, \omega \end{aligned}\right\}.$$

If $\alpha+\beta \equiv \omega+\omega'$, then α cannot be $\omega+\omega'$, because that value would give $\beta = 0$, which is a zero, not an infinity. Hence either $\alpha = \omega$ and then $\beta = \omega'$, or $\alpha = \omega'$ and then $\beta = \omega$. These again are effectively one solution; so that, if $\alpha+\beta \equiv \omega+\omega'$, we have

$$\left.\begin{aligned} \alpha, \beta &= \omega, \omega' \\ \text{and} \qquad a, b &= 0, \omega+\omega' \end{aligned}\right\}.$$

This combination can, by a change of fundamental parallelogram, be made the same as the former; for, taking as new periods

$$2\omega' = 2\omega', \qquad 2\Omega = 2\omega + 2\omega',$$

which give a new fundamental parallelogram, we have $\alpha + \beta \equiv \Omega$, and

$$\alpha, \beta = \omega', \Omega - \omega', \text{ that is, } \omega', \Omega - \omega' + 2\omega'$$

so that $$\left.\begin{array}{l}\alpha, \beta = \omega', \Omega + \omega' \\ a, b = 0, \Omega\end{array}\right\},$$
and

being the same as the former with Ω instead of ω. Hence it is sufficient to retain the first solution alone: and therefore

$$\alpha = \omega', \qquad \beta = \omega' + \omega,$$

$$a = 0, \qquad b = \omega.$$

Hence, by § 116, I., we have

$$\phi(z) = \frac{z(z-\omega)}{(z-\omega')(z-\omega-\omega')} F(z),$$

where $F(z)$ is finite everywhere within the parallelogram.

Again, $\phi(z+\omega')$ has $z=0$ and $z=\omega$ as its irreducible infinities, and it has $z=\omega'$ and $z=\omega+\omega'$ as its irreducible zeros, within the parallelogram of $\phi(z)$; hence

$$\phi(z+\omega') = \frac{(z-\omega')(z-\omega-\omega')}{z(z-\omega)} F_1(z),$$

where $F_1(z)$ is finite everywhere within the parallelogram. Thus

$$\phi(z)\,\phi(z+\omega') = F(z)\,F_1(z),$$

a function which is finite everywhere within the parallelogram; since it is doubly-periodic, it is finite everywhere in the plane and it is therefore a constant and equal to the value at any point. Taking $-\tfrac{1}{2}\omega'$ as the point (which is neither a zero nor an infinity) and remembering that ϕ is an odd function, we have

$$\phi(z)\,\phi(z+\omega') = -\{\phi(\tfrac{1}{2}\omega')\}^2 = \frac{1}{k},$$

k being a constant used to represent the value of $-\{\phi(\tfrac{1}{2}\omega')\}^{-2}$.

Also $$\phi(z+\omega) = \phi(z+\alpha+\beta-2\omega')$$
$$= \phi(z+\alpha+\beta) = -\phi(z),$$

and therefore also $$\phi(\omega - z) = \phi(z).$$

The irreducible zeros of $\phi'(z)$ were obtained in § 117, X. In the present example, those points are $\omega' + \tfrac{1}{2}\omega$, $\omega' + \tfrac{3}{2}\omega$, $\tfrac{1}{2}\omega$, $\tfrac{3}{2}\omega$; so that, as there, we have

$$K\{\phi'(z)\}^2 = \{\phi(z)-\phi(\tfrac{1}{2}\omega)\}\{\phi(z)-\phi(\tfrac{3}{2}\omega)\}\{\phi(z)-\phi(\omega'+\tfrac{1}{2}\omega)\}\{\phi(z)-\phi(\omega'+\tfrac{3}{2}\omega)\},$$

where K is a constant. But

$$\phi(\tfrac{3}{2}\omega) = \phi(2\omega - \tfrac{1}{2}\omega) = \phi(-\tfrac{1}{2}\omega) = -\phi(\tfrac{1}{2}\omega);$$

and
$$\phi(\tfrac{3}{2}\omega + \omega') = \phi(2\omega + 2\omega' - \tfrac{1}{2}\omega - \omega')$$
$$= \phi(-\tfrac{1}{2}\omega - \omega')$$
$$= -\phi(\tfrac{1}{2}\omega + \omega');$$
so that
$$\{\phi'(z)\}^2 = A\left[1 - \left\{\frac{\phi(z)}{\phi(\frac{1}{2}\omega)}\right\}^2\right]\left[1 - \left\{\frac{\phi(z)}{\phi(\frac{1}{2}\omega + \omega')}\right\}^2\right],$$
where A is a new constant, evidently equal to $\{\phi'(0)\}^2$. Now, as we know the periods, the irreducible zeros and the irreducible infinities of the function $\phi(z)$, it is completely determinate save as to a constant factor. To determine this factor we need only know the value of $\phi(z)$ for any particular finite value of z. Let the factor be determined by the condition
$$\phi(\tfrac{1}{2}\omega) = 1;$$
then, since
$$\phi(\tfrac{1}{2}\omega)\,\phi(\tfrac{1}{2}\omega + \omega') = \frac{1}{k}$$
by a preceding equation, we have
$$\phi(\tfrac{1}{2}\omega + \omega') = \frac{1}{k};$$
and then
$$\{\phi'(z)\}^2 = \{\phi'(0)\}^2[1 - \{\phi(z)\}^2][1 - k^2\{\phi(z)\}^2]$$
$$= \mu^2[1 - \{\phi(z)\}^2][1 - k^2\{\phi(z)\}^2].$$
Hence, since $\phi(z)$ is an odd function, we have
$$\phi(z) = \operatorname{sn}(\mu z).$$
Evidently $2\mu\omega$, $2\mu\omega' = 4K$, $2iK'$, where K and K' have the ordinary significations. The simplest case arises when $\mu = 1$.

121. Before proceeding to the deduction of the properties of even functions of z which are doubly-periodic, it is desirable to obtain the addition-theorem for ϕ, that is, the expression of $\phi(y+z)$ in terms of functions of y alone and z alone.

When $\phi(y+z)$ is regarded as a function of z, which is necessarily of the second order, it is (§ 119, XV.) of the form
$$\frac{M + N\phi'(z)}{L},$$
where M and L are of degree in $\phi(z)$ not higher than 2 and N is independent of z. Moreover $y + z = \alpha$ and $y + z = \beta$ are the irreducible simple infinities of $\phi(y+z)$; so that L, as a function of z, may be expressed in the form
$$\{\phi(z) - \phi(\alpha - y)\}\{\phi(z) - \phi(\beta - y)\},$$
and therefore
$$\phi(y+z) = \frac{P + Q\phi(z) + R\{\phi(z)\}^2 + S\phi'(z)}{\{\phi(z) - \phi(\alpha - y)\}\{\phi(z) - \phi(\beta - y)\}},$$

where P, Q, R, S are independent of z but they may be functions of y. Now

$$\phi(\alpha - y) = \phi(\omega' - y) = -\frac{1}{k\phi(y)},$$

and
$$\phi(\beta - y) = \phi(\omega' + \omega - y) = \frac{1}{k\phi(\omega - y)} = \frac{1}{k\phi(y)};$$

so that the denominator of the expression for $\phi(y+z)$ is

$$\{\phi(z)\}^2 - \frac{1}{k^2\{\phi(y)\}^2}.$$

Since $\phi(z)$ is an odd function, $\phi'(z)$ is even; hence

$$\phi(y - z) = \frac{P - Q\phi(z) + R\{\phi(z)\}^2 + S\phi'(z)}{\{\phi(z)\}^2 - \dfrac{1}{k^2\{\phi(y)\}^2}},$$

and therefore
$$\phi(y+z) - \phi(y-z) = \frac{2Q\phi(z)}{\{\phi(z)\}^2 - \dfrac{1}{k^2\{\phi(y)\}^2}}.$$

Differentiating with regard to z and then making $z = 0$, we have

$$2\phi'(y) = \frac{2Q\phi'(0)}{-\dfrac{1}{k^2\{\phi(y)\}^2}},$$

so that, substituting for Q we have

$$\phi(y+z) - \phi(y-z) = \frac{1}{\phi'(0)} \frac{2\phi(z)\phi'(y)}{1 - k^2\{\phi(y)\}^2\{\phi(z)\}^2}.$$

Interchanging y and z and noting that $\phi(y-z) = -\phi(z-y)$, we have

$$\phi(y+z) + \phi(y-z) = \frac{1}{\phi'(0)} \frac{2\phi(y)\phi'(z)}{1 - k^2\{\phi(y)\}^2\{\phi(z)\}^2},$$

and therefore
$$\phi(y+z)\phi'(0) = \frac{\phi(z)\phi'(y) + \phi(y)\phi'(z)}{1 - k^2\{\phi(y)\}^2\{\phi(z)\}^2}$$

which is the addition-theorem required.

Ex. If $f(u)$ be a doubly-periodic function of the second order with infinities b_1, b_2, and $\phi(u)$ a doubly-periodic function of the second order with infinities a_1, a_2 such that, in the vicinity of a_i (for $i = 1, 2$), we have

$$\phi(u) = \frac{(-1)^i \lambda}{u - a_i} + p_i + q_i(u - a_i) + \ldots\ldots,$$

then
$$\frac{f'(a_1) + f'(a_2)}{f(a_1) - f(a_2)} = -\frac{1}{\lambda}\{\phi(b_1) + \phi(b_2) - p_1 - p_2\},$$

the periods being the same for both functions. Verify the theorem when the functions are sn u and sn $(u+v)$. (Math. Trip. Part II., 1891.)

Prove also that, for the function $\phi(u)$, the coefficients p_1 and p_2 are equal. (Burnside.)

122. The preceding discussion of uneven doubly-periodic functions having two simple irreducible infinities is a sufficient illustration of the method of procedure. That, which now follows, relates to doubly-periodic functions with one irreducible infinity of the second degree; and it will be used to deduce some of the leading properties of Weierstrass's σ-function (of § 57) and of functions which arise from it.

The definition of the σ-function is

$$\sigma(z) = z \prod_{-\infty}^{\infty} \prod_{-\infty}^{\infty} \left\{ \left(1 - \frac{z}{\Omega}\right) e^{\frac{z}{\Omega} + \frac{1}{2}\frac{z^2}{\Omega^2}} \right\},$$

where $\Omega = 2m\omega + 2m'\omega'$, the ratio of $\omega' : \omega$ not being purely real, and the infinite product is extended over all terms that are given by assigning to m and to m' all positive and negative integral values from $+\infty$ to $-\infty$, excepting only simultaneous zero values. It has been proved (and it is easy to verify quite independently) that, when $\sigma(z)$ is regarded as the product of the primary factors

$$\left(1 - \frac{z}{\Omega}\right) e^{\frac{z}{\Omega} + \frac{1}{2}\frac{z^2}{\Omega^2}},$$

the doubly-infinite product converges uniformly and unconditionally for all values of z in the finite part of the plane; therefore the function which it represents can, in the vicinity of any point c in the plane, be expanded in a converging series of positive powers of $z - c$, but the series will only express the function in the domain of c. The series, however, can be continued over the whole plane.

It is at once evident that $\sigma(z)$ is not a doubly-periodic function, for it has no infinity in any finite part of the plane.

It is also evident that $\sigma(z)$ is an odd function. For a change of sign in z in a primary factor only interchanges that factor with the one which has equal and opposite values of m and of m', so that the product of the two factors is unaltered. Hence the product of all the primary factors, being independent of the nature of the infinite limits, is an even function; when z is associated as a factor, the function becomes uneven and it is $\sigma(z)$.

The first derivative, $\sigma'(z)$, is therefore an even function; and it is not infinite for any point in the finite part of the plane.

It will appear that, though $\sigma(z)$ is not periodic, it is connected with functions that have 2ω and $2\omega'$ for periods; and therefore the plane will be divided up into parallelograms. When the whole plane is divided up, as in § 105, into parallelograms, the adjacent sides of which are vectorial representations of 2ω and $2\omega'$, the function $\sigma(z)$ has one, and only one, zero in each parallelogram; each such zero is simple, and their aggregate is given by $z = \Omega$. The parallelogram of reference can be chosen so that a zero of $\sigma(z)$ does not lie upon its boundary; and, except where explicit account is taken of the alternative, we shall assume that the argument of ω' is greater than the argument of ω, so that the real part* of $\omega'/i\omega$ is positive.

123. We now proceed to obtain other expressions for $\sigma(z)$, and particularly, in the knowledge that it can be represented by a converging series in the vicinity of any point, to obtain a useful expression in the form of a series, converging in the vicinity of the origin.

Since $\sigma(z)$ is represented by an infinite product that converges uniformly and unconditionally for all finite values of z, its logarithm is equal to the sum of the logarithms of its factors, so that

$$\log \sigma(z) = \log z + \sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} \left\{ \frac{z}{\Omega} + \frac{1}{2}\frac{z^2}{\Omega^2} + \log\left(1 - \frac{z}{\Omega}\right) \right\},$$

where the series on the right-hand side extends to the same combinations of m and m' as the infinite product for z, and, when it is regarded as a sum of functions $\frac{z}{\Omega} + \frac{1}{2}\frac{z^2}{\Omega^2} + \log\left(1 - \frac{z}{\Omega}\right)$, the series converges uniformly and unconditionally, except for points $z = \Omega$. This expression is valid for $\log \sigma(z)$ over the whole plane.

Now let these additive functions be expanded, as in § 82. In the immediate vicinity of the origin, we have

$$\frac{z}{\Omega} + \frac{1}{2}\frac{z^2}{\Omega^2} + \log\left(1 - \frac{z}{\Omega}\right)$$

$$= -\frac{1}{3}\frac{z^3}{\Omega^3} - \frac{1}{4}\frac{z^4}{\Omega^4} - \frac{1}{5}\frac{z^5}{\Omega^5} - \dots,$$

a series which converges uniformly and unconditionally in that vicinity. Then the double series in the expression for $\log \sigma(z)$ becomes

$$-\sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} \left\{ \sum_{r=3}^{\infty} \frac{1}{r}\frac{z^r}{\Omega^r} \right\},$$

and as this new series converges uniformly and unconditionally for points in the vicinity of $z = 0$, we can, as in § 82, take it in the form

$$-\sum_{r=3}^{\infty} \frac{z^r}{r} \left\{ \sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} \Omega^{-r} \right\},$$

which will also, for such values of z, converge uniformly and unconditionally.

In § 56, it was proved that each of the coefficients

$$\sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} \Omega^{-r},$$

for $r = 3, 4, \dots$, is finite, and has a value independent of the nature of the infinite limits in the summation. When we make the positive infinite limit for m numerically equal to the negative infinite limit for m, and likewise for

* This quantity is often denoted by $\Re\left(\frac{\omega'}{i\omega}\right)$.

m', then each of these coefficients determined by an odd index r vanishes, and therefore it vanishes in general. We then have

$$\log \sigma(z) = \log z - \tfrac{1}{4} z^4 \Sigma\Sigma \Omega^{-4} - \tfrac{1}{6} z^6 \Sigma\Sigma \Omega^{-6} - \tfrac{1}{8} z^8 \Sigma\Sigma \Omega^{-8} - \ldots,$$

a series which converges uniformly and unconditionally in the vicinity of the origin.

The coefficients, which occur, involve ω and ω', two independent constants. It is convenient to introduce two other magnitudes, g_2 and g_3, defined by the equations

$$g_2 = 60 \Sigma\Sigma \Omega^{-4}, \quad g_3 = 140 \Sigma\Sigma \Omega^{-6},$$

so that g_2 and g_3 are evidently independent of one another; then all the remaining coefficients are functions* of g_2 and g_3. We thus have

$$\log \sigma(z) = \log z - \frac{1}{240} g_2 z^4 - \frac{1}{840} g_3 z^6 - \ldots - \frac{1}{2n} z^{2n} \Sigma\Sigma \Omega^{-2n} - \ldots,$$

and therefore $$\sigma(z) = z e^{-\frac{1}{240} g_2 z^4 - \frac{1}{840} g_3 z^6 - \ldots},$$

where the series in the index, containing only even powers of z, converges uniformly and unconditionally in the vicinity of the origin.

It is sufficiently evident that this expression for $\sigma(z)$ is an effective representation only in the vicinity of the origin; for points in the vicinity of any other zero of $\sigma(z)$, say c, a similar expression in powers of $z - c$ instead of in powers of z would be obtained.

124. From the first form of the expression for $\log \sigma(z)$, we have

$$\frac{\sigma'(z)}{\sigma(z)} = \frac{1}{z} + \sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} \left(\frac{1}{\Omega} + \frac{z}{\Omega^2} + \frac{1}{z - \Omega} \right),$$

where the quantity in the bracket on the right-hand side is to be regarded as an element of summation, being derived from the primary factor in the product-expression for $\sigma(z)$.

We write $$\zeta(z) = \frac{\sigma'(z)}{\sigma(z)},$$

so that $\zeta(z)$ is, by § 122, an odd function, a result also easily derived from the foregoing equation; and so

$$\zeta(z) = \frac{1}{z} + \Sigma\Sigma \left(\frac{1}{\Omega} + \frac{z}{\Omega^2} + \frac{1}{z - \Omega} \right).$$

This expression for $\zeta(z)$ is valid over the whole plane.

Evidently $\zeta(z)$ has simple infinities given by

$$z = \Omega,$$

for all values of m and of m' between $+\infty$ and $-\infty$, including simultaneous zeros. There is only one infinity in each parallelogram, and it is simple; for the function is the logarithmic derivative of $\sigma(z)$, which has no infinity and

* See *Quart. Journ.*, vol. xxii., pp. 4, 5. The magnitudes g_2 and g_3 are often called the *invariants*.

only one zero (a simple zero) in the parallelogram. Hence $\zeta(z)$ is not a doubly-periodic function.

For points, which are in the immediate vicinity of the origin, we have

$$\zeta(z)=\frac{d}{dz}\left[\log z-\frac{1}{240}g_2z^4-\frac{1}{840}g_3z^6-\ldots-\frac{1}{2n}z^{2n}\Sigma\Sigma\Omega^{-2n}-\ldots\right]$$

$$=\frac{1}{z}-\frac{1}{60}g_2z^3-\frac{1}{140}g_3z^5-\ldots-z^{2n-1}\Sigma\Sigma\Omega^{-2n}-\ldots;$$

but, as in the case of $\sigma(z)$, this is an effective representation of $\zeta(z)$ only in the vicinity of the origin; and a different expression would be used for points in the vicinity of any other infinity.

We again introduce a new function $\wp(z)$ defined by the equation

$$\wp(z)=-\frac{d\zeta(z)}{dz}=-\frac{d^2}{dz^2}\{\log\sigma(z)\}.$$

Because ζ is an odd function, $\wp(z)$ is an even function; and

$$\wp(z)=\frac{1}{z^2}-\sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty}\left\{\frac{1}{\Omega^2}-\frac{1}{(z-\Omega)^2}\right\}=\frac{1}{z^2}+\sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty}\left\{\frac{1}{(z-\Omega)^2}-\frac{1}{\Omega^2}\right\},$$

where the quantity in the bracket is to be regarded as an element of summation. This expression for $\wp(z)$ is valid over the whole plane. Evidently $\wp(z)$ has infinities, each of the second degree, given by $z=\Omega$, for all values of m and of m' between $+\infty$ and $-\infty$, including simultaneous zeros; and there is one, and only one, of these infinities in each parallelogram. One of these infinities is the origin; using the expression which represents $\log\sigma(z)$ in the immediate vicinity of the origin, we have

$$\wp(z)=-\frac{d^2}{dz^2}\left[\log z-\frac{1}{240}g_2z^4-\frac{1}{840}g_3z^6-\ldots\right]$$

$$=\frac{1}{z^2}+\frac{1}{20}g_2z^2+\frac{1}{28}g_3z^4+\ldots+(2n-1)z^{2n-2}\Sigma\Sigma\Omega^{-2n}+\ldots,$$

for points z in the immediate vicinity of the origin. A corresponding expression exists for $\wp(z)$ in the vicinity of any other infinity.

125. The importance of the function $\wp(z)$ is due to the following theorem:—

The function $\wp(z)$ is doubly-periodic, the periods being 2ω and $2\omega'$.

We have
$$\wp(z)=\frac{1}{z^2}+\sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty}\left\{\frac{1}{(z-\Omega)^2}-\frac{1}{\Omega^2}\right\},$$

where the doubly-infinite summation excludes simultaneous zero values, and the expression is valid over the whole plane. Hence

$$\wp(z+2\omega)=\frac{1}{(z+2\omega)^2}+\sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty}\left\{\frac{1}{(z+2\omega-\Omega)^2}-\frac{1}{\Omega^2}\right\},$$

so that

$$\wp(z+2\omega)-\wp(z)=\frac{1}{(z+2\omega)^2}-\frac{1}{z^2}+\sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty}\left\{\frac{1}{(z+2\omega-\Omega)^2}-\frac{1}{(z-\Omega)^2}\right\},$$

obtained by combining together the elements of the summation in $\wp(z+2\omega)$ and $\wp(z)$. The two terms, not included in the summation, can be included, if we remove the numerical restriction as to non-admittance of simultaneous zero values for m and m'; and then

$$\wp(z+2\omega)-\wp(z)=\sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty}\left\{\frac{1}{(z+2\omega-\Omega)^2}-\frac{1}{(z-\Omega)^2}\right\},$$

where now the summation is for all values of m and of m' from $+\infty$ to $-\infty$. Let q denote the infinite limit of m, and p that of m'. Then terms in the first fraction, for $\Omega=2(m\omega+m'\omega')$, are the same as terms in the second for $\Omega=2(m-1)\omega+2m'\omega'$; cancelling these, we have

$$\wp(z+2\omega)-\wp(z)=\sum_{m'=-p}^{m'=p}\left[\frac{1}{\{z+2(q+1)\omega-2m'\omega'\}^2}-\frac{1}{(z-2q\omega-2m'\omega')^2}\right],$$

where q is infinite. But

$$\sum_{n=-\infty}^{n=\infty}\frac{1}{(c-n\pi)^2}=\frac{1}{\sin^2 c},$$

and therefore

$$\sum_{m'=-p=-\infty}^{m'=p=\infty}\frac{1}{\{z+2(q+1)\omega-2m'\omega'\}^2}=\frac{\pi^2}{4\omega'^2}\frac{1}{\sin^2\left\{\dfrac{z+2(q+1)\omega}{2\omega'}\pi\right\}},$$

if p be infinitely great compared with q. This condition may be assumed for the present purpose, because the value of $\wp(z)$ is independent of the nature of the infinite limits in the summation and is therefore unaffected by such a limitation.

Now $$\sin\left[\{z+2(q+1)\omega\}\frac{\pi}{2\omega'}\right]=\frac{1}{2i}\left[e^{\frac{\pi zi}{2\omega'}+\frac{\pi\omega i}{\omega'}(q+1)}-e^{-\frac{\pi zi}{2\omega'}-\frac{\pi\omega i}{\omega'}(q+1)}\right].$$

The fraction $\dfrac{\omega i}{\omega'}$ has a real part. In the exponent it is multiplied by $q+1$, that is, by an infinite quantity; so that the real part of the index of the exponential is infinite, either positive or negative. Thus either the first term is infinite and the second zero, or vice versa; in either case, $\sin\left[\{z+2(q+1)\omega\}\dfrac{\pi}{2\omega'}\right]$ is infinite, and therefore

$$\sum_{m'=-p=-\infty}^{m'=p=\infty}\frac{1}{\{z+2(q+1)\omega-2m'\omega'\}^2}=0.$$

Similarly for the other sum. Hence

$$\wp(z+2\omega)-\wp(z)=0.$$

In the same way it may be shewn that

$$\wp(z+2\omega')-\wp(z)=0;$$

therefore $\wp(z)$ is doubly-periodic in 2ω and $2\omega'$.

Now in any parallelogram whose adjacent sides are 2ω and $2\omega'$, there is only one infinity and it is of multiplicity two; hence, by § 116, Prop. III., Cor. 3, 2ω and $2\omega'$ determine a primitive parallelogram for $\wp(z)$.

We shall assume the parallelogram of reference chosen so as to include the origin.

126. The function $\wp(z)$ is thus of the second order and the first class.

Since its irreducible infinity is of the second degree, the only irreducible infinity of $\wp'(z)$ is of the third degree, being the origin; and the function $\wp'(z)$ is odd.

The zeros of $\wp'(z)$ are thus ω, ω', and $(\omega + \omega')$; or, if we introduce a new quantity ω'' defined by the equation

$$\omega'' = \omega + \omega',$$

the zeros of $\wp'(z)$ are $\omega, \omega', \omega''$.

We take

$$\wp(\omega) = e_1, \qquad \wp(\omega'') = e_2, \qquad \wp(\omega') = e_3, \qquad \wp(z) = \wp;$$

and then, by § 119, Prop. XIV., Cor., we have

$$\wp'^2 = A(\wp - e_1)(\wp - e_2)(\wp - e_3),$$

where A is some constant. To determine the equation more exactly, we substitute the expression of $\wp$ in the vicinity of the origin. Then

$$\wp = \frac{1}{z^2} + \frac{1}{20} g_2 z^2 + \frac{1}{28} g_3 z^4 + \dots$$

so that

$$\wp' = -\frac{2}{z^3} + \frac{1}{10} g_2 z + \frac{1}{7} g_3 z^3 + \dots.$$

When substitution is made, it is necessary to retain in the expansion all terms up to z^0 inclusive. We then have, for $\wp'^2$, the expression

$$\frac{4}{z^6} - \frac{2}{5}\frac{g_2}{z^2} - \frac{4}{7} g_3 + \dots;$$

and for $A(\wp - e_1)(\wp - e_2)(\wp - e_3)$, the expression

$$A\left[\frac{1}{z^6} + \frac{3}{20}\frac{g_2}{z^2} + \frac{3}{28} g_3 + \dots \right.$$
$$\left. - (e_1 + e_2 + e_3)\left(\frac{1}{z^4} + \frac{1}{10} g_2 + \dots\right) + (e_1 e_2 + e_2 e_3 + e_3 e_1)\left(\frac{1}{z^2} + \dots\right) - e_1 e_2 e_3\right].$$

When we equate coefficients in these two expressions, we find

$$A = 4,$$

$$e_1 + e_2 + e_3 = 0, \qquad e_1 e_2 + e_2 e_3 + e_3 e_1 = -\tfrac{1}{4} g_2, \qquad e_1 e_2 e_3 = \tfrac{1}{4} g_3;$$

therefore the differential equation satisfied by $\wp$ is

$$\wp'^2 = 4(\wp - e_1)(\wp - e_2)(\wp - e_3)$$
$$= 4\wp^3 - g_2 \wp - g_3.$$

Evidently
$$\wp'' = 6\wp^2 - \tfrac{1}{2}g_2,$$
$$\wp''' = 12\wp\wp',$$
and so on; and it is easy to verify that the $2n$th derivative of $\wp$ is a rational integral algebraical function of $\wp$ of degree $n+1$ and that the $(2n+1)$th derivative of $\wp$ is the product of $\wp'$ by a rational integral algebraical function of degree n.

The differential equation can be otherwise obtained, by dependence on Cor. 2, Prop. V. of § 116. We have, by differentiation of $\wp'$,
$$\wp'' = \frac{6}{z^4} + \frac{1}{10}g_2 + \frac{3}{7}g_3z^2 + \ldots$$
for points in the vicinity of the origin; and also
$$\wp^2 = \frac{1}{z^4} + \frac{1}{10}g_2 + \frac{1}{14}g_3z^2 + \ldots.$$
Hence $\wp''$ and $\wp^2$ have the same irreducible infinities in the same degree and their fractional parts are essentially the same: they are homoperiodic and therefore they are equivalent to one another. It is easy to see that $\wp'' - 6\wp^2$ is equal to a function which, being finite in the vicinity of the origin, is finite in the parallelogram of reference and therefore, as it is doubly-periodic, is finite over the whole plane. It therefore has a constant value, which can be obtained by taking the value at any point; the value of the function for $z=0$ is $-\tfrac{1}{2}g_2$ and therefore
$$\wp'' - 6\wp^2 = -\tfrac{1}{2}g_2,$$
so that
$$\wp'' = 6\wp^2 - \tfrac{1}{2}g_2,$$
the integration of which, with determination of the constant of integration, leads to the former equation.

This form, involving the second derivative, is a convenient one by which to determine a few more terms of the expansion in the vicinity of the origin: and it is easy to shew that
$$\wp = \frac{1}{z^2} + \frac{1}{20}g_2z^2 + \frac{1}{28}g_3z^4 + \frac{1}{1200}g_2^2z^6 + \frac{3}{6160}g_2g_3z^8 + \ldots,$$
from which some theorems relating to the sums $\Sigma\Sigma\Omega^{-2n}$ can be deduced*.

Ex. If c_n be the coefficient of z^{2n-2} in the expansion of $\wp(z)$ in the vicinity of the origin, then
$$c_n = \frac{3}{(2n+1)(n-3)} \sum_{r=2}^{r=n-2} c_r c_{n-r}. \qquad \text{(Weierstrass.)}$$

We have
$$\wp'^2 = 4\wp^3 - g_2\wp - g_3;$$
the function $\wp'$ is odd and in the vicinity of the origin we have
$$\wp' = -\frac{2}{z^3} + \ldots;$$

* See a paper by the author, *Quart. Journ.*, vol. xxii, (1887), pp. 1—43, where other references are given and other applications of the general theorems are made.

hence, representing by $-(4\wp^3-g_2\wp-g_3)^{\frac{1}{2}}$ that branch of the function which is negative for large real values, we have

$$\frac{d\wp}{dz}=-(4\wp^3-g_2\wp-g_3)^{\frac{1}{2}},$$

and therefore

$$z=\int_{\wp}\frac{d\wp}{(4\wp^3-g_2\wp-g_3)^{\frac{1}{2}}}.$$

The upper limit is determined by the fact that when $z=0$, $\wp=\infty$; so that

$$\begin{aligned} z&=\int_{\wp}^{\infty}\frac{d\wp}{(4\wp^3-g_2\wp-g_3)^{\frac{1}{2}}}\\ &=\int_{\wp}^{\infty}\frac{d\wp}{\{4(\wp-e_1)(\wp-e_2)(\wp-e_3)\}^{\frac{1}{2}}}.\end{aligned}$$

This is, as it should be, an integral with a doubly-infinite series of values. We have, by Ex. 6 of § 104,

$$\omega_1=\omega\ =\int_{e_1}^{\infty}\frac{d\wp}{(4\wp^3-g_2\wp-g_3)^{\frac{1}{2}}},$$

$$\omega_2=\omega''=\int_{e_2}^{\infty}\frac{d\wp}{(4\wp^3-g_2\wp-g_3)^{\frac{1}{2}}},$$

$$\omega_3=\omega'\ =\int_{e_3}^{\infty}\frac{d\wp}{(4\wp^3-g_2\wp-g_3)^{\frac{1}{2}}},$$

with the relation $\omega''=\omega+\omega'$.

127. We have seen that $\wp(z)$ is doubly-periodic, so that

$$\wp(z+2\omega)=\wp(z),$$

and therefore

$$\frac{d\zeta(z+2\omega)}{dz}=\frac{d\zeta(z)}{dz};$$

hence integrating $\zeta(z+2\omega)=\zeta(z)+A$.

Now ζ is an odd function; hence, taking $z=-\omega$ which is not an infinity of ζ, we have

$$A=2\zeta(\omega)=2\eta$$

say, where η denotes $\zeta(\omega)$; and therefore

$$\zeta(z+2\omega)-\zeta(z)=2\eta,$$

which is a constant.

Similarly $\zeta(z+2\omega')-\zeta(z)=2\eta'$,

where $\eta'=\zeta(\omega')$ and is constant.

Hence combining the results, we have

$$\zeta(z+2m\omega+2m'\omega')-\zeta(z)=2m\eta+2m'\eta',$$

where m and m' are any integers.

It is evident that η and η' cannot be absorbed into ζ; so that ζ is not a periodic function, a result confirmatory of the statement in § 124.

There is, however, a *pseudo-periodicity* of the function ζ: its characteristic is the reproduction of the function with an added constant for an added period. This form is only one of several simple forms of pseudo-periodicity which will be considered in the next chapter.

128. But, though $\zeta(z)$ is not periodic, functions which are periodic can be constructed by its means.

Thus, if $\phi(z) = A\,\zeta(z-a) + B\,\zeta(z-b) + C\,\zeta(z-c) + \ldots,$

then
$$\phi(z+2\omega) - \phi(z) = \Sigma A\,\{\zeta(z-a+2\omega) - \zeta(z-a)\}$$
$$= 2\eta\,(A + B + C + \ldots),$$
and
$$\phi(z+2\omega') - \phi(z) = 2\eta'\,(A + B + C + \ldots),$$
so that, subject to the condition
$$A + B + C + \ldots = 0,$$
$\phi(z)$ is a doubly-periodic function.

Again, we know that, within the fundamental parallelogram, ζ has a single irreducible infinity and that the infinity is simple; hence the irreducible infinities of the function $\phi(z)$ are $z = a, b, c, \ldots$, and each is a simple infinity. The condition $A + B + C + \ldots = 0$ is merely the condition of Prop. III., § 116, that the 'integral residue' of the function is zero.

Conversely, a doubly-periodic function with m assigned infinities can be expressed in terms of ζ and its derivatives. Let a_1 be an irreducible infinity of Φ of degree n, and suppose that the fractional part of Φ for expansion in the immediate vicinity of a_1 is
$$\frac{A_1}{z-a_1} + \frac{B_1}{(z-a_1)^2} + \ldots + \frac{K_1}{(z-a_1)^n}.$$
Then
$$\Phi(z) - \left[A_1\zeta(z-a_1) - B_1\zeta'(z-a_1) + \frac{C_1}{2!}\zeta''(z-a_1) - \ldots \right.$$
$$\left. + (-1)^n \frac{K_1}{(n-1)!}\frac{d^{n-1}}{dz^{n-1}}\zeta(z-a_1)\right]$$
is not infinite for $z = a_1$.

Proceeding similarly for each of the irreducible infinities, we have a function
$$\Phi(z) - \sum_{r=1}^{m}\left[A_r\zeta(z-a_r) - B_r\zeta'(z-a_r) + \frac{C_r}{2!}\zeta''(z-a_r) - \ldots\right],$$
which is not infinite for any of the points $z = a_1, a_2, \ldots$. But because $\Phi(z)$ is doubly-periodic, we have
$$A_1 + A_2 + \ldots + A_n = 0,$$
and therefore the function
$$\sum_{r=1}^{m} A_r\zeta(z-a_r)$$

is doubly-periodic. Moreover, all the derivatives of any order of each of the functions ζ are doubly-periodic; hence the foregoing function is doubly-periodic.

The function has been shewn to be not infinite at the points $a_1, a_2, \ldots$, and therefore it has no infinities in the fundamental parallelogram; consequently, being doubly-periodic, it has no infinities in the plane and it is a constant, say C. Hence we have

$$\Phi(z) = C + \sum_{r=1}^{m} A_r \zeta(z - a_r) - \sum_{r=1}^{m} B_r \frac{d\zeta(z - a_r)}{dz} + \frac{1}{2\,!} \sum_{r=1}^{m} C_r \frac{d^2\zeta(z - a_r)}{dz^2} - \ldots,$$

with the condition $\sum_{r=1}^{m} A_r = 0$, which is satisfied because $\Phi(z)$ is doubly-periodic.

This is the required expression* for $\Phi(z)$ in terms of the function ζ and its derivatives; it is evidently of especial importance when the indefinite integral of a doubly-periodic function is required.

129. Constants η and η', connected with ω and ω', have been introduced by the pseudo-periodicity of $\zeta(z)$; the relation, contained in the following proposition, is necessary and useful:—

The constants η, η', ω, ω' are connected by the relation

$$\eta\omega' - \eta'\omega = \pm \tfrac{1}{2}\pi i,$$

the + or − sign being taken according as the real part of $\omega'/\omega i$ is positive or negative.

A fundamental parallelogram having an angular point at z_0 is either of the form (i) in fig. 34, in which case $\Re\left(\dfrac{\omega'}{\omega i}\right)$ is positive; or of the form (ii), in which case $\Re\left(\dfrac{\omega'}{\omega i}\right)$ is negative. Evidently a description of the parallelogram $ABCD$ in (i) will give for an integral the same result (but with an opposite sign) as a description of the parallelogram in (ii) for the same integral in the direction $ABCD$ in that figure.

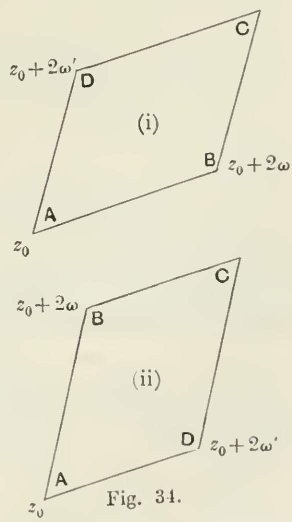

Fig. 34.

We choose the fundamental parallelogram, so that it may contain the origin in the included area. The origin is the only infinity of ζ which can be within the area: along the boundary ζ is always finite.

Now since

$$\zeta(z + 2\omega) - \zeta(z) = 2\eta,$$
$$\zeta(z + 2\omega') - \zeta(z) = 2\eta',$$

* See Hermite, *Ann. de Toulouse*, t. ii, (1888), C, pp. 1—12.

the integral of $\zeta(z)$ round $ABCD$ in (i), fig. 34, is (§ 116, Prop. II., Cor.)

$$\int_A^D 2\eta dz - \int_A^B 2\eta' dz,$$

the integrals being along the lines AD and AB respectively, that is, the integral is

$$4(\eta\omega' - \eta'\omega).$$

But as the origin is the only infinity within the parallelogram, the path of integration $ABCDA$ can be deformed so as to be merely a small curve round the origin. In the vicinity of the origin, we have

$$\zeta(z) = \frac{1}{z} - \frac{1}{60} g_2 z^3 - \frac{1}{140} g_3 z^5 - \dots,$$

and therefore, as the integrals of all terms except the first vanish when taken round this curve, we have

$$\int \zeta(z)\, dz = \int \frac{dz}{z}$$

$$= 2\pi i.$$

Hence $$4(\eta\omega' - \eta'\omega) = 2\pi i,$$

and therefore $$\eta\omega' - \eta'\omega = \tfrac{1}{2}\pi i.$$

This is the result as derived from (i), fig. 34, that is, when $\Re\left(\frac{\omega'}{i\omega}\right)$ is positive.

When (ii), fig. 34, is taken account of, the result is the same except that, when the circuit passes from z_0 to $z_0 + 2\omega$, then to $z_0 + 2\omega + 2\omega'$, then to $z_0 + 2\omega'$ and then to z_0, it passes in the negative direction round the parallelogram. The value of the integral along the path $ABCDA$ is the same as before, viz., $4(\eta\omega' - \eta'\omega)$; when the path is deformed into a small curve round the origin, the value of the integral is $\int \frac{dz}{z}$ taken negatively, and therefore it is $-2\pi i$: hence

$$\eta\omega' - \eta'\omega = -\tfrac{1}{2}\pi i.$$

Combining the results, we have

$$\eta\omega' - \eta'\omega = \pm \tfrac{1}{2}\pi i,$$

according as $\Re\left(\frac{\omega'}{\omega i}\right)$ is positive or negative.

Corollary. If there be a change to any other fundamental parallelogram, determined by 2Ω and $2\Omega'$, where

$$\Omega = p\omega + q\omega', \qquad \Omega' = p'\omega + q'\omega',$$

p, q, p', q' being integers such that $pq' - p'q = \pm 1$, and if H, H' denote $\zeta(\Omega)$, $\zeta(\Omega')$, then

$$H = p\eta + q\eta', \qquad H' = p'\eta + q'\eta';$$

therefore $$H\Omega' - H'\Omega = \pm \tfrac{1}{2}\pi i,$$

according as the real part of $\frac{\Omega'}{i\Omega}$ is positive or negative.

130. It has been seen that $\zeta(z)$ is pseudo-periodic; there is also a pseudo-periodicity for $\sigma(z)$, but of a different kind. We have

$$\zeta(z+2\omega)=\zeta(z)+2\eta,$$

that is,

$$\frac{\sigma'(z+2\omega)}{\sigma(z+2\omega)}=\frac{\sigma'(z)}{\sigma(z)}+2\eta,$$

and therefore

$$\sigma(z+2\omega)=Ae^{2\eta z}\sigma(z),$$

where A is a constant. To determine A, we make $z=-\omega$, which is not a zero or an infinity of $\sigma(z)$; then, since $\sigma(z)$ is an odd function, we have

$$-Ae^{-2\eta\omega}=1,$$

so that

$$\sigma(z+2\omega)=-e^{2\eta(z+\omega)}\sigma(z).$$

Hence

$$\sigma(z+4\omega)=-e^{2\eta(z+3\omega)}\sigma(z+2\omega)$$
$$=e^{2\eta(2z+4\omega)}\sigma(z);$$

and similarly

$$\sigma(z+2m\omega)=(-1)^m e^{2\eta(mz+m^2\omega)}\sigma(z).$$

Proceeding in the same way from

$$\zeta(z+2\omega')=\zeta(z)+2\eta',$$

we find

$$\sigma(z+2m'\omega')=(-1)^{m'}e^{2\eta'(m'z+m'^2\omega')}\sigma(z).$$

Then

$$\sigma(z+2m\omega+2m'\omega')=(-1)^m e^{2\eta(mz+m^2\omega+2mm'\omega')}\sigma(z+2m'\omega')$$
$$=(-1)^{m+m'}e^{2z(m\eta+m'\eta')+2\eta m^2\omega+4\eta mm'\omega'+2\eta'm'^2\omega'}\sigma(z)$$
$$=(-1)^{m+m'}e^{2(m\eta+m'\eta')(z+m\omega+m'\omega')+2mm'(\eta\omega'-\eta'\omega)}\sigma(z).$$

But

$$\eta\omega'-\eta'\omega=\pm\tfrac{1}{2}\pi i,$$

so that

$$e^{2mm'(\eta\omega'-\eta'\omega)}=e^{\pm mm'\pi i}=(-1)^{mm'},$$

and therefore

$$\sigma(z+2m\omega+2m'\omega')=(-1)^{mm'+m+m'}e^{2(m\eta+m'\eta')(z+m\omega+m'\omega')}\sigma(z),$$

which is the law of change of $\sigma(z)$ for increase of z by integral multiples of the periods.

Evidently $\sigma(z)$ is not a periodic function, a result confirmatory of the statement in § 122. But there is a pseudo-periodicity the characteristic of which is the reproduction, for an added period, of the function with an exponential factor the index being linear in the variable. This is another of the forms of pseudo-periodicity which will be considered in the next chapter.

131. But though $\sigma(z)$ is not periodic, we can by its means construct functions which are periodic in the pseudo-periods of $\sigma(z)$.

By the result in the last section, we have

$$\frac{\sigma(z-\alpha+2m\omega+2m'\omega')}{\sigma(z-\beta+2m\omega+2m'\omega')}=\frac{\sigma(z-\alpha)}{\sigma(z-\beta)}e^{2(m\eta+m'\eta')(\beta-\alpha)};$$

and therefore, if $\phi(z)$ denote

$$\frac{\sigma(z-\alpha_1)\,\sigma(z-\alpha_2)\ldots\ldots\sigma(z-\alpha_n)}{\sigma(z-\beta_1)\,\sigma(z-\beta_2)\ldots\ldots\sigma(z-\beta_n)},$$

then $$\phi(z+2m\omega+2m'\omega')=e^{2(m\eta+m'\eta')(\Sigma\beta_r-\Sigma\alpha_r)}\,\phi(z),$$

so that $\phi(z)$ is doubly-periodic in 2ω and $2\omega'$ provided

$$\Sigma\beta_r-\Sigma\alpha_r=0.$$

Now the zeros of $\phi(z)$, regarded as a product of σ-functions, are $\alpha_1, \alpha_2, \ldots, \alpha_n$ and the points homologous with them; and the infinities are $\beta_1, \beta_2, \ldots, \beta_n$ and the points homologous with them. It may happen that the points α and β are not all in the parallelogram of reference; if the irreducible points homologous with them be $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$, then

$$\Sigma a_r \equiv \Sigma b_r \ (\text{mod. } 2\omega,\ 2\omega'),$$

and the new points are the irreducible zeros and the irreducible infinities of $\phi(z)$. This result, we know from Prop. III., § 116, must be satisfied.

It is naturally assumed that no one of the points α is the same as, or is homologous with, any one of the points β: the order of the doubly-periodic function would otherwise be diminished by 1.

If any α be repeated, then that point is a repeated zero of $\phi(z)$; similarly if any β be repeated, then that point is a repeated infinity of $\phi(z)$. In every case, the sum of the irreducible zeros must be congruent with the sum of the irreducible infinities in order that the above expression for $\phi(z)$ may be doubly-periodic.

Conversely, if a doubly-periodic function $\phi(z)$ be required with m assigned irreducible zeros a and m assigned irreducible infinities b, which are subject to the congruence

$$\Sigma a \equiv \Sigma b \ (\text{mod. } 2\omega,\ 2\omega'),$$

we first find points α and β homologous with a and with b respectively such that

$$\Sigma\alpha=\Sigma\beta.$$

Then the function $$\frac{\sigma(z-\alpha_1)\ldots\ldots\sigma(z-\alpha_m)}{\sigma(z-\beta_1)\ldots\ldots\sigma(z-\beta_m)}$$

has the same zeros and the same infinities as $\phi(z)$, and is homoperiodic with it; and therefore, by § 116, IV.,

$$\phi(z)=A\,\frac{\sigma(z-\alpha_1)\ldots\ldots\sigma(z-\alpha_m)}{\sigma(z-\beta_1)\ldots\ldots\sigma(z-\beta_m)},$$

where A is a quantity independent of z.

Ex. 1. Consider $\wp'(z)$. It has the origin for an infinity of the third degree and all the remaining infinities are reducible to the origin; and its three irreducible zeros are ω, ω', ω''. Moreover, since $\omega''=\omega'+\omega$, we have $\omega+\omega'+\omega''$ congruent with but not equal to zero. We therefore choose other points so that the sum of the zeros may be actually the same as the sum of the infinities, which is zero; the simplest choice is to take ω, ω', $-\omega''$. Hence

$$\wp'(z) = A\,\frac{\sigma(z-\omega)\,\sigma(z-\omega')\,\sigma(z+\omega'')}{\sigma^3(z)},$$

where A is a constant. To determine A, consider the expansions in the immediate vicinity of the origin; then

$$-\frac{2}{z^3}+\ldots\ldots = A\,\frac{\sigma(-\omega)\,\sigma(-\omega')\,\sigma(\omega'')}{z^3}+\ldots\ldots,$$

so that

$$\wp'(z) = -2\,\frac{\sigma(z-\omega)\,\sigma(z-\omega')\,\sigma(z+\omega'')}{\sigma(\omega)\,\sigma(\omega')\,\sigma(\omega'')\,\sigma^3(z)}.$$

Another method of arranging zeros, so that their sum is equal to that of the infinities, is to take $-\omega$, $-\omega'$, ω''; and then we should find

$$\wp'(z) = 2\,\frac{\sigma(z+\omega)\,\sigma(z+\omega')\,\sigma(z-\omega'')}{\sigma(\omega)\,\sigma(\omega')\,\sigma(\omega'')\,\sigma^3(z)}.$$

This result can, however, be deduced from the preceding form merely by changing the sign of z.

Ex. 2. Consider the function

$$A\,\frac{\sigma(u+v)\,\sigma(u-v)}{\sigma^2(u)},$$

where v is any quantity and A is independent of u. It is, quà function of u, doubly-periodic; and it has $u=0$ as an infinity of the second degree, all the infinities being homologous with the origin. Hence the function is homoperiodic with $\wp(u)$ and it has the same infinities as $\wp(u)$: thus the two are equivalent, so that

$$A\,\frac{\sigma(u+v)\,\sigma(u-v)}{\sigma^2(u)} = B\wp(u) - C,$$

where B and C are independent of u. The left-hand side vanishes if $u=v$; hence $C=B\wp(v)$, and therefore

$$A'\,\frac{\sigma(u+v)\,\sigma(u-v)}{\sigma^2(u)} = \wp(u) - \wp(v),$$

where A' is a new quantity independent of u. To determine A' we consider the expansions in the vicinity of $u=0$; we have

$$\frac{A'\sigma(v)\,\sigma(-v)}{u^2}+\ldots\ldots = \frac{1}{u^2}+\ldots\ldots,$$

so that

$$-A'\sigma^2(v) = 1,$$

and therefore

$$\frac{\sigma(u+v)\,\sigma(u-v)}{\sigma^2(u)\,\sigma^2(v)} = \wp(v) - \wp(u),$$

a formula of very great importance.

Ex. 3. Taking logarithmic derivatives with regard to u of the two sides of the last equation, we have

$$\zeta(u+v)+\zeta(u-v)-2\zeta(u) = -\frac{\wp'(u)}{\wp(v)-\wp(u)};$$

and, similarly, taking them with regard to v, we have

$$\zeta(u+v)-\zeta(u-v)-2\zeta(v) = \frac{\wp'(v)}{\wp(v)-\wp(u)};$$

whence

$$\zeta(u+v)-\zeta(u)-\zeta(v) = \tfrac{1}{2}\,\frac{\wp'(v)-\wp'(u)}{\wp(v)-\wp(u)},$$

giving the special value of the left-hand side as (§ 128) a doubly-periodic function. It is also the addition-theorem, so far as there is an addition-theorem, for the ζ-function.

Ex. 4. We can, by differentiation, at once deduce the addition-theorem for $\wp(u+v)$. Evidently

$$\wp(u+v)=\wp(u)-\tfrac{1}{2}\frac{d}{du}\left\{\frac{\wp'(v)-\wp'(z)}{\wp(v)-\wp(z)}\right\},$$

which is only one of many forms: one of the most useful is

$$\wp(u+v)=-\wp(u)-\wp(v)+\tfrac{1}{4}\left\{\frac{\wp'(z)-\wp'(v)}{\wp(z)-\wp(v)}\right\}^2,$$

which can be deduced from the preceding form.

The result can be used to modify the expression for a general doubly-periodic function $\Phi(z)$ obtained in § 128. We have

$$\begin{aligned}\sum_{r=1}^{m} A_r\zeta(z-a_r)&=\sum_{r=1}^{m} A_r\left\{\zeta(z)-\zeta(a_r)-\tfrac{1}{2}\frac{\wp'(a_r)+\wp'(z)}{\wp(a_r)-\wp(z)}\right\}\\&=\left(\sum_{r=1}^{m} A_r\right)\zeta(z)-\sum_{r=1}^{m} A_r\zeta(a_r)+\tfrac{1}{2}\sum_{r=1}^{m} A_r\frac{\wp'(z)+\wp'(a_r)}{\wp(z)-\wp(a_r)}\\&=-\sum_{r=1}^{m} A_r\zeta(a_r)+\tfrac{1}{2}\sum_{r=1}^{m} A_r\frac{\wp'(z)+\wp'(a_r)}{\wp(z)-\wp(a_r)}.\end{aligned}$$

Each derivative of ζ can be expressed either as an integral algebraical function of $\wp(z-a_r)$ or as the product of $\wp'(z-a_r)$ by such a function; and by the use of the addition-theorem these can be expressed in the form

$$\frac{M+N\wp'(z)}{L},$$

where L, M, N are rational integral algebraical functions of $\wp(z)$. Hence the function $\Phi(z)$ can be expressed in the same form, the simplest case being when all its infinities are simple, and then

$$\begin{aligned}\Phi(z)&=C+\sum_{r=1}^{m} A_r\zeta(z-a_r)\\&=C-\sum_{r=1}^{m} A_r\zeta(a_r)+\tfrac{1}{2}\sum_{r=1}^{m} A_r\frac{\wp'(z)+\wp'(a_r)}{\wp(z)-\wp(a_r)}\\&=B+\tfrac{1}{2}\sum_{r=1}^{m} A_r\frac{\wp'(z)+\wp'(a_r)}{\wp(z)-\wp(a_r)},\end{aligned}$$

with the condition $\sum_{r=1}^{m} A_r=0$.

Ex. 5. The function $\wp(z)-e_1$ is an even function, doubly-periodic in 2ω and $2\omega'$ and having $z=0$ for an infinity of the second degree; it has only a single infinity of the second degree in a fundamental parallelogram.

Again, $z=\omega$ is a zero of the function; and, since $\wp'(\omega)=0$ but $\wp''(\omega)$ is not zero, it is a double zero of $\wp(z)-e_1$. All the zeros are therefore reducible to $z=\omega$; and the function has only a single zero of the second degree in a fundamental parallelogram.

Taking then the parallelogram of reference so as to include the points $z=0$ and $z=\omega$, we have

$$\wp(z)-e_1=\frac{(z-\omega)^2}{z^2}Q(z),$$

where $Q(z)$ has no zero and no infinity for points within the parallelogram.

Again, for $\wp(z+\omega)-e_1$, the irreducible zero of the second degree within the parallelogram is given by $z+\omega\equiv\omega$, that is, it is $z=0$; and the irreducible infinity of the second degree within the parallelogram is given by $z+\omega\equiv 0$, that is, it is $z=\omega$. Hence we have

$$\wp(z+\omega)-e_1=\frac{z^2}{(z-\omega)^2}Q_1(z),$$

where $Q_1(z)$ has no zero and no infinity for points within the parallelogram.

Hence $$\{\wp(z)-e_1\}\{\wp(z+\omega)-e_1\}=Q(z)\,Q_1(z),$$

that is, it is a function which has no zero and no infinity for points within the parallelogram of reference. Being doubly-periodic, it therefore has no zero and no infinity anywhere in the plane; it consequently is a constant, which is the value for any point. Taking the special value $z=\omega'$, we have $\wp(\omega')=e_3$, and $\wp(\omega'+\omega)=e_2$; and therefore

$$\{\wp(z)-e_1\}\{\wp(z+\omega)-e_1\}=(e_3-e_1)(e_2-e_1).$$

Similarly $$\{\wp(z)-e_2\}\{\wp(z+\omega'')-e_2\}=(e_1-e_2)(e_3-e_2),$$

and $$\{\wp(z)-e_3\}\{\wp(z+\omega')-e_3\}=(e_2-e_3)(e_1-e_3).$$

It is possible to derive at once from these equations the values of the $\wp$-function for the quarter-periods.

Note. In the preceding chapter some theorems were given which indicated that functions, which are doubly-periodic in the same periods, can be expressed in terms of one another: in particular cases, care has occasionally to be exercised to be certain that the periods of the functions are the same, especially when transformations of the variables are effected. For instance, since $\wp(z)$ has the origin for an infinity and sn u has it for a zero, it is natural to express the one in terms of the other. Now $\wp(z)$ is an even function, and sn u is an odd function; hence the relation to be obtained will be expected to be one between $\wp(z)$ and $\text{sn}^2 u$. But one of the periods of $\text{sn}^2 u$ is only one-half of the corresponding period of sn u; and so the period-parallelogram is changed. The actual relation* is

$$\wp(z)-e_3=(e_1-e_3)\,\text{sn}^{-2}u,$$

where $u=(e_1-e_3)^{\frac{1}{2}}z$ and $k^2=(e_2-e_3)/(e_1-e_3)$.

Again, with the ordinary notation of Jacobian elliptic functions, the periods of sn z are $4K$ and $2iK'$, those of dn z are $2K$ and $4iK'$, and those of cn z are $4K$ and $2K+2iK'$. The squares of these three functions are homoperiodic in $2K$ and $2iK'$; they are each of the second order, and they have the same infinities. Hence $\text{sn}^2 z$, $\text{cn}^2 z$, $\text{dn}^2 z$ are equivalent to one another (§ 116, V.).

But such cases belong to the detailed development of the theory of particular classes of functions, rather than to what are merely illustrations of the general propositions.

132. As a last illustration giving properties of the functions just considered, the derivatives of an elliptic function with regard to the periods will be obtained.

Let $\phi(z)$ be any function, doubly-periodic in 2ω and $2\omega'$ so that

$$\phi(z+2m\omega+2m'\omega')=\phi(z),$$

the coefficients in ϕ implicitly involve ω and ω'. Let ϕ_1, ϕ_2, and ϕ' respectively denote $\partial\phi/\partial\omega$, $\partial\phi/\partial\omega'$, $\partial\phi/\partial z$; then

$$\phi_1(z+2m\omega+2m'\omega')+2m\phi'(z+2m\omega+2m'\omega')=\phi_1(z),$$

$$\phi_2(z+2m\omega+2m'\omega')+2m'\phi'(z+2m\omega+2m'\omega')=\phi_2(z),$$

$$\phi'(z+2m\omega+2m'\omega')=\phi'(z).$$

* Halphen, *Fonctions Elliptiques*, t. i, pp. 23—25.

Multiplying by ω, ω', z respectively and adding, we have

$$\omega\phi_1(z+2m\omega+2m'\omega')+\omega'\phi_2(z+2m\omega+2m'\omega')$$
$$+(z+2m\omega+2m'\omega')\,\phi'(z+2m\omega+2m'\omega')$$
$$=\omega\phi_1(z)+\omega'\phi_2(z)+z\phi'(z).$$

Hence, *if* $$f(z)=\omega\phi_1(z)+\omega'\phi_2(z)+z\phi'(z),$$
then $f(z)$ *is a function doubly-periodic in the periods of* ϕ.

Again, multiplying by η, η', $\zeta(z)$, adding, and remembering that

$$\zeta(z+2m\omega+2m'\omega')=\zeta(z)+2m\eta+2m'\eta',$$

we have

$$\eta\phi_1(z+2m\omega+2m'\omega')+\eta'\phi_2(z+2m\omega+2m'\omega')$$
$$+\zeta(z+2m\omega+2m'\omega')\,\phi'(z+2m\omega+2m'\omega')$$
$$=\eta\phi_1(z)+\eta'\phi_2(z)+\zeta(z)\,\phi'(z).$$

Hence, *if* $$g(z)=\eta\phi_1(z)+\eta'\phi_2(z)+\zeta(z)\,\phi'(z),$$
then $g(z)$ *is a function doubly-periodic in the periods of* ϕ.

In what precedes, the function $\phi(z)$ is any function, doubly-periodic in 2ω, $2\omega'$; one simple and useful case occurs when $\phi(z)$ is taken to be the function $\wp(z)$. Now

$$\wp(z)=\frac{1}{z^2}+\frac{1}{20}g_2z^2+\frac{1}{28}g_3z^4+\frac{1}{1200}g_2^2z^6+\dots,$$

and $$\zeta(z)=\frac{1}{z}-\frac{1}{60}g_2z^3-\frac{1}{140}g_3z^5-\frac{1}{8400}g_2^2z^7-\dots;$$

hence, in the vicinity of the origin, we have

$$\omega\frac{\partial\wp}{\partial\omega}+\omega'\frac{\partial\wp}{\partial\omega'}+z\frac{\partial\wp}{\partial z}=-\frac{2}{z^2}+\text{even integral powers of } z^2$$
$$=-2\wp,$$

since both functions are doubly-periodic and the terms independent of z vanish for both functions. It is easy to see that this equation merely expresses the fact that $\wp$, which is equal to

$$\frac{1}{z^2}+\Sigma\Sigma'\left\{\frac{1}{(z-\Omega)^2}-\frac{1}{\Omega^2}\right\},$$

is homogeneous of degree -2 in z, ω, ω'.

Similarly

$$\eta\frac{\partial\wp}{\partial\omega}+\eta'\frac{\partial\wp}{\partial\omega'}+\zeta(z)\frac{\partial\wp}{\partial z}=-\frac{2}{z^4}+\frac{2}{15}g_2+\text{even integral powers of } z.$$

But, in the vicinity of the origin,

$$\frac{\partial^2\wp}{\partial z^2}=\frac{6}{z^4}+\frac{1}{10}g_2+\text{even integral powers of } z,$$

so that

$$\eta \frac{\partial \wp}{\partial \omega} + \eta' \frac{\partial \wp}{\partial \omega'} + \zeta(z) \frac{\partial \wp}{\partial z} + \tfrac{1}{3} \frac{\partial^2 \wp}{\partial z^2} = \tfrac{1}{6} g_2 + \text{even integral powers of } z.$$

The function on the left-hand side is doubly-periodic: it has no infinity at the origin and therefore none in the fundamental parallelogram; it therefore has no infinities in the plane. It is thus constant and equal to its value anywhere, say at the origin. This value is $\tfrac{1}{6} g_2$, and therefore

$$\eta \frac{\partial \wp}{\partial \omega} + \eta' \frac{\partial \wp}{\partial \omega'} + \zeta(z) \frac{\partial \wp}{\partial z} = -\tfrac{1}{3} \frac{\partial^2 \wp}{\partial z^2} + \tfrac{1}{6} g_2$$
$$= -2\wp^2 + \tfrac{1}{3} g_2.$$

This equation, when combined with

$$\omega \frac{\partial \wp}{\partial \omega} + \omega' \frac{\partial \wp}{\partial \omega'} + z \frac{\partial \wp}{\partial z} = -2\wp,$$

gives the value of $\dfrac{\partial \wp}{\partial \omega}$ *and* $\dfrac{\partial \wp}{\partial \omega'}$.

The equations are identically satisfied. Equating the coefficients of z^2 in the expansions, which are valid in the vicinity of the origin, we have

$$\left.\begin{aligned} \omega \frac{\partial g_2}{\partial \omega} + \omega' \frac{\partial g_2}{\partial \omega'} &= -4g_2 \\ \eta \frac{\partial g_2}{\partial \omega} + \eta' \frac{\partial g_2}{\partial \omega'} &= -6g_3 \end{aligned}\right\};$$

and equating the coefficients of z^4 in the same expansions, we have

$$\left.\begin{aligned} \omega \frac{\partial g_3}{\partial \omega} + \omega' \frac{\partial g_3}{\partial \omega'} &= -6g_3 \\ \eta \frac{\partial g_3}{\partial \omega} + \eta' \frac{\partial g_3}{\partial \omega'} &= -\tfrac{1}{3} g_2^2 \end{aligned}\right\}.$$

Hence for any function u, which involves ω and ω' and therefore implicitly involves g_2 and g_3, we have

$$\omega \frac{\partial u}{\partial \omega} + \omega' \frac{\partial u}{\partial \omega'} = -\left(4g_2 \frac{\partial u}{\partial g_2} + 6g_3 \frac{\partial u}{\partial g_3}\right),$$

$$\eta \frac{\partial u}{\partial \omega} + \eta' \frac{\partial u}{\partial \omega'} = -\tfrac{1}{2}\left(12g_3 \frac{\partial u}{\partial g_2} + \tfrac{2}{3} g_2^2 \frac{\partial u}{\partial g_3}\right).$$

Since $\wp$ is such a function, we have

$$4g_2 \frac{\partial \wp}{\partial g_2} + 6g_3 \frac{\partial \wp}{\partial g_3} - z \frac{\partial \wp}{\partial z} = 2\wp,$$

$$12g_3 \frac{\partial \wp}{\partial g_2} + \tfrac{2}{3} g_2^2 \frac{\partial \wp}{\partial g_3} - 2\zeta(z) \frac{\partial \wp}{\partial z} = 4\wp^2 - \tfrac{2}{3} g_2,$$

being *the equations which determine the derivatives of* $\wp$ *with regard to the invariants* g_2 *and* g_3.

The latter equation, integrated twice, leads to

$$\frac{\partial^2\sigma}{\partial z^2} - 12g_3\frac{\partial\sigma}{\partial g_2} - \frac{2}{3}g_2^2\frac{\partial\sigma}{\partial g_3} + \frac{1}{12}g_2z^2\sigma = 0,$$

a differential equation satisfied by $\sigma(z)$*.

133. The foregoing investigations give some of the properties of doubly-periodic functions of the second order, whether they be uneven and have two simple irreducible infinities, or even and have one double irreducible infinity.

If a function U of the second order have a repeated infinity at $z = \gamma$, then it is determined by an equation of the form

$$U'^2 = 4a^2[(U-\lambda)(U-\mu)(U-\nu)]^{\frac{1}{2}},$$

or, taking $U - \frac{1}{3}(\lambda+\mu+\nu) = Q$, the equation is

$$Q'^2 = 4a^2[(Q-e_1)(Q-e_2)(Q-e_3)]^{\frac{1}{2}},$$

where $e_1 + e_2 + e_3 = 0$. Taking account of the infinities, we have

$$Q = \wp(az - a\gamma);$$

and therefore

$$\begin{aligned} U - \tfrac{1}{3}(\lambda+\mu+\nu) &= \wp(az - a\gamma) \\ &= -\wp(az) - \wp(a\gamma) + \tfrac{1}{4}\left\{\frac{\wp'(az)+\wp'(a\gamma)}{\wp(az)-\wp(a\gamma)}\right\}^2, \end{aligned}$$

by Ex. 4, p. 262. The right-hand side cannot be an odd function; hence *an odd function of the second order cannot have a repeated infinity.* Similarly, by taking reciprocals of the functions, it follows that *an odd function of the second order cannot have a repeated zero.*

It thus appears that the investigations in §§ 120, 121 are sufficient for the included range of properties of odd functions. We now proceed to obtain the general equations of even functions. Every such function can (by § 118, XIII., Cor. I.) be expressed in the form $\{a\wp(z)+b\} \div \{c\wp(z)+d\}$, and its equations could thence be deduced from those of $\wp(z)$; but, partly for uniformity, we shall adopt the same method as in § 120 for odd functions. And, as already stated (p. 251), the separate class of functions of the second order that are neither even nor odd, will not be discussed.

134. Let, then, $\phi(z)$ denote an even doubly-periodic function of the second order (it may be either of the first class or of the second class) and let 2ω, $2\omega'$ be its periods; and denote $2\omega + 2\omega'$ by $2\omega''$. Then

$$\phi(z) = \phi(-z),$$

since the function is even; and since

$$\begin{aligned} \phi(\omega+z) &= \phi(-\omega-z) \\ &= \phi(2\omega-\omega-z) \\ &= \phi(\omega-z), \end{aligned}$$

* For this and other deductions from these equations, see Frobenius und Stickelberger, *Crelle*, t. xcii, (1882), pp. 311—327; Halphen, *Traité des fonctions elliptiques*, t. i, (1886), chap. IX.; and a memoir by the author, quoted on p. 254, note.

it follows that $\phi(\omega+z)$—and, similarly, $\phi(\omega'+z)$ and $\phi(\omega''+z)$ are even functions.

Now $\phi(\omega+z)$, an even function, has two irreducible infinities, and is periodic in 2ω, $2\omega'$; also $\phi(z)$, an even function, has two irreducible infinities and is periodic in 2ω, $2\omega'$. There is therefore a relation between $\phi(z)$ and $\phi(\omega+z)$, which, by § 118, Prop. XIII., Cor. I., is of the first degree in $\phi(z)$ and of the first degree in $\phi(\omega+z)$; thus it must be included in

$$B\phi(z)\phi(\omega+z)-C\phi(z)-C'\phi(\omega+z)+A=0.$$

But $\phi(z)$ is periodic in 2ω; hence, on writing $z+\omega$ for z in the equation, it becomes

$$B\phi(\omega+z)\phi(z)-C\phi(\omega+z)-C'\phi(z)+A=0;$$

thus
$$C=C'.$$

If B be zero, then C may not be zero, for the relation cannot become evanescent: it is of the form

$$\phi(z)+\phi(\omega+z)=A'\quad\ldots\ldots\ldots\ldots(1).$$

If B be not zero, then the relation is

$$\phi(\omega+z)=\frac{C\phi(z)-A}{B\phi(z)-C}\quad\ldots\ldots\ldots\ldots(2).$$

Treating $\phi(\omega'+z)$ in the same way, we find that the relation between it and $\phi(z)$ is

$$F\phi(z)\phi(\omega'+z)-D\phi(z)-D\phi(\omega'+z)+E=0,$$

so that, if F be zero, the relation is of the form

$$\phi(z)+\phi(\omega'+z)=E'\quad\ldots\ldots\ldots\ldots(1)',$$

and, if F be not zero, the relation is of the form

$$\phi(\omega'+z)=\frac{D\phi(z)-E}{F\phi(z)-D}\quad\ldots\ldots\ldots\ldots(2)'.$$

Four cases thus arise, viz., the coexistence of (1) with (1)′, of (1) with (2)′, of (2) with (1)′, and of (2) with (2)′. These will be taken in order.

I.: the coexistence of (1) with (1)′. From (1) we have

$$\phi(\omega'+z)+\phi(\omega''+z)=A',$$

so that
$$\phi(z)+\phi(\omega+z)+\phi(\omega'+z)+\phi(\omega''+z)=2A'.$$

Similarly, from (1)′,

$$\phi(z)+\phi(\omega'+z)+\phi(\omega+z)+\phi(\omega''+z)=2E';$$

so that $A'=E'$, and then

$$\phi(\omega+z)=\phi(\omega'+z),$$

whence $\omega\sim\omega'$ is a period, contrary to the initial hypothesis that 2ω and $2\omega'$ determine a fundamental parallelogram. Hence equations (1) and (1)′ cannot coexist.

II.: the coexistence of (1) with (2)′. From (1) we have

$$\phi(\omega''+z)=A'-\phi(\omega'+z)$$
$$=\frac{(A'F-D)\phi(z)-(A'D-E)}{F\phi(z)-D},$$

on substitution from (2)′. From (2)′ we have

$$\phi(\omega''+z)=\frac{D\phi(\omega+z)-E}{F\phi(\omega+z)-D}$$
$$=\frac{(A'D-E)-D\phi(z)}{A'F-D-F\phi(z)},$$

on substitution from (1). The two values of $\phi(\omega''+z)$ must be the same, whence

$$A'F-D=D,$$

which relation establishes the periodicity of $\phi(z)$ in $2\omega''$, when it is considered as given by either of the two expressions which have been obtained. We thus have

$$A'F=2D;$$

and then, by (1), we have

$$\phi(z)-\frac{D}{F}+\phi(\omega+z)-\frac{D}{F}=0;$$

and, by (2)′, we have

$$\left\{\phi(z)-\frac{D}{F}\right\}\left\{\phi(\omega'+z)-\frac{D}{F}\right\}=\frac{D^2-EF}{F^2}.$$

If a new even function be introduced, doubly-periodic in the same periods having the same infinities and defined by the equation

$$\phi_1(z)=\phi(z)-\frac{D}{F},$$

the equations satisfied by $\phi_1(z)$ are

$$\left.\begin{array}{r}\phi_1(\omega+z)+\phi_1(z)=0\\ \phi_1(\omega'+z)\,\phi_1(z)=\text{constant}\end{array}\right\}.$$

To the detailed properties of such functions we shall return later; meanwhile it may be noticed that these equations are, in form, the same as those satisfied by an odd function of the second order.

III.: the coexistence of (2) with (1)′. This case is similar to II., with the result that, if an even function be introduced, doubly-periodic in the same periods having the same infinities and defined by the equation

$$\phi_2(z)=\phi(z)-\frac{C}{B},$$

the equations satisfied by $\phi_2(z)$ are

$$\left.\begin{array}{r}\phi_2(\omega'+z)+\phi_2(z)=0\\ \phi_2(\omega+z)\,\phi_2(z)=\text{constant}\end{array}\right\}.$$

It is, in fact, merely the previous case with the periods interchanged.

IV.: the coexistence of (2) with (2)$'$. From (2) we have

$$\phi(\omega''+z)=\frac{C\phi(\omega'+z)-A}{B\phi(\omega'+z)-C}$$
$$=\frac{(CD-AF)\phi(z)-(CE-AD)}{(BD-CF)\phi(z)-(BE-CD)},$$

on substitution from (2)$'$. Similarly from (2)$'$, after substitution from (2), we have

$$\phi(\omega''+z)=\frac{(CD-BE)\phi(z)+(CE-AD)}{(CF-BD)\phi(z)+(CD-AF)}.$$

The two values must be the same; hence

$$CD-AF=-(CD-BE),$$

which indeed is the condition that each of the expressions for $\phi(\omega''+z)$ should give a function periodic in $2\omega''$. Thus

$$AF+BE=2CD.$$

One case may be at once considered and removed, viz. if C and D vanish together. Then since, by the hypothesis of the existence of (2) and of (2)$'$, neither B nor F vanishes, we have

$$\frac{A}{B}=-\frac{E}{F},$$

so that $$\phi(\omega+z)=-\frac{A}{B\phi(z)}=\frac{E}{F\phi(z)}=-\phi(\omega'+z),$$

and then the relations are $\phi(\omega+z)+\phi(\omega'+z)=0$,

or, what is the same thing, $$\left.\begin{aligned}\phi(z)+\phi(\omega''+z)&=0\\ \phi(z)\phi(\omega+z)&=\text{constant}\end{aligned}\right\}.$$
and

This case is substantially the same as that of II. and III., arising merely from a modification (§ 109) of the fundamental parallelogram, into one whose sides are determined by 2ω and $2\omega''$.

Hence we may have (2) coexistent with (2)$'$ provided

$$AF+BE=2CD;$$

C and D do not both vanish, and neither B nor F vanishes.

IV. (1). Let neither C nor D vanish; and for brevity write

$$\phi(\omega+z)=\phi_1,\quad \phi(\omega''+z)=\phi_2,\quad \phi(\omega'+z)=\phi_3,\quad \phi(z)=\phi.$$

Then the equations in IV. are

$$B\phi\phi_1-C(\phi+\phi_1)+A=0,$$
$$F\phi\phi_3-D(\phi+\phi_3)+E=0.$$

Now a doubly-periodic function, with given zeros and given infinities, is determinate save as to an arbitrary constant factor. We therefore introduce an arbitrary factor λ, so that

$$\phi=\lambda\psi,$$

and then taking $$\frac{C}{B\lambda}=c_1,\quad \frac{D}{F\lambda}=c_3,$$

we have
$$(\psi - c_1)(\psi_1 - c_1) = c_1^2 - \frac{A}{B\lambda^2},$$
$$(\psi - c_3)(\psi_3 - c_3) = c_3^2 - \frac{E}{F\lambda^2}.$$

The arbitrary quantity λ is at our disposal: we introduce a new quantity c_2, defined by the equation
$$\frac{A}{B\lambda^2} = c_1(c_2 + c_3) - c_2c_3,$$
and therefore at our disposal. But since
$$AF + BE = 2CD,$$
we have
$$\frac{A}{B\lambda^2} + \frac{E}{F\lambda^2} = 2\frac{C}{B\lambda}\frac{D}{F\lambda} = 2c_1c_3,$$
and therefore
$$\frac{E}{F\lambda^2} = c_3(c_1 + c_2) - c_1c_2.$$
Hence the foregoing equations are
$$(\psi - c_1)(\psi_1 - c_1) = (c_1 - c_2)(c_1 - c_3),$$
$$(\psi - c_3)(\psi_3 - c_3) = (c_3 - c_1)(c_3 - c_2).$$
The equation for ϕ_2, that is $\phi(\omega'' + z)$, is
$$\phi_2 = \frac{L\phi - M}{N\phi - L},$$
where $L = CD - BE = AF - CD, \quad M = AD - CE, \quad N = CF - BD,$

so that
$$AN + BM = 2CL.$$
As before, one particular case may be considered and removed. If N be zero, so that
$$\frac{C}{B} = \frac{D}{F} = \alpha$$
say, and
$$\frac{A}{B} + \frac{E}{F} = 2\frac{CD}{BF} = 2\alpha^2,$$
then we find
$$\phi + \phi_2 = \phi_1 + \phi_3 = 2\alpha,$$
or taking a function
$$\chi = \phi - \alpha,$$
the equation becomes
$$\left.\begin{aligned}\chi(z) + \chi(\omega'' + z) = 0.\\ \text{The other equations then become}\\ \chi(z)\chi(\omega + z) = \alpha^2 - \frac{A}{B}\\ \chi(z)\chi(\omega' + z) = \alpha^2 - \frac{E}{F}\end{aligned}\right\},$$
and therefore they are similar to those in Cases II. and III.

If N be not zero, then it is easy to shew that
$$N = BF\lambda(c_1 - c_3),$$
$$L = BF\lambda^2(c_1 - c_3)c_2,$$
$$M = BF\lambda^3(c_1 - c_3)(c_2c_1 + c_2c_3 - c_1c_3);$$

and then the equation connecting ϕ and ϕ_2 changes to

$$(\psi - c_2)(\psi_2 - c_2) = (c_2 - c_1)(c_2 - c_3)$$

which, with

$$\left.\begin{aligned}(\psi - c_2)(\psi_2 - c_2) &= (c_2 - c_1)(c_2 - c_3)\\ (\psi - c_1)(\psi_1 - c_1) &= (c_1 - c_2)(c_1 - c_3)\\ (\psi - c_3)(\psi_3 - c_3) &= (c_3 - c_1)(c_3 - c_2)\end{aligned}\right\},$$

are relations between ψ, ψ_1, ψ_2, ψ_3, where the quantity c_2 is at our disposal.

IV. (2). These equations have been obtained on the supposition that neither C nor D is zero. If either vanish, let it be C: then D does not vanish; and the equations can be expressed in the form

$$\phi\phi_1 = \frac{E}{F},$$

$$\left(\phi - \frac{D}{F}\right)\left(\phi_3 - \frac{D}{F}\right) = \frac{D^2 - EF}{F^2},$$

$$\left(\phi - \frac{E}{D}\right)\left(\phi_2 - \frac{E}{D}\right) = -\frac{E(D^2 - EF)}{FD^2}.$$

We therefore obtain the following theorem:

If ϕ be an even function doubly-periodic in 2ω and $2\omega'$ and of the second order, and if all functions equivalent to ϕ in the form $R\phi + S$ (where R and S are constants) be regarded as the same as ϕ, then either the function satisfies the system of equations

$$\left.\begin{aligned}\phi(z) + \phi(\omega\ + z) &= 0\\ \phi(z)\quad \phi(\omega' + z) &= H\\ \phi(z)\quad \phi(\omega'' + z) &= -H\end{aligned}\right\}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\text{I})^*,$$

where H is a constant; or it satisfies the system of equations

$$\left.\begin{aligned}\{\phi(z) - c_1\}\{\phi(\omega\ + z) - c_1\} &= (c_1 - c_2)(c_1 - c_3)\\ \{\phi(z) - c_3\}\{\phi(\omega' + z) - c_3\} &= (c_3 - c_1)(c_3 - c_2)\\ \{\phi(z) - c_2\}\{\phi(\omega'' + z) - c_2\} &= (c_2 - c_1)(c_2 - c_3)\end{aligned}\right\}\ \ldots\ldots\ldots\ldots(\text{II}),$$

where of the three constants c_1, c_2, c_3 one can be arbitrarily assigned.

We shall now very briefly consider these in turn.

135. So far as concerns the former class of equations satisfied by an even doubly-periodic function, viz.,

$$\left.\begin{aligned}\phi(z) + \phi(\omega\ + z) &= 0\\ \phi(z)\quad \phi(\omega' + z) &= H\end{aligned}\right\},$$

we proceed initially as in (§ 120) the case of an odd function. We have the further equations

$$\phi(z) = \phi(-z),$$

$$\phi(\omega + z) = \phi(\omega - z),\quad \phi(\omega' + z) = \phi(\omega' - z).$$

* The systems obtained by the interchange of ω, ω', ω'' among one another in the equations are not substantially distinct from the form adopted for the system I.; the apparent difference can be removed by an appropriate corresponding interchange of the periods.

Taking $z = -\frac{1}{2}\omega$, the first gives

$$\phi(\tfrac{1}{2}\omega) + \phi(\tfrac{1}{2}\omega) = 0,$$

so that $\frac{1}{2}\omega$ is either a zero or an infinity.

If $\frac{1}{2}\omega$ be a zero, then

$$\phi(\tfrac{3}{2}\omega) = \phi(\omega + \tfrac{1}{2}\omega) = -\phi(\tfrac{1}{2}\omega) \text{ by the first equation}$$
$$= 0,$$

so that $\frac{1}{2}\omega$ and $\frac{3}{2}\omega$ are zeros. And then, by the second equation,

$$\omega' + \tfrac{1}{2}\omega, \quad \omega' + \tfrac{3}{2}\omega$$

are infinities.

If $\frac{1}{2}\omega$ be an infinity, then in the same way $\frac{3}{2}\omega$ is also an infinity; and then $\omega' + \frac{1}{2}\omega$, $\omega' + \frac{3}{2}\omega$ are zeros. Since these amount merely to interchanging zeros and infinities, which is the same functionally as taking the reciprocal of the function, we may choose either arrangement. We shall take that which gives $\frac{1}{2}\omega$, $\frac{3}{2}\omega$ as the zeros; and $\omega' + \frac{1}{2}\omega$, $\omega' + \frac{3}{2}\omega$ as the infinities.

The function ϕ is evidently of the second class, in that it has two distinct simple irreducible infinities.

Because $\omega' + \frac{1}{2}\omega$, $\omega' + \frac{3}{2}\omega$ are the irreducible infinities of $\phi(z)$, the four zeros of $\phi'(z)$ are, by § 117, the irreducible points homologous with ω'', $\omega'' + \omega$, $\omega'' + \omega'$, $\omega'' + \omega''$, that is, the irreducible zeros of $\phi'(z)$ are 0, ω, ω', ω''. Moreover

$$\phi(0) + \phi(\omega) = 0,$$
$$\phi(\omega') + \phi(\omega'') = 0,$$

by the first of the equations of the system; hence the relation between $\phi(z)$ and $\phi'(z)$ is

$$\phi'^2(z) = A\{\phi(z) - \phi(0)\}\{\phi(z) - \phi(\omega)\}\{\phi(z) - \phi(\omega')\}\{\phi(z) - \phi(\omega'')\}$$
$$= A\{\phi^2(0) - \phi^2(z)\}\{\phi^2(\omega') - \phi^2(z)\}.$$

Since the origin is neither a zero nor an infinity of $\phi(z)$, let

$$\phi(z) = \phi(0)\,\phi_1(z)$$

so that $\phi_1(0)$ is unity and $\phi_1'(0)$ is zero; then

$$\phi_1'^2(z) = \lambda^2\{1 - \phi_1^2(z)\}\{\mu - \phi_1^2(z)\}$$

the differential equation determining $\phi_1(z)$.

The character of the function depends upon the value of μ and the constant of integration. The function may be compared with $\operatorname{cn} u$, by taking 2ω, $2\omega' = 4K$, $2K + 2iK'$; and with $\dfrac{1}{\operatorname{dn} u}$, by taking 2ω, $2\omega' = 2K$, $4iK'$, which (§ 131, note) are the periods of these (even) Jacobian elliptic functions.

We may deal even more briefly with the even function characterised by the second class of equations in § 134. One of the quantities c_1, c_2, c_3 being at our disposal, we choose it so that

$$c_1 + c_2 + c_3 = 0;$$

and then the analogy with the equations of Weierstrass's $\wp$-function is complete (see § 133).

CHAPTER XII.

Pseudo-Periodic Functions.

136. Most of the functions in the last two Chapters are of the type called doubly-periodic, that is, they are reproduced when their arguments are increased by integral multiples of two distinct periods. But, in §§ 127, 130, functions of only a pseudo-periodic type have arisen: thus the ζ-function satisfies the equation

$$\zeta(z + m2\omega + m'2\omega') = \zeta(z) + m2\eta + m'2\eta',$$

and the σ-function the equation

$$\sigma(z + m2\omega + m'2\omega') = (-1)^{mm'+m+m'} e^{2(m\eta+m'\eta')(z+m\omega+m'\omega')} \sigma(z).$$

These are instances of the most important classes: and the distinction between the two can be made even less by considering the function $e^{\zeta(z)} = \xi(z)$, when we have

$$\xi(z + m2\omega + m'2\omega') = e^{2m\eta} e^{2m'\eta'} \xi(z).$$

In the case of the ξ-function an increase of the argument by a period leads to the reproduction of the function multiplied by an exponential factor that is constant, and in the case of the σ-function a similar change of the argument leads to the reproduction of the function multiplied by an exponential factor having its index of the form $az + b$.

Hence, when an argument is subject to periodic increase, there are three simple classes of functions of that argument.

First, if a function $f(z)$ satisfy the equations

$$f(z + 2\omega) = f(z), \quad f(z + 2\omega') = f(z),$$

it is strictly periodic: it is sometimes called *a doubly-periodic function of the first kind*. The general properties of such functions have already been considered.

Secondly, if a function $F(z)$ satisfy the equations

$$F(z + 2\omega) = \mu F(z), \quad F(z + 2\omega') = \mu' F(z),$$

where μ and μ' are constants, it is pseudo-periodic: it is called *a doubly-periodic function of the second kind.* The first derivative of the logarithm of such a function is a doubly-periodic function of the first kind.

Thirdly, if a function $\phi(z)$ satisfy the equations

$$\phi(z+2\omega) = e^{az+b}\phi(z), \quad \phi(z+2\omega') = e^{a'z+b'}\phi(z),$$

where a, b, a', b' are constants, it is pseudo-periodic: it is called *a doubly-periodic function of the third kind.* The second derivative of the logarithm of such a function is a doubly-periodic function of the first kind.

The equations of definition for functions of the third kind can be modified. We have

$$\begin{aligned}\phi(z+2\omega+2\omega') &= e^{a(z+2\omega')+b+a'z+b'}\phi(z)\\ &= e^{a'(z+2\omega)+b'+az+b}\phi(z),\end{aligned}$$

whence

$$a'\omega - a\omega' = -m\pi i,$$

where m is an integer. Let a new function $E(z)$ be introduced, defined by the equation

$$E(z) = e^{\lambda z^2+\mu z}\phi(z);$$

then λ and μ can be chosen so that $E(z)$ satisfies the equations

$$E(z+2\omega) = E(z), \quad E(z+2\omega') = e^{Az+B}E(z).$$

From the last equations, we have

$$\begin{aligned}E(z+2\omega+2\omega') &= e^{A(z+2\omega)+B}E(z)\\ &= e^{Az+B}E(z),\end{aligned}$$

so that $2A\omega$ is an integral multiple of $2\pi i$.

Also we have

$$\begin{aligned}E(z+2\omega) &= e^{\lambda(z+2\omega)^2+\mu(z+2\omega)}\phi(z+2\omega)\\ &= e^{4\lambda z\omega+4\lambda\omega^2+2\mu\omega+az+b}E(z),\end{aligned}$$

so that

$$4\lambda\omega + a = 0,$$

and

$$4\lambda\omega^2 + 2\mu\omega + b \equiv 0 \pmod{2\pi i}.$$

Similarly,

$$\begin{aligned}E(z+2\omega') &= e^{\lambda(z+2\omega')^2+\mu(z+2\omega')}\phi(z+2\omega')\\ &= e^{4\lambda z\omega'+4\lambda\omega'^2+2\mu\omega'+a'z+b'}E(z),\end{aligned}$$

so that

$$4\lambda\omega' + a' = A,$$

and

$$4\lambda\omega'^2 + 2\mu\omega' + b' \equiv B \pmod{2\pi i}.$$

From the two equations, which involve λ and not μ, we have

$$\begin{aligned}A\omega &= a'\omega - a\omega'\\ &= -m\pi i,\end{aligned}$$

agreeing with the result with $2A\omega$ is an integral multiple of $2\pi i$.

And from the two equations, which involve μ, we have, on the elimination of μ and on substitution for λ and A,

$$b'\omega - b\omega' - a\omega'(\omega' - \omega) \equiv B\omega \pmod{2\pi i}.$$

If A be zero, then $E(z)$ is a doubly-periodic function of the first kind when e^B is unity, and it is a doubly-periodic function of the second kind when e^B is not unity. Hence A, and therefore m, may be assumed to be different from zero for functions of the third kind. Take a new function $\Phi(z)$ such that

$$\Phi(z) = E\left(z - \frac{B}{A}\right) = E\left(z + \frac{B\omega}{m\pi i}\right);$$

then $\Phi(z)$ satisfies the equations

$$\Phi(z + 2\omega) = \Phi(z), \quad \Phi(z + 2\omega') = e^{-\frac{m\pi i}{\omega}z}\Phi(z),$$

which will be taken as the *canonical equations defining a doubly-periodic function of the third kind.*

Ex. Obtain the values of λ, μ, A, B for the Weierstrassian function $\sigma(z)$.

We proceed to obtain some properties of these two classes of functions which, for brevity, will be called *secondary-periodic* functions and *tertiary-periodic* functions respectively.

Doubly-Periodic Functions of the Second Kind.

For the secondary-periodic functions the chief sources of information are

Hermite, *Comptes Rendus*, t. liii, (1861), pp. 214--228, ib., t. lv, (1862), pp. 11—18, 85—91; *Sur quelques applications des fonctions elliptiques*, §§ I—III, separate reprint (1885) from *Comptes Rendus*; "Note sur la théorie des fonctions elliptiques" in Lacroix, vol. ii, (6th edition, 1885), pp. 484—491; *Cours d'Analyse*, (4^{me} éd.), pp. 227—234.

Mittag-Leffler, *Comptes Rendus*, t. xc, (1880), pp. 177—180.

Frobenius, *Crelle*, t. xciii, (1882), pp. 53—68.

Brioschi, *Comptes Rendus*, t. xcii, (1881), pp. 325—328.

Halphen, *Traité des fonctions elliptiques*, t. i, pp. 225—238, 411—426, 438—442, 463.

137. In the case of the periodic functions of the first kind it was proved that they can be expressed by means of functions of the second order in the same period—these being the simplest of such functions. It will now be proved that a similar result holds for secondary-periodic functions, defined by the equations

$$F(z + 2\omega) = \mu F(z), \quad F(z + 2\omega') = \mu' F(z).$$

Take a function
$$G(z) = \frac{\sigma(z+a)}{\sigma(z)\,\sigma(a)} e^{\lambda z};$$

then we have
$$G(z + 2\omega) = \frac{\sigma(z + a + 2\omega)}{\sigma(a)\,\sigma(z + 2\omega)} e^{\lambda z + 2\lambda\omega}$$
$$= e^{2\eta a + 2\lambda\omega} G(z),$$

and
$$G(z + 2\omega') = e^{2\eta' a + 2\lambda\omega'} G(z).$$

The quantities a and λ being unrestricted, we choose them so that

$$\mu = e^{2\eta a + 2\lambda\omega}, \quad \mu' = e^{2\eta' a + 2\lambda\omega'};$$

and then $G(z)$, a known function, satisfies the same equation as $F(z)$.

Let u denote a quantity independent of z, and consider the function

$$f(z) = F(z)\, G(u - z).$$

We have

$$\begin{aligned} f(z + 2\omega) &= F(z + 2\omega)\, G(u - z - 2\omega) \\ &= \mu F(z) \frac{1}{\mu} G(u - z) \\ &= f(z); \end{aligned}$$

and similarly $f(z + 2\omega') = f(z)$,

so that $f(z)$ is a doubly-periodic function of the first kind with 2ω and $2\omega'$ for its periods.

The sum of the residues of $f(z)$ is therefore zero. To express this sum, we must obtain the fractional part of the function for expansion in the vicinity of each of the (accidental) singularities of $f(z)$, that lie within the parallelogram of periods. The singularities of $f(z)$ are those of $G(u - z)$ and those of $F(z)$.

Choosing the parallelogram of reference so that it may contain u, we have $z = u$ as the only singularity of $G(u - z)$ and it is of the first order, so that, since

$$G(\zeta) = \frac{1}{\zeta} + \text{positive integral powers of } \zeta$$

in the vicinity of $\zeta = 0$, we have, in the vicinity of u,

$$\begin{aligned} f(z) &= \{F(u) + \text{positive integral powers of } u - z\} \left\{\frac{1}{u - z} + \text{positive powers}\right\} \\ &= -\frac{F(u)}{z - u} + \text{positive integral powers of } z - u; \end{aligned}$$

hence the residue of $f(z)$ for u is $-F(u)$.

Let $z = c$ be a pole of $F(z)$ in the parallelogram of order $n + 1$; and, in the vicinity of c, let

$$F(z) = \frac{C_1}{z - c} + C_2 \frac{d}{dz}\left(\frac{1}{z - c}\right) + \ldots + C_{n+1} \frac{d^n}{dz^n}\left(\frac{1}{z - c}\right) + \text{positive integral powers.}$$

Then in that vicinity

$$G(u - z) = G(u - c) - (z - c) \frac{d}{du} G(u - c) + \frac{(z - c)^2}{2\,!} \frac{d^2}{du^2} G(u - c) - \ldots,$$

and therefore the coefficient of $\frac{1}{z - c}$ in the expansion of $f(z)$ for points in the vicinity of c is

$$C_1 G(u - c) + C_2 \frac{d}{du} G(u - c) + C_3 \frac{d^2}{du^2} G(u - c) + \ldots + C_{n+1} \frac{d^n}{du^n} G(u - c),$$

which is therefore the residue of $f(z)$ for c.

This being the form of the residue of $f(z)$ for each of the poles of $F(z)$, then, since the sum of the residues is zero, we have

$$-F(u) + \Sigma \left[C_1 G(u - c) + C_2 \frac{d}{du} G(u - c) + \ldots + C_{n+1} \frac{d^n}{du^n} G(u - c)\right] = 0,$$

or, changing the variable,

$$F(z) = \Sigma \left[C_1 G(z-c) + C_2 \frac{d}{dz} G(z-c) + \ldots + C_{n+1} \frac{d^n}{dz^n} G(z-c) \right],$$

where the summation extends over all the poles of $F(z)$ within that parallelogram of periods in which z lies. This result is due to Hermite.

138. It has been assumed that a and λ, parameters in G, are determinate, an assumption that requires μ and μ' to be general constants: their values are given by

$$\eta a + \omega\lambda = \tfrac{1}{2}\log\mu, \quad \eta' a + \omega'\lambda = \tfrac{1}{2}\log\mu',$$

and, therefore, since $\eta\omega' - \eta'\omega = \pm \tfrac{1}{2} i\pi$, we have

$$\left.\begin{aligned} \pm i\pi a &= \quad \omega' \log\mu - \omega \log\mu' \\ \pm i\pi\lambda &= -\eta' \log\mu + \eta \log\mu' \end{aligned}\right\}.$$

Now λ may vanish without rendering $G(z)$ a null function. If a vanish (or, what is the same thing, be an integral combination of the periods), then $G(z)$ is an exponential function multiplied by an infinite constant when λ does not vanish, and it ceases to be a function when λ does vanish. These cases must be taken separately.

First, let a and λ vanish*; then both μ and μ' are unity, the function F is doubly-periodic of the first kind; but the expression for F is not determinate, owing to the form of G. To render it determinate, consider λ as zero and a as infinitesimal, to be made zero ultimately. Then

$$\begin{aligned} G(z) &= \frac{\sigma(z) + a\sigma'(z) + \ldots}{a\sigma(z)} (1 + \text{positive integral powers of } a) \\ &= \frac{1}{a} + \zeta(z) + \text{positive powers of } a. \end{aligned}$$

Since a is infinitesimal, μ and μ' are very nearly unity. When the function F is given, the coefficients $C_1, C_2, \ldots$ may be affected by a, so that for any one we have

$$C_k = b_k + a\gamma_k + \text{higher powers of } a,$$

where γ_k is finite; and b_k is the actual value for the function which is strictly of the first kind, so that

$$\Sigma b_1 = 0,$$

the summation being extended over the poles of the function. Then retaining only a^{-1} and a^0, we have

$$\begin{aligned} &\Sigma \left[C_1 G(u-c) + C_2 \frac{d}{du} G(u-c) + \ldots\ldots\ldots\ldots + C_{n+1} \frac{d^n}{du^n} G(u-c) \right] \\ &= \Sigma \frac{b_1}{a} + \Sigma\gamma_1 + \Sigma \left[b_1 \zeta(u-c) + b_2 \frac{d}{du} \zeta(u-c) + \ldots + b_{n+1} \frac{d^n}{du^n} \zeta(u-c) \right] \\ &= C_0 + \Sigma \left[b_1 \zeta(u-c) + \ldots + b_{n+1} \frac{d^n}{du^n} \zeta(u-c) \right], \end{aligned}$$

* This case is discussed by Hermite (l.c., p. 275).

where C_0, equal to $\Sigma\gamma_1$, is a constant and the term in $\frac{1}{a}$ vanishes. This expression, with the condition $\Sigma b_1 = 0$, is the value of $F(u)$ or, changing the variables, we have

$$F(z) = C_0 + \Sigma\left[b_1\zeta(z-c) + b_2\frac{d}{dz}\zeta(z-c) + \ldots + b_{n+1}\frac{d^n}{dz^n}\zeta(z-c)\right],$$

with the condition $\Sigma b_1 = 0$, a result agreeing with the one formerly (§ 128) obtained.

When F is not given, but only its infinities are assigned arbitrarily, then $\Sigma C = 0$ because F is to be a doubly-periodic function of the first kind; the term $\frac{1}{a}\Sigma C$ vanishes, and we have the same expression for $F(z)$ as before.

Secondly, let a vanish* but not λ, so that μ and μ' have the forms

$$\mu = e^{2\lambda\omega}, \quad \mu' = e^{2\lambda\omega'}.$$

We take a function

$$g(z) = e^{\lambda z}\zeta(z);$$

then

$$\begin{aligned} g(z-2\omega) &= \mu^{-1}e^{\lambda z}\zeta(z-2\omega) \\ &= \mu^{-1}e^{\lambda z}\{\zeta(z) - 2\eta\} \\ &= \mu^{-1}\{g(z) - 2\eta e^{\lambda z}\}, \end{aligned}$$

and

$$g(z-2\omega') = \mu'^{-1}\{g(z) - 2\eta' e^{\lambda z}\}.$$

Introducing a new function $H(z)$ defined by the equation

$$H(z) = F(z)\,g(u-z),$$

we have

$$H(z+2\omega) = H(z) - 2\eta e^{\lambda(u-z)}F(z),$$

and

$$H(z+2\omega') = H(z) - 2\eta' e^{\lambda(u-z)}F(z).$$

Consider a parallelogram of periods which contains the point u; then, if Θ be the sum of the residues of $H(z)$ for poles in this parallelogram, we have

$$2\pi i\Theta = \int H(z)\,dz,$$

the integral being taken positively round the parallelogram. But, by § 116, Prop. II. Cor., this integral is

$$4e^{\lambda u}\left\{\omega\eta'\int_0^1 e^{-\lambda(p+2\omega t)}F(p+2\omega t)\,dt - \omega'\eta\int_0^1 e^{-\lambda(p+2\omega' t)}F(p+2\omega' t)\,dt\right\},$$

where p is the corner of the parallelogram and each integral is taken for real values of t from 0 to 1. Each of the integrals is a constant, so far as concerns u; and therefore we may take

$$\Theta = -Ae^{\lambda u},$$

the quantity inside the above bracket being denoted by $-\frac{1}{2}i\pi A$.

The residue of $H(z)$ for $z = u$, arising from the simple pole of $g(u-z)$, is $-F(u)$ as in § 137.

If $z = c$ be an accidental singularity of $F(z)$ of order $n+1$, so that, in the vicinity of $z = c$,

$$F(z) = C_1\frac{1}{z-c} + C_2\frac{d}{dz}\left(\frac{1}{z-c}\right) + \ldots + C_{n+1}\frac{d^n}{dz^n}\left(\frac{1}{z-c}\right) + P(z-c),$$

* This is discussed by Mittag-Leffler, (l.c., p. 275).

then the residue of $H(z)$ for $z=c$ is

$$C_1 g(u-c) + C_2 \frac{d}{du} g(u-c) + \ldots + C_{n+1} \frac{d^n}{du^n} g(u-c);$$

and similarly for all the other accidental singularities of $F(z)$. Hence

$$-F(u) + \Sigma \left\{C_1 + C_2 \frac{d}{du} + \ldots + C_{n+1} \frac{d^n}{du^n}\right\} g(u-c) = -Ae^{\lambda u},$$

or
$$F(z) = Ae^{\lambda z} + \Sigma \left\{C_1 + C_2 \frac{d}{dz} + \ldots + C_{n+1} \frac{d^n}{dz^n}\right\} g(z-c),$$

where the summation extends over all the accidental singularities of $F(z)$ in a parallelogram of periods which contains z, and $g(z)$ is the function $e^{\lambda z}\zeta(z)$. This result is due to Mittag-Leffler.

Since $\mu = e^{2\lambda\omega}$ and

$$g(z-c+2\omega) = \mu g(z-c) + 2\eta\mu e^{\lambda(z-c)},$$

we have

$$\begin{aligned}\mu F(z) = F(z+2\omega) \\ &= \mu Ae^{\lambda z} + \Sigma \left\{C_1 + C_2 \frac{d}{dz} + \ldots + C_{n+1} \frac{d^n}{dz^n}\right\} \mu g(z-c) \\ &\quad + 2\eta\mu e^{\lambda z} \Sigma (C_1 + C_2\lambda + \ldots + C_{n+1}\lambda^n) e^{-\lambda c};\end{aligned}$$

and therefore
$$\Sigma (C_1 + C_2\lambda + \ldots + C_{n+1}\lambda^n) e^{-\lambda c} = 0,$$

the summation extending over all the accidental singularities of $F(z)$. The same equation can be derived through $\mu' F(z) = F(z+2\omega')$.

Again ΣC_1 is the sum of the residues in a parallelogram of periods, and therefore

$$2\pi i \Sigma C_1 = \int F(z)\, dz,$$

the integral being taken positively round it. If p be one corner, the integral is

$$2\omega(1-\mu') \int_0^1 F(p+2\omega t)\, dt - 2\omega'(1-\mu) \int_0^1 F(p+2\omega' t)\, dt,$$

each integral being for real variables of t.

Hermite's special form can be derived from Mittag-Leffler's by making λ vanish.

Note. Both Hermite and Mittag-Leffler, in their investigations, have used the notation of the Jacobian theory of elliptic functions, instead of dealing with general periodic functions. The forms of their results are as follows, using as far as possible the notation of the preceding articles.

I. When the function is defined by the equations

$$F(z+2K) = \mu F(z), \quad F(z+2iK') = \mu' F(z),$$

then
$$F(z) = \Sigma \left\{C_1 + C_2 \frac{d}{dz} + \ldots + C_{n+1} \frac{d^n}{dz^n}\right\} G(z-c),$$

where
$$G(z) = \frac{H'(0)\, H(z+\omega)}{H(z)\, H(\omega)} e^{\lambda z},$$

(the symbol H denoting the Jacobian H-function), and the constants ω and λ are determined by the equations

$$\mu = e^{2\lambda K}, \quad \mu' = e^{-\frac{i\pi\omega}{K} + 2\lambda i K'}.$$

II. If both λ and ω be zero, so that $F(z)$ is a doubly-periodic function of the first kind, then

$$F(z) = C_0 + \Sigma \left\{b_1 + b_2 \frac{d}{dz} + \dots + b_{n+1} \frac{d^n}{dz^n}\right\} \frac{H'(z-c)}{H(z-c)},$$

with the condition $\Sigma b_1 = 0$.

III. If ω be zero, but not λ, then

$$F(z) = Ae^{\lambda z} + \Sigma \left\{C_1 + C_2 \frac{d}{dz} + \dots + C_{n+1} \frac{d^n}{dz^n}\right\} g(z-c),$$

where
$$g(z) = \frac{H'(z)}{H(z)} e^{\lambda z},$$

the constants being subject to the condition

$$\Sigma (C_1 + C_2\lambda + \dots + C_{n+1}\lambda^n) e^{-\lambda c} = 0,$$

and the summations extending to all the accidental singularities of $F(z)$ in a parallelogram of periods containing the variable z.

139. Reverting now to the function $F(z)$ we have $G(z)$, defined as

$$\frac{\sigma(z+a)}{\sigma(z)\,\sigma(a)} e^{\lambda z},$$

when a and λ are properly determined, satisfying the equations

$$G(z + 2\omega) = \mu G(z), \quad G(z + 2\omega') = \mu' G(z).$$

Hence $\Omega(z) = F(z)/G(z)$ is a doubly-periodic function of the first kind; and therefore the number of its irreducible zeros is equal to the number of its irreducible infinities, and their sums (proper account being taken of multiplicity) are congruent to one another with moduli 2ω and $2\omega'$.

Let $c_1, c_2, \dots, c_m$ be the set of infinities of $F(z)$ in the parallelogram of periods containing the point z; and let $\gamma_1, \dots, \gamma_\mu$ be the set of zeros of $F(z)$ in the same parallelogram, an infinity of order n or a zero of order n occurring n times in the respective sets. The only zero of $G(z)$ in the parallelogram is congruent with $-a$, and its only infinity is congruent with 0, each being simple. Hence the $m+1$ irreducible infinities of $\Omega(z)$ are congruent with

$$-a, c_1, c_2, \dots, c_m,$$

and its $\mu + 1$ irreducible zeros are congruent with

$$0, \gamma_1, \gamma_2, \dots, \gamma_\mu;$$

and therefore
$$m + 1 = \mu + 1,$$
$$-a + \Sigma c \equiv \Sigma\gamma.$$

From the first it follows* that *the number of infinities of a doubly-periodic function of the second kind in a parallelogram of periods is equal to the number of its zeros, and that the excess of the sum of the former over the sum of the latter is congruent with*

$$\pm\left(\frac{\omega'}{\pi i}\log\mu - \frac{\omega}{\pi i}\log\mu'\right),$$

the sign being the same as that of $\Re\left(\frac{\omega'}{i\omega}\right)$.

The result just obtained renders it possible to derive another expression for $F(z)$, substantially due to Hermite. Consider a function

$$F_1(z) = \frac{\sigma(z-\gamma_1)\,\sigma(z-\gamma_2)\ldots\sigma(z-\gamma_m)}{\sigma(z-c_1)\,\sigma(z-c_2)\ldots\sigma(z-c_m)}\,e^{\rho z},$$

where ρ is a constant. Evidently $F_1(z)$ has the same zeros and the same infinities, each in the same degree, as $F(z)$. Moreover

$$F_1(z+2\omega) = F_1(z)\,e^{2\eta(\Sigma c-\Sigma\gamma)+2\rho\omega},$$

$$F_1(z+2\omega') = F_1(z)\,e^{2\eta'(\Sigma c-\Sigma\gamma)+2\rho\omega'}.$$

If, then, we choose points c and γ, such that

$$\Sigma c - \Sigma\gamma = a,$$

and we take $\rho = \lambda$, where a and λ are the constants of $G(z)$, then

$$F_1(z+2\omega) = \mu F_1(z), \quad F_1(z+2\omega') = \mu' F_1(z).$$

The function $F_1(z)/F(z)$ is a doubly-periodic function of the first kind and by the construction of $F_1(z)$ it has no zeros and no infinities in the finite part of the plane: it is therefore a constant. Hence

$$F(z) = A\,\frac{\sigma(z-\gamma_1)\,\sigma(z-\gamma_2)\ldots\sigma(z-\gamma_m)}{\sigma(z-c_1)\,\sigma(z-c_2)\ldots\sigma(z-c_m)}\,e^{\lambda z},$$

where $\Sigma c - \Sigma\gamma = a$, and a and λ are determined as for the function $G(z)$.

140. One of the most important applications of secondary doubly-periodic functions is that which leads to the solution of Lamé's equation in the cases when it can be integrated by means of uniform functions. This equation is subsidiary to the solution of the general equation, characteristic of the potential of an attracting mass at a point in free space; and it can be expressed either in the form

$$\frac{d^2w}{dz^2} = (Ak^2\,\mathrm{sn}^2 z + B)\,w,$$

or in the form

$$\frac{d^2w}{dz^2} = \{A\wp(z) + B\}\,w,$$

* Frobenius, *Crelle*, xciii, pp. 55—68, a memoir which contains developments of the properties of the function $G(z)$. The result appears to have been noticed first by Brioschi, (*Comptes Rendus*, t. xcii, p. 325), in discussing a more limited form.

according to the class of elliptic functions used. In order that the integral may be uniform, the constant A must be $n(n+1)$, where n is a positive integer; this value of A, moreover, is the value that occurs most naturally in the derivation of the equation. The constant B can be taken arbitrarily.

The foregoing equation is one of a class, the properties of which have been established* by Picard, Floquet, and others. Without entering into their discussion, the following will suffice to connect them with the secondary periodic function.

Let two independent special solutions be $g(z)$ and $h(z)$, uniform functions of z; every solution is of the form $\alpha g(z) + \beta h(z)$, where α and β are constants. The equation is unaltered when $z + 2\omega$ is substituted for z; hence $g(z + 2\omega)$ and $h(z + 2\omega)$ are solutions, so that we must have

$$g(z+2\omega) = Ag(z) + Bh(z), \quad h(z+2\omega) = Cg(z) + Dh(z),$$

where, as the functions are determinate, A, B, C, D are determinate constants, such that $AD - BC$ is different from zero.

Similarly, we obtain equations of the form

$$g(z+2\omega') = A'g(z) + B'h(z), \quad h(z+2\omega') = C'g(z) + D'h(z).$$

Using both equations to obtain $g(z + 2\omega + 2\omega')$ in the same form, we have

$$BC' = B'C, \quad AB' + BD' = A'B + B'D;$$

and similarly, for $h(z + 2\omega + 2\omega')$, we have

$$CA' + DC' = C'A + D'C, \quad BC' = B'C;$$

therefore
$$\frac{C}{B} = \frac{C'}{B'} = \delta, \quad \frac{A-D}{B} = \frac{A'-D'}{B'} = \epsilon.$$

Let a solution
$$F(z) = ag(z) + bh(z)$$
be chosen, so as to give

$$F(z+2\omega) = \mu F(z), \quad F(z+2\omega') = \mu' F(z),$$

if possible. The conditions for the first are

$$\frac{aA+bC}{a} = \frac{aB+bD}{b} = \mu,$$

so that $a/b \,(=\xi)$ must satisfy the equation

$$A - D = \xi B - \frac{C}{\xi};$$

and the conditions for the second are

$$\frac{aA'+bC'}{a} = \frac{aB'+bD'}{b} = \mu',$$

* Picard, *Comptes Rendus*, t. xc, (1880), pp. 128—131, 293—295; *Crelle*, t. xc, (1880), pp. 281—302.

Floquet, *Comptes Rendus*, t. xcviii, (1884), pp. 82—85; *Ann. de l'Éc. Norm. Sup.*, 3^me^ Sér., t. i, (1884), pp. 181—238.

so that ξ must satisfy the equation

$$A' - D' = \xi B' - \frac{C'}{\xi}.$$

These two equations are the same, being

$$\xi^2 - \epsilon\xi - \delta = 0.$$

Let ξ_1 and ξ_2 be the roots of this equation which, in general, are unequal; and let μ_1, μ_1' and μ_2, μ_2' be the corresponding values of μ, μ'. Then two functions, say $F_1(z)$ and $F_2(z)$, are determined: they are independent of one another, so therefore are $g(z)$ and $h(z)$; and therefore every solution can be expressed in terms of them. Hence *a linear differential equation of the second order, having coefficients that are doubly-periodic functions of the first kind, can generally be integrated by means of doubly-periodic functions of the second kind.*

It therefore follows that Lamé's equation, which will be taken in the form

$$\frac{1}{w}\frac{d^2w}{dz^2} = n(n+1)\wp(z) + B,$$

can be integrated by means of secondary doubly-periodic functions.

141. Let $z = c$ be an accidental singularity of w of order m; then, for points z in the immediate vicinity of c, we have

$$w = \frac{A}{(z-c)^m}\{1 + p(z-c) + q(z-c)^2 + \dots\},$$

and therefore

$$\frac{1}{w}\frac{d^2w}{dz^2} = \frac{m+m^2}{(z-c)^2} - \frac{2mp}{z-c} + \text{positive powers of } z-c.$$

Since this is equal to $\qquad n(n+1)\wp(z) + B$

it follows that c must be congruent to zero and that m, a positive integer, must be n. Moreover, $p = 0$. Hence *the accidental singularities of w are congruent to zero, and each is of order n.*

The secondary periodic function, which has no accidental singularities except those of order n congruent to $z = 0$, has n irreducible zeros. Let them be $-a_1, -a_2, \dots, -a_n$; then the form of the function is

$$w = \frac{\sigma(z+a_1)\,\sigma(z+a_2)\dots\sigma(z+a_n)}{\sigma^n(z)}\,e^{\rho z}.$$

Hence
$$\frac{1}{w}\frac{dw}{dz} = \rho - n\zeta(z) + \sum_{r=1}^{n}\zeta(z+a_r),$$

or, taking $\rho = -\Sigma\zeta(a_r)$, we have

$$\frac{1}{w}\frac{dw}{dz} = \sum_{r=1}^{n}\{\zeta(z+a_r) - \zeta(z) - \zeta(a_r)\},$$

and therefore
$$\frac{1}{w}\frac{d^2w}{dz^2} - \frac{1}{w^2}\left(\frac{dw}{dz}\right)^2 = n\wp(z) - \sum_{r=1}^{n}\wp(z+a_r).$$

But, by Ex. 3, § 131, we have

$$\frac{1}{w^2}\left(\frac{dw}{dz}\right)^2 = \frac{1}{4}\left\{\sum_{r=1}^{n} \frac{\wp'(a_r)-\wp'(z)}{\wp(a_r)-\wp(z)}\right\}^2$$

$$= \tfrac{1}{4}\sum_{r=1}^{n}\left\{\frac{\wp'(a_r)-\wp'(z)}{\wp(a_r)-\wp(z)}\right\}^2 + \tfrac{1}{2}\sum_{r=1}^{n}\sum_{s=1}^{n}\left\{\frac{\wp'(a_r)-\wp'(z)}{\wp(a_r)-\wp(z)}\cdot\frac{\wp'(a_s)-\wp'(z)}{\wp(a_s)-\wp(z)}\right\}$$

$$= \sum_{r=1}^{n}\{\wp(z+a_r)+\wp(z)+\wp(a_r)\} + \tfrac{1}{2}\sum_{r=1}^{n}\sum_{s=1}^{n}\left\{\frac{\wp'(a_r)-\wp'(z)}{\wp(a_r)-\wp(z)}\cdot\frac{\wp'(a_s)-\wp'(z)}{\wp(a_s)-\wp(z)}\right\},$$

by Ex. 4, § 131. Thus

$$\frac{1}{w}\frac{d^2w}{dz^2} = 2n\wp(z) + \sum_{r=1}^{n}\wp(a_r) + \tfrac{1}{2}\sum_{r=1}^{n}\sum_{s=1}^{n}\left\{\frac{\wp'(a_r)-\wp'(z)}{\wp(a_r)-\wp(z)}\cdot\frac{\wp'(a_s)-\wp'(z)}{\wp(a_s)-\wp(z)}\right\}.$$

Now

$$\frac{\wp'(a_r)-\wp'(z)}{\wp(a_r)-\wp(z)}\cdot\frac{\wp'(a_s)-\wp'(z)}{\wp(a_s)-\wp(z)}$$

$$= \frac{4\wp^3(z) - g_2\wp(z) - g_3 + \wp'(a_r)\wp'(a_s) - \{\wp'(a_r)+\wp'(a_s)\}\wp'(z)}{\{\wp(z)-\wp(a_r)\}\{\wp(z)-\wp\{(a_s)\}}$$

$$= 4\{\wp(z)+\wp(a_r)+\wp(a_s)\} + \frac{A\{\wp'(a_r)-\wp'(z)\}}{\wp(z)-\wp(a_r)} + \frac{B\{\wp'(a_s)-\wp'(z)\}}{\wp(z)-\wp(a_s)},$$

where $$A = \frac{\wp'(a_r)+\wp'(a_s)}{\wp(a_r)-\wp(a_s)} = -B.$$

Let the constants a be such that

$$\left.\begin{array}{l} \dfrac{\wp'(a_1)+\wp'(a_2)}{\wp(a_1)-\wp(a_2)} + \dfrac{\wp'(a_1)+\wp'(a_3)}{\wp(a_1)-\wp(a_3)} + \ldots = 0 \\ \dfrac{\wp'(a_2)+\wp'(a_1)}{\wp(a_2)-\wp(a_1)} + \dfrac{\wp'(a_2)+\wp'(a_3)}{\wp(a_2)-\wp(a_3)} + \ldots = 0 \\ \vdots \qquad\qquad \vdots \qquad\qquad \vdots \quad \vdots \end{array}\right\},$$

n equations of which only $n-1$ are independent, because the sum of the n left-hand sides vanishes. Then in the double summation the coefficient of each of the fractions $\dfrac{\wp'(a_r)-\wp'(z)}{\wp(z)-\wp(a_r)}$ is zero; and so

$$\sum_{r=1}^{n}\sum_{s=1}^{n}\left\{\frac{\wp'(a_r)-\wp'(z)}{\wp(a_r)-\wp(z)}\,\frac{\wp'(a_s)-\wp'(z)}{\wp(a_s)-\wp(z)}\right\} = 2n(n-1)\wp(z) + 4(n-1)\sum_{r=1}^{n}\wp(a_r),$$

and therefore $$\frac{1}{w}\frac{d^2w}{dz^2} = n(n+1)\wp(z) + (2n-1)\sum_{r=1}^{n}\wp(a_r).$$

Hence *it follows that*

$$F(z) = w_1 = \frac{\sigma(z+a_1)\,\sigma(z+a_2)\ldots\sigma(z+a_n)}{\sigma^n(z)}\, e^{-z\sum_{r=1}^{n}\zeta(a_r)}$$

satisfies Lamé's equation, provided the n constants a be determined by the preceding equations and by the relation

$$B = (2n-1)\sum_{r=1}^{n}\wp(a_r).$$

Evidently the equation is unaltered when $-z$ is substituted for z; and therefore

$$F(-z) = w_2 = \frac{\sigma(z-a_1)\,\sigma(z-a_2)\ldots\sigma(z-a_n)}{\sigma^n(z)}\, e^{z\sum_{r=1}^{n}\zeta(a_r)}$$

is another solution. Every solution is of the form

$$MF(z) + NF(-z),$$

where M and N are arbitrary constants.

Corollary. The simplest cases are when $n = 1$ and $n = 2$.

When $n = 1$, the equation is

$$\frac{1}{w}\frac{d^2w}{dz^2} = 2\wp(z) + B;$$

there is only a single constant a determined by the single equation

$$B = \wp(a),$$

and the general solution is

$$w = M\frac{\sigma(z+a)}{\sigma(z)}e^{-z\zeta(a)} + N\frac{\sigma(z-a)}{\sigma(z)}e^{z\zeta(a)}.$$

When $n = 2$, the equation is

$$\frac{1}{w}\frac{d^2w}{dz^2} = 6\wp(z) + B.$$

The general solution is

$$w = M\frac{\sigma(z+a)\,\sigma(z+b)}{\sigma^2(z)}e^{-z\zeta(a)-z\zeta(b)} + N\frac{\sigma(z-a)\,\sigma(z-b)}{\sigma^2(z)}e^{z\zeta(a)+z\zeta(b)},$$

where a and b are determined by the conditions

$$\frac{\wp'(a) + \wp'(b)}{\wp(a) - \wp(b)} = 0, \qquad \wp(a) + \wp(b) = \tfrac{1}{3}B.$$

Rejecting the solution $a+b\equiv 0$, we have a and b determined by the equations

$$\wp(a) + \wp(b) = \tfrac{1}{3}B, \quad \wp(a)\,\wp(b) = \tfrac{1}{9}B^2 - \tfrac{1}{4}g_2.$$

For a full discussion of Lamé's equation and for references to the original sources of information, see Halphen, *Traité des fonctions elliptiques*, t. ii, chap. XII., in particular, pp. 495 et seq.

Ex. When Lamé's equation has the form

$$\frac{1}{w}\frac{d^2w}{dz^2} = n(n+1)\,k^2\,\mathrm{sn}^2 z - h,$$

obtain the solution for $n=1$, in terms of the Jacobian Theta-Functions,

$$w = A\frac{H(z+\omega)}{\Theta(z)}e^{-z\frac{\Theta'(\omega)}{\Theta(\omega)}} + B\frac{H(z-\omega)}{\Theta(z)}e^{z\frac{\Theta'(\omega)}{\Theta(\omega)}},$$

where ω is determined by the equation $\mathrm{dn}^2\omega = h - k^2$; and discuss in particular the solution when h has the values $1+k^2$, 1, k^2.

Obtain the solution for $n=2$ in the form

$$w = A\frac{d}{dz}\left[\frac{H(z+\omega)}{\Theta(z)}e^{\left\{\lambda - \frac{\Theta'(\omega)}{\Theta(\omega)}\right\}z}\right] + B\frac{d}{dz}\left[\frac{H(z-\omega)}{\Theta(z)}e^{-\left\{\lambda - \frac{\Theta'(\omega)}{\Theta(\omega)}\right\}z}\right],$$

where λ and ω are given by the equations

$$\lambda^2 = \frac{(2k^2 \operatorname{sn}^2 a - 1 - k^2)(2k^2 \operatorname{sn}^2 a - 1)(2 \operatorname{sn}^2 a - 1)}{3k^2 \operatorname{sn}^4 a - 2(1+k^2)\operatorname{sn}^2 a + 1},$$

$$\operatorname{sn}^2 \omega = \frac{\operatorname{sn}^4 a \,(2k^2 \operatorname{sn}^2 a - 1 - k^2)}{3k^2 \operatorname{sn}^4 a - 2(1+k^2)\operatorname{sn}^2 a + 1},$$

and a is derived from h by the relation

$$h = 4(1+k^2) - 6k^2 \operatorname{sn}^2 a.$$

Deduce the three solutions that occur when λ is zero, and the two solutions that occur when λ is infinite. (Hermite.)

Doubly-Periodic Functions of the Third Kind.

142. The equations characteristic of a doubly-periodic function $\Phi(z)$ of the third kind are

$$\Phi(z+2\omega) = \Phi(z), \quad \Phi(z+2\omega') = e^{-\frac{m\pi i}{\omega} z}\,\Phi(z),$$

where m is an integer different from zero.

Obviously the number of zeros in a parallelogram is a constant, as well as the number of infinities. Let a parallelogram, chosen so that its sides contain no zero and no infinity of $\Phi(z)$, have p, $p+2\omega$, $p+2\omega'$ for three of its angular points; and let $a_1, a_2, \ldots, a_l$ be the zeros and $c_1, \ldots, c_m$ be the infinities, multiplicity of order being represented by repetitions. Then using $\Psi(z)$ to denote $\dfrac{d}{dz}\{\log \Phi(z)\}$, we have, as the equations characteristic of $\Psi(z)$,

$$\Psi(z+2\omega) = \Psi(z), \quad \Psi(z+2\omega') = \Psi(z) - \frac{m\pi i}{\omega};$$

and for points in the parallelogram

$$\Psi(z) = \sum_{r=1}^{l} \frac{1}{z-a_r} - \sum_{s=1}^{n} \frac{1}{z-c_s} + H(z),$$

where $H(z)$ has no infinity within the parallelogram. Hence

$$2\pi i(l-n) = \int \Psi(z)\,dz,$$

the integral being taken round the parallelogram: by using the Corollary to Prop. II. in § 116, we have

$$2\pi i(l-n) = -\int_p^{p+2\omega} -\left(\frac{m\pi i}{\omega}\right) dz = 2m\pi i,$$

so that

$$l = n + m:$$

or *the algebraical excess of the number of irreducible zeros over the number of irreducible infinities is equal to* m.

Again, since

$$\frac{z}{z-\mu} = 1 + \frac{\mu}{z-\mu},$$

we have

$$\Sigma \frac{a}{z-a} - \Sigma \frac{c}{z-c} + l - n = z\Psi(z) - zH(z),$$

and therefore

$$2\pi i(\Sigma a - \Sigma c) = \int z\Psi(z)\,dz,$$

the integral being taken round the parallelogram. As before, this gives

$$2\pi i\,(\Sigma a - \Sigma c) = \int_p^{p+2\omega'} 2\omega\Psi(z)\,dz - \int_p^{p+2\omega}\left\{2\omega'\Psi(z) - \frac{m\pi i}{\omega}(z+2\omega')\right\}dz.$$

The former integral is

$$2\omega\int_p^{p+2\omega'}\frac{\Phi'(z)}{\Phi(z)}\,dz$$

$$= 2\omega\left(-\frac{m\pi i}{\omega}p\right) = -2m\pi i p,$$

for the side of the parallelogram contains* no zero and no infinity of $\Phi(z)$.

The latter integral, with its own sign, is

$$-2\omega'\int_p^{p+2\omega}\frac{\Phi'(z)}{\Phi(z)}\,dz + \frac{m\pi i}{\omega}\int_p^{p+2\omega}(z+2\omega')\,dz$$

$$= 0 + \frac{m\pi i}{2\omega}\{(p+2\omega+2\omega')^2 - (p+2\omega')^2\}$$

$$= 2m\pi i\,(p+\omega+2\omega').$$

Hence
$$\Sigma a - \Sigma c = m\,(\omega+2\omega'),$$
giving *the excess of the sum of the zeros over the sum of the infinities in any parallelogram chosen so as to contain the variable z and to have no one of its sides passing through a zero or an infinity of the function.*

These will be taken as the irreducible zeros and the irreducible infinities: all others are congruent with them.

All these results are obtained through the theorem II. of § 116, which assumes that the argument of ω' is greater than the argument of ω or, what is the equivalent assumption (§ 129), that

$$\eta\omega' - \eta'\omega = \tfrac{1}{2}\pi i.$$

143. Taking the function, naturally suggested for the present class by the corresponding function for the former class, we introduce a function

$$\phi(z) = e^{\lambda z^2+\mu z}\frac{\sigma(z-a_1)\,\sigma(z-a_2)\ldots\sigma(z-a_l)}{\sigma(z-c_1)\,\sigma(z-c_2)\ldots\sigma(z-c_n)},$$

where the a's and the c's are connected by the relations

$$\Sigma a - \Sigma c = m\,(\omega+2\omega'),\quad l-n=m.$$

Then $\phi(z)$ satisfies the equations characteristic of doubly-periodic functions of the third kind, if

$$\begin{cases} 0 = 4\lambda\omega + 2m\eta, \\ k\,.\,2\pi i = 4\lambda\omega^2 + 2m\eta\omega + 2\mu\omega + m\pi i - 2m\eta\,(\omega+2\omega')\,; \end{cases}$$

$$\begin{cases} -\dfrac{m\pi i}{\omega} = 4\lambda\omega' + 2m\eta', \\ k'\,.\,2\pi i = 4\lambda\omega'^2 + 2m\eta'\omega' + 2\mu\omega' + m\pi i - 2m\eta'\,(\omega+2\omega'), \end{cases}$$

* Both in this integral and in the next, which contain parts of the form $\int\frac{dw}{w}$, there is, as in Prop. VII., § 116, properly an additive term of the form $2\kappa\pi i$, where κ is an integer; but, as there, both terms can be removed by modification of the position of the parallelogram, and this modification is supposed, in the proof, to have been made.

k and k' being disposable integers. These are uniquely satisfied by taking

$$\lambda = -\frac{1}{2}\frac{m\eta}{\omega},$$

$$\mu = \frac{1}{2}\frac{m\pi i}{\omega} + m(\eta + 2\eta'),$$

with $$k = 0, \quad k' = m.$$

Assuming the last two, the values of λ and μ are thus obtained so as to make $\phi(z)$ a doubly-periodic function of the third kind.

Now let $a_1, \ldots, a_l$ be chosen as the irreducible zeros of $\Phi(z)$ and $c_1, \ldots, c_n$ as the irreducible infinities of $\Phi(z)$, which is possible owing to the conditions to which they were subjected. Then $\Phi(z)/\phi(z)$ is a doubly-periodic function of the first kind: it has no zeros and no infinities in the parallelogram of periods and therefore none in the whole plane; it is therefore a constant, so that

$$\Phi(z) = Ae^{-\frac{1}{2}\frac{\eta}{\omega}mz^2 + \left\{\frac{1}{2}\frac{\pi i}{\omega} + (\eta + 2\eta')\right\}mz} \frac{\sigma(z - a_1)\,\sigma(z - a_2)\ldots\sigma(z - a_l)}{\sigma(z - c_1)\,\sigma(z - c_2)\ldots\sigma(z - c_n)},$$

a representation of $\Phi(z)$ in terms of known quantities.

Ex. Had the representation been effected by means of the Jacobian Theta-Functions which would replace $\sigma(z)$ by $H(z)$, then the term in z^2 in the exponential would be absent.

144. No limitation on the integral value of m, except that it must not vanish, has been made: and the form just obtained holds for all values. Equivalent expressions in the form of sums of functions can be constructed: but there is then a difference between the cases of m positive and m negative.

If m be positive, being the excess of the number of irreducible zeros over the number of irreducible infinities, the function is said to be of positive class m; it is evident that there are suitable functions without any irreducible infinities—they are integral functions.

When m is negative ($= -n$), the function is said to be of negative class n; but there are no corresponding integral functions.

145. First, *let m be positive.*

i. If the function have no accidental singularities, it can be expressed in the form

$$Ae^{\lambda z^2 + \mu z}\,\sigma(z - a_1)\,\sigma(z - a_2)\ldots\sigma(z - a_m),$$

with appropriate values of λ and μ.

ii. If the function have n irreducible accidental singularities, then it has $m + n$ irreducible zeros. We proceed to shew that the function can be expressed by means of similar functions of positive class m, with a single accidental singularity.

Using λ and μ to denote

$$-\tfrac{1}{2}\frac{m\eta}{\omega} \text{ and } \tfrac{1}{2}\frac{m\pi i}{\omega} + m(\eta + 2\eta'),$$

which are the constants in the exponential factor common to all functions of the same class, consider a function, of positive class m with a single accidental singularity, in the form

$$\psi_m(z, u) = e^{\lambda(z^2-u^2)+\mu(z-u)} \frac{\sigma(z-b_1)\,\sigma(z-b_2)\ldots\sigma(z-b_{m+1})}{\sigma(u-b_1)\,\sigma(u-b_2)\ldots\sigma(u-b_{m+1})} \frac{1}{\sigma(z-u)},$$

where $b_1, b_2, \ldots, b_m$ are arbitrary constants, of sum s, and

$$\begin{aligned} m(\omega + 2\omega') &= b_{m+1} + b_1 + b_2 + \ldots b_m - u \\ &= b_{m+1} + s - u. \end{aligned}$$

The function ψ_m satisfies the equations

$$\psi_m(z + 2\omega, u) = \psi_m(z, u), \quad \psi_m(z + 2\omega', u) = e^{-\frac{m\pi zi}{\omega}} \psi_m(z, u);$$

regarded as a function of z, it has u for its sole accidental singularity, evidently simple.

The function $\dfrac{1}{\psi_m(z, u)}$ can be expressed in the form

$$e^{\lambda(u^2-z^2)+\mu(u-z)} \frac{\sigma(u-z)\,\sigma(u-b_1)\ldots\sigma(u-b_m)}{\sigma(z-b_1)\ldots\ldots\ldots\ldots\sigma(z-b_m)} \frac{\sigma\{s - m(\omega+2\omega')\}}{\sigma\{u - z - s + m(\omega + 2\omega')\}}.$$

Regarded as a function of u, it has $z, b_1, \ldots, b_m$ for zeros and $z + s - m(\omega + 2\omega')$ for its sole accidental singularity, evidently simple: also

$$z + b_1 + \ldots + b_m - \{z + s - m(\omega + 2\omega')\} = m(\omega + 2\omega').$$

Hence owing to the values of λ and μ, it follows that $\dfrac{1}{\psi_m(z, u)}$, when regarded as a function of u, satisfies all the conditions that establish a doubly-periodic function of the third kind of positive class m, so that

$$\frac{1}{\psi_m(z, u + 2\omega)} = \frac{1}{\psi_m(z, u)},$$

$$\frac{1}{\psi_m(z, u + 2\omega')} = e^{-\frac{m\pi zi}{\omega}} \frac{1}{\psi_m(z, u)};$$

and therefore

$$\psi_m(z, u + 2\omega) = \psi_m(z, u), \quad \psi_m(z, u + 2\omega') = e^{\frac{m\pi zi}{\omega}} \psi_m(z, u).$$

Evidently $\psi_m(z, u)$ regarded as a function of u is of negative class m: its infinities and its sole zero can at once be seen from the form

$$\psi_m(z, u) = e^{\lambda(z^2-u^2)+\mu(z-u)} \frac{\sigma(z-b_1)\ldots\sigma(z-b_m)\,\sigma\{u - z - s + m(\omega + 2\omega')\}}{\sigma(u-z)\,\sigma(u-b_1)\ldots\sigma(u-b_m)\,\sigma\{s - m(\omega + 2\omega')\}}.$$

Each of the infinities is simple. In the vicinity of $u = z$, the expansion of the function is

$$\frac{-1}{u-z} + \text{positive integral powers of } u - z:$$

and, in the vicinity of $u = b_r$, it is

$$\frac{G_r(z)}{u - b_r} + \text{positive integral powers of } u - b_r,$$

where $G_r(z)$ denotes

$$e^{\lambda(z^2 - b_r^2) + \mu(z - b_r)} \frac{\sigma(z-b_1)\ldots\sigma(z-b_{r-1})\sigma(z-b_{r+1})\ldots\sigma(z-b_m)\sigma\{z+s-b_r-m(\omega+2\omega')\}}{\sigma(b_r-b_1)\ldots\sigma(b_r-b_{r-1})\sigma(b_r-b_{r+1})\ldots\sigma(b_r-b_m)\sigma\{s-m(\omega+2\omega')\}},$$

and is therefore an integral function of z of positive class m.

Let $\Phi(u)$ be a doubly-periodic function of the third kind, of positive class m; and let its irreducible accidental singularities, that is, those which occur in a parallelogram containing the point u, be α_1 of order $1 + \mu_1$, α_2 of order $1 + \mu_2$, and so on. In the immediate vicinity of a point α_r, let

$$\Phi(u) = \left(A_r - B_r \frac{d}{du} + C_r \frac{d^2}{du^2} - \ldots \pm M_r \frac{d^{\mu_r}}{du^{\mu_r}}\right) \frac{1}{u - \alpha_r} + P_r(u - \alpha_r).$$

Then proceeding as in the case of the secondary doubly-periodic functions (§ 137), we construct a function

$$F(u) = \Phi(u)\, \psi_m(z, u).$$

We at once have $\quad F(u + 2\omega) = F(u) = F(u + 2\omega'),$

so that $F(u)$ is a doubly-periodic function of the first kind; hence the sum of its residues for all the poles in a parallelogram of periods is zero.

For the infinities of $F(u)$, which arise through the factor $\psi_m(z, u)$, we have as the residue for $u = z$

$$-\Phi(z),$$

and as the residue for $u = b_r$, where $r = 1, 2, \ldots, m$,

$$\Phi(b_r)\, G_r(z).$$

In the vicinity of α_r, we have

$$\psi_m(z, u) = \psi_m(z, \alpha_r) + (u - \alpha_r)\psi_m'(z, \alpha_r) + \frac{(u - \alpha_r)^2}{2!}\psi_m''(z, \alpha_r) + \ldots,$$

where dashes imply differentiation of $\psi_m(z, u)$ with regard to u, after which u is made equal to α_r; so that in $\Phi(u)\,\psi_m(z, u)$ the residue for $u = \alpha_r$, where $r = 1, 2, \ldots$, is

$$E_r(z) = A_r \psi_m(z, \alpha_r) + B_r \psi_m'(z, \alpha_r) + C_r \psi_m''(z, \alpha_r) + \ldots + M_r \psi_m^{(\mu_r)}(z, \alpha_r).$$

Hence we have

$$-\Phi(z) + \sum_{r=1}^{m} \Phi(b_r)\, G_r(z) + \sum_{s=1} E_s(z) = 0,$$

and therefore $\quad \Phi(z) = \sum_{s=1} E_s(z) + \sum_{r=1}^{m} \Phi(b_r)\, G_r(z),$

giving *the expression of $\Phi(z)$ by means of doubly-periodic functions of the third kind, which are of positive class m and have either no accidental singularity or only one and that a simple singularity.*

The m quantities $b_1, \ldots, b_m$ are arbitrary; the simplest case which occurs is when the m zeros of $\Phi(z)$ are different and are chosen as the values of $b_1, \ldots, b_m$. The value of $\Phi(z)$ is then

$$\Phi(z) = \sum_{s=1} E_s(z),$$

where the summation extends to all the irreducible accidental singularities; while, if there be the further simplification that all the accidental singularities are simple, then

$$\Phi(z) = A_1 \psi_m(z, \alpha_1) + A_2 \psi_m(z, \alpha_2) + \ldots,$$

the summation extending to all the irreducible simple singularities.

The quantity $\psi_m(z, \alpha_r)$, which is equal to

$$e^{\lambda(z^2-\alpha_r^2)+\mu(z-\alpha_r)} \frac{\sigma(z-b_1)\ldots\sigma(z-b_m)\,\sigma\{z+\Sigma b - m(\omega+2\omega') - \alpha_r\}}{\sigma(\alpha_r-b_1)\ldots\sigma(\alpha_r-b_m)\,\sigma\{\Sigma b - m(\omega+2\omega')\}\,\sigma(z-\alpha_r)},$$

and is subsidiary to the construction of the function $E(z)$, *is called the simple element of positive class* m.

In the general case, the portion

$$\Sigma\, \Phi(b_r)\, G_r(z)$$

gives an integral function of z, and the portion $\sum\limits_{s=1} E_s(z)$ gives a fractional function of z.

146. Secondly, *let m be negative and equal to* $-n$. The equations satisfied by $\Phi(z)$ are

$$\Phi(z+2\omega) = \Phi(z), \qquad \Phi(z+2\omega') = e^{\frac{n\pi z i}{\omega}}\, \Phi(z),$$

and the number of irreducible singularities is greater by n than the number of irreducible zeros.

One expression for $\Phi(z)$ is at once obtained by forming its reciprocal, which satisfies the equations

$$\frac{1}{\Phi(z+2\omega)} = \frac{1}{\Phi(z)}, \qquad \frac{1}{\Phi(z+2\omega')} = e^{-\frac{n\pi z i}{\omega}} \frac{1}{\Phi(z)},$$

and is therefore of the class just considered: the value of $\frac{1}{\Phi(z)}$ is of the form

$$\Sigma\, E_s(z) + \Sigma\, A_r G_r(z).$$

For purposes of expansion, however, this is not a convenient form as it gives only the reciprocal of $\Phi(z)$.

To represent the function, Appell constructed the element

$$\chi_n(z, y) = \frac{\pi}{2\omega} \sum_{s=-\infty}^{s=\infty} e^{\frac{n s \pi i}{\omega}\{y+(s-1)\omega'\}} \cot \frac{\pi(z-y-2s\omega')}{2\omega},$$

which, since the real part of $\omega'/\omega i$ is positive, converges for all values of z and y, except those for which

$$z \equiv y \pmod{2\omega,\ 2\omega'}.$$

For each of these values one term of the series, and therefore the series itself, becomes infinite of the first order.

Evidently
$$\chi_n(z, y + 2\omega) = \chi_n(z, y),$$
$$\chi_n(z, y + 2\omega') = e^{-\frac{n\pi y i}{\omega}} \chi_n(z, y);$$

therefore in the present case

$$\Omega(y) = \Phi(y)\, \chi_n(z, y),$$

regarded as a function of y, is a doubly-periodic function of the first kind.

Hence the sum of the residues of its irreducible accidental singularities is zero.

When the parallelogram is chosen, which includes z, these singularities are

(i) $y = z$, arising through $\chi_n(z, y)$;

(ii) the singularities of $\Phi(y)$, which are at least n in number, and are $n + l$ when Φ has l irreducible zeros.

The expansion of $\chi_n(z, y)$, in powers of $y - z$, in the vicinity of the point z, is

$$\frac{-1}{y - z} + \text{positive integral powers of } y - z;$$

therefore the residue of $\Omega(y)$ is

$$-\Phi(z).$$

Let α_r be any irreducible singularity, and in the vicinity of α_r let $\Phi(y)$ denote

$$\left(A_r - B_r \frac{d}{dy} + C_r \frac{d^2}{dy^2} + \dots \pm P_r \frac{d^p}{dy^p}\right) \frac{1}{y - \alpha_r}$$
$$+ \text{positive integral powers of } y - \alpha_r,$$

where the series of negative powers is finite because the singularity is accidental; then the residue of $\Omega(y)$ is

$$A_r \chi_n(z, \alpha_r) + B_r \chi_n'(z, \alpha_r) + C_r \chi_n''(z, \alpha_r) + \dots + P_r \chi_n^{(p)}(z, \alpha_r),$$

where $\chi_n^{(\lambda)}(z, \alpha_r)$ is the value of

$$\frac{d^\lambda \chi_n(z, y)}{dy^\lambda}$$

when $y = \alpha_r$ after differentiation. Similarly for the residues of other singularities: and so, as their sum is zero, we have

$$\Phi(z) = \Sigma \{A_r \chi_n(z, \alpha_r) + B_r \chi_n'(z, \alpha_r) + \dots + P_r \chi_n^{(p)}(z, \alpha_r)\},$$

the summation extending over all the singularities.

The simplest case occurs when all the $N\ (>n)$ singularities α are accidental and of the first order; the function $\Phi(z)$ can then be expressed in the form

$$A_1 \chi_n(z, \alpha_1) + A_2 \chi_n(z, \alpha_2) + \ldots + A_N \chi_n(z, \alpha_N).$$

The quantity $\chi_n(z, \alpha)$, which is equal to

$$\frac{\pi}{2\omega} \sum_{s=-\infty}^{s=\infty} e^{\frac{ns\pi i}{\omega}\{(s-1)\omega'+\alpha\}} \cot \frac{\pi(z-\alpha-2s\omega')}{2\omega},$$

is called the *simple element for the expression of a doubly-periodic function of the third kind of negative class n.*

Ex. Deduce the result

$$\frac{2K}{\pi} \frac{\operatorname{cn} u}{\operatorname{sn} u} = \sum_{s=-\infty}^{s=\infty} (-1)^s \cot \left\{\frac{\pi(u+2siK')}{2K}\right\}.$$

147. The function $\chi_n(z, y)$ can be used also as follows. Since $\chi_m(z, y)$, quà function of y, satisfies the equations

$$\chi_m(z, y+2\omega) = \chi_m(z, y),$$
$$\chi_m(z, y+2\omega') = e^{-\frac{m\pi y i}{\omega}} \chi_m(z, y),$$

which are the same equations as are satisfied by a function of y of positive class m, therefore $\chi_m(\alpha, z)$, which is equal to

$$\frac{\pi}{2\omega} \sum_{s=-\infty}^{s=\infty} e^{\frac{ms\pi i}{\omega}\{z+(s-1)\omega'\}} \cot \frac{\pi(\alpha-z-2s\omega')}{2\omega},$$

being a function of z, satisfies the characteristic equations of § 142; and, in the vicinity of $z=\alpha$,

$$\chi_m(\alpha, z) = \frac{-1}{z-\alpha} + \text{positive integral powers of } z-\alpha.$$

If then we take the function $\Phi(z)$ of § 145, in the case when it has simple singularities at $\alpha_1, \alpha_2, \ldots$ and is of positive class m, then

$$\Phi(z) + A_1 \chi_m(\alpha_1, z) + A_2 \chi_m(\alpha_2, z) + \ldots$$

is a function of positive class m without any singularities: it is therefore equal to an integral function of positive class m, say to $G(z)$, where

$$G(z) = Ae^{\lambda z^2+\mu z} \sigma(z-a_1) \ldots \sigma(z-a_m),$$

so that $$\Phi(z) = G(z) - A_1 \chi_m(\alpha_1, z) - A_2 \chi_m(\alpha_2, z) - \ldots.$$

Ex. As a single example, consider a function of negative class 2, and let it have no zero within the parallelogram of reference. Then for the function, in the canonical product-form of § 143, the two irreducible infinities are subject to the relation

$$c_1 + c_2 = 2(\omega + 2\omega'),$$

and the function is $$\Phi(z) = Ke^{\frac{\eta}{\omega}z^2 - \left(\frac{\pi i}{\omega}+2\eta+4\eta'\right)z} \frac{1}{\sigma(z-c_1)\,\sigma(z-c_2)}.$$

The simple elements to express $\Phi(z)$ as a sum are

$$\chi_2(z, c_1) = \frac{\pi}{2\omega} \sum_{-\infty}^{\infty} e^{\frac{2s\pi i}{\omega}\{(s-1)\omega' + c_1\}} \cot \frac{\pi}{2\omega}(z - c_1 - 2s\omega'),$$

$$\chi_2(z, c_2) = \frac{\pi}{2\omega} \sum_{-\infty}^{\infty} e^{\frac{2s\pi i}{\omega}\{(s-1)\omega' + 2\omega + 4\omega' - c_1\}} \cot \frac{\pi}{2\omega}(z + c_1 - 2\omega - 4\omega' - 2s\omega')$$

$$= \frac{\pi}{2\omega} e^{\frac{4\pi i}{\omega}(c_1 - \omega')} \sum_{-\infty}^{\infty} e^{\frac{2r\pi i}{\omega}\{(r-1)\omega' - c_1\}} \cot \frac{\pi}{2\omega}(z + c_1 - 2r\omega')$$

after an easy reduction,

$$= e^{\frac{4\pi i}{\omega}(c_1 - \omega')} \chi_2(z, -c_1).$$

The residue of $\Phi(z)$ for c_1, which is a simple singularity, is

$$A_1 = K e^{\frac{\eta}{\omega} c_1^2 - \left(\frac{\pi i}{\omega} + 2\eta + 4\eta'\right) c_1} \frac{1}{\sigma(c_1 - c_2)};$$

and for c_2, also a simple singularity, it is

$$A_2 = K e^{\frac{\eta}{\omega} c_2^2 - \left(\frac{\pi i}{\omega} + 2\eta + 4\eta'\right) c_2} \frac{1}{\sigma(c_2 - c_1)},$$

so that

$$\frac{A_1}{A_2} = -e^{\frac{\pi i}{\omega}(c_1 - c_2)} = -e^{\frac{2\pi i}{\omega}(c_1 - 2\omega')}.$$

Hence the expression for $\Phi(z)$ as a sum, which is

$$A_1 \chi_2(z, c_1) + A_2 \chi_2(z, c_2),$$

becomes

$$A_1 \{\chi_2(z, c_1) - e^{\frac{2\pi i}{\omega} c_1} \chi_2(z, -c_1)\};$$

that is, it is a constant multiple of

$$e^{-\frac{\pi i}{\omega} c_1} \chi_2(z, c_1) - e^{\frac{\pi i}{\omega} c_1} \chi_2(z, -c_1).$$

Again,

$$\Phi(z) = K e^{\frac{\eta}{\omega} z^2 - \left(\frac{\pi i}{\omega} + 2\eta + 4\eta'\right) z} \frac{1}{\sigma(z - c_1)\,\sigma(z + c_1 - 2\omega - 4\omega')}$$

$$= -K e^{\frac{\eta}{\omega} z^2 - \left(\frac{\pi i}{\omega} + 2\eta + 4\eta'\right) z + 2(\eta + 2\eta')(z + c_1 - \omega - 2\omega')} \frac{1}{\sigma(z - c_1)\,\sigma(z + c_1)}$$

$$= L e^{\frac{\eta}{\omega} z^2 - \frac{\pi i z}{\omega}} \frac{\sigma(2c_1)}{\sigma(z - c_1)\,\sigma(z + c_1)},$$

on changing the constant factor. Hence it is possible to determine L so that

$$\Phi(z) = e^{-\frac{\pi i}{\omega} c_1} \chi_2(z, c_1) - e^{\frac{\pi i}{\omega} c_1} \chi_2(z, -c_1).$$

Taking the residues of the two sides for $z = c_1$, we have

$$L e^{\frac{\eta}{\omega} c_1^2 - \frac{\pi i}{\omega} c_1} = e^{-\frac{\pi i}{\omega} c_1},$$

and therefore finally we have

$$e^{\frac{\eta}{\omega}(z^2 - c^2) - \frac{\pi i z}{\omega}} \frac{\sigma(2c)}{\sigma(z - c)\,\sigma(z + c)} = e^{-\frac{\pi i c}{\omega}} \chi_2(z, c) - e^{\frac{\pi i c}{\omega}} \chi_2(z, -c)$$

$$= \frac{\pi}{2\omega} \sum_{-\infty}^{\infty} e^{2s(s-1)\pi i \frac{\omega'}{\omega}} \left\{ e^{(2s-1)\frac{\pi i c_1}{\omega}} \cot \frac{\pi}{2\omega}(z - c_1 - 2s\omega') - e^{-(2s-1)\frac{\pi i c_1}{\omega}} \cot \frac{\pi}{2\omega}(z + c_1 - 2s\omega') \right\},$$

the right-hand side of which admits of further modification if desired.

Many examples of such developments in trigonometrical series are given by Hermite*, Biehler†, Halphen‡, Appell§, and Krause||.

148. We shall not further develop the theory of these uniform doubly-periodic functions of the third kind. It will be found in the memoirs of Appell§ to whom it is largely due; and in the treatises of Halphen**, and of Rausenberger††.

It need hardly be remarked that the classes of uniform functions of a single variable which have been discussed form only a small proportion of functions reproducing themselves save as to a factor when the variable is subjected to homographic substitutions, of which a very special example is furnished by linear additive periodicity. Thus there are the various classes of pseudo-automorphic functions, (§ 305) called Thetafuchsian by Poincaré, their characteristic equation being

$$\Theta\left(\frac{\alpha z+\beta}{\gamma z+\delta}\right)=(\gamma z+\delta)^{2m}\,\Theta(z),$$

for all the substitutions of the group determining the function: and other classes are investigated in the treatises which have just been quoted.

The following examples relate to particular classes of pseudo-periodic functions.

Ex. 1. Shew that, if $F(z)$ be a uniform function satisfying the equations

$$F(z+2\omega')=F(z),$$

$$F\left(z+\frac{2\omega}{m}\right)=bF(z),$$

where b is a primitive mth root of unity, then $F(z)$ can be expressed in the form

$$\Sigma\left(A_0+A_1\frac{d}{dz}+\ldots\ldots+A_n\frac{d^n}{dz^n}\right)f(z-a),$$

where $f(z)$ denotes the function

$$\zeta(z)+b\zeta\left(z-\frac{2\omega}{m}\right)+b^2\zeta\left(z-\frac{4\omega}{m}\right)+\ldots\ldots+b^{m-1}\zeta\left(z-\frac{2m\omega-2\omega}{m}\right);$$

and prove that $\int F(z)\,dz$ can be expressed in the form of a doubly-periodic function together with a sum of logarithms of doubly-periodic functions with constant coefficients.

(Goursat.)

* *Comptes Rendus*, t. lv, (1862), pp. 11—18.

† *Sur les developpements en séries des fonctions doublement périodiques de troisième espèce*, (Thèse, Paris, Gauthier-Villars, 1879).

‡ *Traité des fonctions elliptiques*, t. i, chap. XIII.

§ *Annales de l'Éc. Norm. Sup.*, 3me Sér., t. i, pp. 135—164, t. ii, pp. 9—36, t. iii, pp. 9—42.

|| *Math. Ann.*, t. xxx, (1887), pp. 425—436, 516—534.

** *Traité des fonctions elliptiques*, t. i, chap. XIV.

†† *Lehrbuch der Theorie der periodischen Functionen*, (Leipzig, Teubner, 1884), where further references are given.

Ex. 2. Shew that, if a pseudo-periodic function be defined by the equations

$$f(z+2\omega)=f(z)+\lambda,$$
$$f(z+2\omega')=f(z)+\lambda',$$

and if, in the parallelogram of periods containing the point z, it have infinities c, ... such that in their immediate vicinity

$$f(z)=\left\{C_1+C_2\frac{d}{dz}+\ldots\ldots+C_{n+1}\frac{d^n}{dz^n}\right\}\frac{1}{z-c}+P(z-c),$$

then $f(z)$ can be expressed in the form

$$\frac{\lambda'\eta-\lambda\eta'}{\pm i\pi}z+A+\Sigma\left\{C_1+C_2\frac{d}{dz}+\ldots\ldots+C_{n+1}\frac{d^n}{dz^n}\right\}\zeta(z-c),$$

the summation extending over all the infinities of $f(z)$ in the above parallelogram of periods, and the constants C_1, ... being subject to the condition

$$\pm i\pi\Sigma C_1=\lambda\omega'-\lambda'\omega.$$

Deduce an expression for a doubly-periodic function $\phi(z)$ of the third kind, by assuming

$$f(z)=\frac{\phi'(z)}{\phi(z)}.$$

(Halphen.)

Ex. 3. If $S(z)$ be a given doubly-periodic function of the first kind, then a pseudo-periodic function $F(z)$, which satisfies the equations

$$F(z+2\omega)=F(z),$$
$$F(z+2\omega')=e^{\frac{n\pi iz}{\omega}}S(z)F(z),$$

where n is an integer, can be expressed in the form

$$F(z)=Ae^{\int^z\left\{\frac{S'(z)}{S(z)}+\frac{n\pi i}{\omega}\right\}\pi(z)\,dz},$$

where A is a constant and $\pi(z)$ denotes

$$\frac{\eta z}{i\pi}+G+\Sigma\left(B_r+C_r\frac{d}{dz}+D_r\frac{d^2}{dz^2}+\ldots\ldots\right)\zeta(z-b_r),$$

the summation extending over all points b_r and the constants B_r being subject to the relation

$$\Sigma B_r=-\frac{\omega}{i\pi}.$$

Explain how the constants b, G and B can be determined. (Picard.)

Ex. 4. Shew that the function $F(z)$ defined by the equation

$$F(z)=\sum_{n=-\infty}^{n=\infty}z^{2^{n+1}}(1-z^{2^n})^2,$$

for values of $|z|$, which are <1, satisfies the equation

$$F(z^2)=F(z);$$

and that the function
$$F_1(x)=\sum_{n=-\infty}^{n=\infty}\frac{\phi_n(x)-a}{\phi_n^2(x)},$$

where $\phi(x)=x^3-1$, and $\phi_n(x)$, for positive and negative values of n, denotes $\phi[\phi\{\phi\ldots\ldots\phi(x)\}]$, ϕ being repeated n times, and a is the positive root of $a^3-a-1=0$; satisfies the equation

$$F_1(x^3-1)=F_1(x)$$

for real values of the variable.

Discuss the convergence of the series which defines the function $F_1(x)$. (Appell.)

CHAPTER XIII.

Functions possessing an Algebraical Addition-Theorem.

149. We may consider at this stage an interesting set* of important theorems, due to Weierstrass, which are a justification, if any be necessary, for the attention ordinarily (and naturally) paid to functions belonging to the three simplest classes of algebraic, simply-periodic and doubly-periodic functions.

A function $\phi(u)$ is said to possess an algebraical addition theorem, when among the three values of the function for arguments u, v, and $u+v$, where u and v are general and not merely special arguments, an algebraical equation exists† having its coefficients independent of u and v.

150. It is easy to see, from one or two examples, that the function does not need to be a uniform function of the argument. The possibility of multiformity is established in the following proposition:

A function defined by an algebraical equation, the coefficients of which are uniform algebraical functions of the argument, or are uniform simply-periodic functions of the argument, or are uniform doubly-periodic functions of the argument, possesses an algebraical addition-theorem.

* They are placed in the forefront of Schwarz's account of Weierstrass's theory of elliptic functions, as contained in the *Formeln und Lehrsätze zum Gebrauche der elliptischen Functionen;* but they are there stated (§§ 1—3) without proof. The only proof that has appeared is in a memoir by Phragmén, *Acta Math.*, t. vii, (1885), pp. 33—42; and there are some statements (pp. 390—393) in Biermann's *Theorie der analytischen Functionen* relative to the theorems. The proof adopted in the text does not coincide with that given by Phragmén.

† There are functions which possess a kind of algebraical addition-theorem; thus, for instance, the Jacobian Theta-functions are such that $\Theta_\lambda(u+v)\,\Theta_\mu(u-v)$ can be rationally expressed in terms of the Theta-functions having u and v for their arguments. Such functions are, however, naturally excluded from the class of functions indicated in the definition.

Such functions, however, possess what may be called a *multiplication-theorem* for multiplication of the argument by an integer, that is, the set of functions $\Theta(mu)$ can be expressed algebraically in terms of the set of functions $\Theta(u)$. This is an extremely special case of a set of transcendental functions having a multiplication-theorem, which are investigated by Poincaré, *Liouville*, 4me Sér., t. iv, (1890), pp. 313—365.

First, let the coefficients be algebraical functions of the argument u. If the function defined by the equation be U, we have

$$U^m g_0(u) + U^{m-1} g_1(u) + \dots + g_m(u) = 0,$$

where $g_0(u)$, $g_1(u)$, ..., $g_m(u)$ are rational integral algebraical functions of u of degree, say, not higher than n. The equation can be transformed into

$$u^n f_0(U) + u^{n-1} f_1(U) + \dots + f_n(U) = 0,$$

where $f_0(U)$, $f_1(U)$, ..., $f_n(U)$ are rational integral algebraical functions of U of degree not higher than m.

If V denote the function when the argument is v, and W denote it when the argument is $u+v$, then

$$v^n f_0(V) + v^{n-1} f_1(V) + \dots + f_n(V) = 0,$$

and $$(u+v)^n f_0(W) + (u+v)^{n-1} f_1(W) + \dots + f_n(W) = 0.$$

The algebraical elimination of the two quantities u and v between these three equations leads to an algebraical equation between the quantities $f(U)$, $f(V)$ and $f(W)$, that is, to an algebraical equation between U, V, W, say of the form

$$G(U, V, W) = 0,$$

where G denotes an algebraical function, with coefficients independent of u and v. It is easy to prove that G is symmetrical in U and V, and that its degree in each of the three quantities U, V, W is mn^2. The equation $G = 0$ implies that the function U possesses an algebraical addition-theorem.

Secondly, let the coefficients* be uniform simply-periodic functions of the argument u. Let ω denote the period: then, by § 113, each of these functions is a rational algebraical function of $\tan\frac{\pi u}{\omega}$. Let u' denote $\tan\frac{\pi u}{\omega}$; then the equation is of the form

$$U^m g_0(u') + U^{m-1} g_1(u') + \dots + g_m(u') = 0,$$

where the coefficients g are rational algebraical (and can be taken as integral) functions of u'. If p be the highest degree of u' in any of them, then the equation can be transformed into

$$u'^p f_0(U) + u'^{p-1} f_1(U) + \dots + f_p(U) = 0,$$

where $f_0(U)$, $f_1(U)$, ..., $f_p(U)$ are rational integral algebraical functions of U of degree not higher than m.

* The limitation to uniformity for the coefficients has been introduced merely to make the illustration simpler; if in any case they were multiform, the equation would be replaced by another which is equivalent to all possible forms of the first arising through the (finite) multiformity of the coefficients: and the new equation would conform to the specified conditions.

Let v' denote $\tan\frac{\pi v}{\omega}$, and w' denote $\tan\frac{\pi(u+v)}{\omega}$; then the corresponding values of the function are determined by the equations

$$v'^p f_0(V) + v'^{p-1} f_1(V) + \ldots + f_p(V) = 0,$$

and

$$w'^p f_0(W) + w'^{p-1} f_1(W) + \ldots + f_p(W) = 0.$$

The relation between u', v', w' is

$$u'v'w' + u' + v' - w' = 0.$$

The elimination of the three quantities u', v', w' among the four equations leads as before to an algebraical equation

$$G(U, V, W) = 0,$$

where G denotes an algebraical function (now of degree mp^2) with coefficients independent of u and v. The function U therefore possesses an algebraical addition-theorem.

Thirdly, let the coefficients be uniform doubly-periodic functions of the argument u. Let ω and ω' be the two periods; and let $\wp(u)$, the Weierstrassian elliptic function in those periods, be denoted by ξ. Then every coefficient can be expressed in the form

$$\frac{M + N\wp'(u)}{L},$$

where L, M, N are rational integral algebraical functions of ξ of finite degree. Unless each of the quantities N is zero, the form of the equation when these values are substituted for the coefficients is

$$A + B\wp'(u) = 0,$$

so that

$$A^2 = B^2(4\xi^3 - g_2\xi - g_3);$$

and this is of the form

$$U^{2m} g_0(\xi) + U^{2m-1} g_1(\xi) + \ldots + g_{2m}(\xi) = 0,$$

where the coefficients g are rational algebraical (and can be taken as integral) functions of ξ. If q be the highest degree of ξ in any of them, the equation can be transformed into

$$\xi^q f_0(U) + \xi^{q-1} f_1(U) + \ldots + f_q(U) = 0,$$

where the coefficients f are rational integral algebraical functions of U of degree not higher than $2m$.

Let η denote $\wp(v)$ and ζ denote $\wp(u+v)$; then the corresponding values of the function are determined by the equations

$$\eta^q f_0(V) + \eta^{q-1} f_1(V) + \ldots\ldots\ldots + f_q(V) = 0,$$

and

$$\zeta^q f_0(W) + \zeta^{q-1} f_0(W) + \ldots\ldots\ldots + f_q(W) = 0.$$

By using Ex. 4, § 131, it is easy to shew that the relation between ξ, η, ζ is

$$16(\xi+\eta+\zeta)^2(\xi-\eta)^2 - 8(\xi+\eta+\zeta)\{4(\xi^3+\eta^3) - g_2(\xi+\eta) - 2g_3\}$$
$$+ (4\xi^2 + 4\xi\eta + 4\eta^2 - g_2)^2 = 0.$$

The elimination of ξ, η, ζ from the three equations leads as before to an algebraical equation

$$G(U, V, W) = 0,$$

of finite degree and with coefficients independent of u and v. Therefore in this case also the function U possesses an algebraical addition-theorem.

If, however, all the quantities N be zero, the equation defining U is of the form

$$U^m h_0(\xi) + U^{m-1} h_1(\xi) + \ldots + h_m(\xi) = 0,$$

and a similar argument then leads to the inference that U possesses an algebraical addition-theorem.

The proposition is thus completely established.

151. The generalised converse of the preceding proposition now suggests itself: what are the classes of functions of one variable that possess an algebraical addition-theorem? The solution is contained in Weierstrass's theorem:—

An analytical function $\phi(u)$, which possesses an algebraical theorem, is either

(i) *an algebraical function of u; or*

(ii) *an algebraical function of $e^{\frac{i\pi u}{\omega}}$, where ω is a suitably chosen constant; or*

(iii) *an algebraical function of the elliptic function $\wp(u)$, the periods—or the invariants g_2 and g_3—being suitably chosen constants.*

Let U denote $\phi(u)$.

For a given general value of u, the function U may have m values where, for functions in general, there is not a necessary limit to the value of m; it will be proved that, when the function possesses an algebraical addition-theorem, the integer m must be finite.

For a given general value of U, that is, a value of U when its argument is not in the immediate vicinity of a branch-point if there be branch-points, the variable u may have p values, where p may be finite or may be infinite.

Similarly for given general values of v and of V, which will be used to denote $\phi(v)$.

First, let p be finite. Then because u has p values for a given value of U and v has p values for a given value of V, and since neither set is affected by the value of the other function, the sum $u+v$ has p^2 values because any member of the set u can be combined with any member of the set v; and this number p^2 of values of $u+v$ is derived for a given value of U and a given value of V.

Now in forming the function $\phi(u+v)$, which will be denoted by W, we have m values of W for each value of $u+v$ and therefore we have mp^2 values of W for the whole set, that is, for a given value of U and a given value of V.

Hence the equation between U, V, W is of degree* mp^2 in W, necessarily finite when the equation is algebraical; and therefore m is finite.

Because m is finite, U has a finite number m of values for a given value of u; and, because p is finite, u has a finite number p of values for a given value of U. Hence U is determined in terms of u by an algebraical equation of degree m, the coefficients of which are rational integral algebraical functions of degree p; and *therefore U is an algebraic function of u.*

152. Next, let p be infinite; then (see *Note*, p. 303) the system of values may be composed of (i) a single simply-infinite series of values or (ii) a finite number of simply-infinite series of values or (iii) a simply-infinite number of simply-infinite series of values, say, a single doubly-infinite series of values or (iv) a finite number of doubly-infinite series of values or (v) an infinite number of doubly-infinite series of values where, in (v), the infinite number is not restricted to be simply-infinite.

Taking these alternatives in order, we first consider the case where *the p values of u for a given general value of U constitute a single simply-infinite series.* They may be denoted by $f(u, n)$, where n has a simply-infinite series of values and the form of f is such that $f(u, 0) = u$.

Similarly, the p values of v for a given general value of V may be denoted by $f(v, n')$, where n' has a simply-infinite series of values. Then the different values of the argument for the function W are the set of values given by

$$f(u, n) + f(v, n'),$$

for the simply-infinite series of values for n and the similar series of values for n'.

The values thus obtained as arguments of W must all be contained in the series $f(u+v, n'')$, where n'' has a simply-infinite series of values; and, in the present case, $f(u+v, n'')$ cannot contain other values. Hence for some values of n and some values of n', the total aggregate being not finite, the equation

$$f(u, n) + f(v, n') = f(u+v, n'')$$

must hold, for continuously varying values of u and v.

In the first place, an interchange of u and v is equivalent to an interchange of n and n' on the left-hand side; hence n'' is symmetrical in n and n'. Again, we have

$$\frac{\partial f(u, n)}{\partial u} = \frac{\partial f(u+v, n'')}{\partial (u+v)}$$

$$= \frac{\partial f(v, n')}{\partial v},$$

* The degree for special functions may be reduced, as in Cor. 1, Prop. XIII, § 118; but in no case is it increased. Similarly modifications, in the way of finite reductions, may occur in the succeeding cases; but they will not be noticed, as they do not give rise to essential modification in the reasoning.

so that the form of $f(u, n)$ is such that its first derivative with regard to u is independent of u. Let $\theta(n)$ be this value, where $\theta(n)$, independent of u, may be dependent on n; then, since

$$\frac{\partial f(u, n)}{\partial u} = \theta(n),$$

we have
$$f(u, n) = u\theta(n) + \psi(n),$$

$\psi(n)$ being independent of u. Substituting this expression in the former equation, we have the equation

$$u\theta(n) + \psi(n) + v\theta(n') + \psi(n') = (u + v)\,\theta(n'') + \psi(n''),$$

which must be true for all values of u and v; hence

$$\theta(n) = \theta(n'') = \theta(n'),$$

so that $\theta(n)$ is a constant and equal to its value when $n = 0$. But when n is zero, $f(u, 0)$ is u; so that $\theta(0) = 1$ and $\psi(0) = 0$, and therefore

$$f(u, n) = u + \psi(n),$$

where ψ vanishes with n.

The equation defining ψ is

$$\psi(n) + \psi(n') = \psi(n'');$$

for values of n from a singly-infinite series and for values of n' from the same series, that series is reproduced for n''. Since $\psi(n)$ vanishes with n, we take

$$\psi(n) = n\chi(n),$$

and therefore
$$n\chi(n) + n'\chi(n') = n''\chi(n'').$$

Again, when n' vanishes, the required series of values of n'' is given by taking $n'' = n$; and, when n' does not vanish, n'' is symmetrical in n and n', so that we have

$$n'' = n + n' + nn'\lambda,$$

where λ is not infinite for zero or finite values of n or n'. Thus

$$n\chi(n) + n'\chi(n') = (n + n' + nn'\lambda)\,\chi(n + n' + nn'\lambda).$$

Since the left-hand side is the sum of two functions of distinct and independent magnitudes, the form of the equation shews that it can be satisfied only if

$$\lambda = 0, \text{ so that } n'' = n + n';$$

and
$$\chi(n) = \chi(n'') = \chi(n'),$$

so that each is a constant, say ω; then

$$f(u, n) = u + n\omega,$$

which is the form that the series must adopt when the series $f(u + v, n'')$ is obtained by the addition of $f(u, n)$ and $f(v, n')$.

It follows at once that the single series of arguments for W is obtained, as one simply-infinite series, of the form $u+v+n''\omega$. For each of these arguments we have m values of W, and the set of m values of W is the same for all the different arguments; that is, W has m values for a given value of U and a given value of V. Moreover, U has m values for each argument and likewise V; hence, as the equation between U, V, W is of a degree that is necessarily finite because the equation is algebraical, the integer m is finite.

It thus appears that the function U has a finite number m of values for each value of the argument u, and that for a given value of the function the values of the argument form a simply-periodic series represented by $u+n\omega$. But the function $\tan\left(\frac{\pi u}{\omega}\right)$ is such that, for a given value, the values of the argument are represented by the series $u+n\omega$; hence for each value of $\tan\left(\frac{\pi u}{\omega}\right)$ there are m values of U and for each value of U there is one value of $\tan\frac{\pi u}{\omega}$. It therefore follows, by §§ 113, 114, that between U and $\tan\left(\frac{\pi u}{\omega}\right)$ there is an algebraical relation which is of the first degree in $\tan\frac{\pi u}{\omega}$ and the mth degree in U, that is, U is an algebraic function of $\tan\frac{\pi u}{\omega}$. Hence U is an algebraic function also of $e^{\frac{i\pi u}{\omega}}$.

Note. This result is based upon the supposition that the series of arguments, for which a branch of the function has the same value, can be arranged in the form $f(u, n)$, where n has a simply-infinite series of integral values. If, however, there were no possible law of this kind—the foregoing proof shews that, if there be one such law, there is only one such law, with a properly determined constant ω—then the values would be represented by $u_1, u_2, \ldots, u_p$ with p infinite in the limit. In that case, there would be an infinite number of sets of values for $u+v$ of the type $u_\lambda + v_\mu$, where λ and μ might be the same or might be different; each set would give a branch of the function W and then there would be an infinite number of values of W corresponding to one branch of U and one branch of V. The equation between U, V and W would be of infinite degree in W, that is, it would be transcendental and not algebraical. The case is excluded by the hypothesis that the addition-theorem is algebraical, and therefore the equation between U, V and W is algebraical.

153. Next, let there be a number of simply-infinite series of values of the argument of the function, say q, where q is greater than unity and may be either finite or infinite. Let $u_1, u_2, \ldots, u_q$ denote typical members of each series.

Then all the members of the series containing u_1 must be of the form

$f_1(u_1, n)$, for an infinite series of values of the integer n. Otherwise, as in the preceding note, the sum of the values in the series of arguments u and of those in the same series of arguments v would lead to an infinite number of distinct series of values of the argument $u+v$, with a corresponding infinite number of values W; and the relation between U, V, W would cease to be algebraical.

In the same way, the members of the corresponding series containing v_1 must be of the form $f_1(v_1, n')$ for an infinite series of values of the integer n'. Among the combinations

$$f_1(u_1, n) + f_1(v_1, n')$$

the simply-infinite series $f_1(u_1+v_1, n'')$ must occur for an infinite series of values of n''; and therefore, as in the preceding case,

$$f_1(u_1, n) = u_1 + n\omega_1,$$

where ω_1 is an appropriate constant. Further, there is only one series of values for the combination of these two series; it is represented by

$$u_1 + v_1 + n''\omega_1.$$

In the same way, the members of the series containing u_2 can be represented in the form $u_2 + n\omega_2$, where ω_2 is an appropriate constant, which may be (but is not necessarily) the same as ω_1; and the series containing u_2, when combined with the set containing v_2, leads to only a single series represented in the form $u_2 + v_2 + n''\omega_2$. And so on, for all the series in order.

But now since $u_2 + m_2\omega_2$, where m_2 is an integer, is a value of u for a given value of U, it follows that $U(u_2 + m_2\omega_2) = U(u_2)$ identically, each being equal to U. Hence

$$U(u_1 + m_1\omega_1 + m_2\omega_2) = U(u_1 + m_1\omega_1) = U(u_1) = U,$$

and therefore $u_1 + m_1\omega_1 + m_2\omega_2$ is also a value of u for the given value of U, leading to a series of arguments which must be included among the original series or be distributed through them. Similarly $u_1 + \Sigma m_r\omega_r$, where the coefficients m are integers and the constants ω are properly determined, represents a series of values of the variable u, included among the original series or distributed through them. And generally, when account is taken of all the distinct series thus obtained, the aggregate of values of the variable u can be represented in the form $u_\lambda + \Sigma m_r\omega_r$, for $\lambda = 1, 2, \ldots, \kappa$, where κ is some finite or infinite integer.

Three cases arise, (a) when the quantities ω are equal to one another or can be expressed as integral multiples of only one quantity ω, (b) when the quantities ω are equivalent to two quantities Ω_1 and Ω_2 (the ratio of which is not real), so that each quantity ω can be expressed in the form

$$\omega_r = p_{1r}\Omega_1 + p_{2r}\Omega_2,$$

the coefficients p_{1r}, p_{2r} being finite integers; (c) when the quantities ω are not equivalent to only two quantities, such as Ω_1 and Ω_2.

For case (a), each of the κ infinite series of values u can be expressed in the form $u_\lambda + p\omega$, for $\lambda = 1, 2, \ldots, \kappa$ and integral values of p.

First, let κ be finite, so that the original integer q is finite. Then the values of the argument for W are of the type

$$u_\lambda + p\omega + v_\mu + p'\omega,$$

that is,

$$u_\lambda + v_\mu + p''\omega,$$

for all combinations of λ and μ and for integral values of p''. There are thus κ^2 series of values, each series containing a simply-infinite number of terms of this type.

For each of the arguments in any one of these infinite series, W has m values; and the set of m values is the same for all the arguments in one and the same infinite series. Hence W has $m\kappa^2$ values for all the arguments in all the series taken together, that is, for a given value of U and a given value of V. The relation between U, V, W is therefore of degree $m\kappa^2$, necessarily finite when the equation is algebraical; hence m is finite.

It thus appears that the function U has a finite number m of values for each value of the argument u, and that for a given value of the function there are a finite number κ of distinct series of values of the argument of the form $u + p\omega$, ω being the same for all the series. But the function $\tan \frac{\pi u}{\omega}$ has one value for each value of u and the series $u+p\omega$ represents the series of values of u for a given value of $\tan \frac{\pi u}{\omega}$. It therefore follows that there are m values of U for each value of $\tan \frac{\pi u}{\omega}$ and that there are κ values of $\tan \frac{\pi u}{\omega}$ for each value of U; and therefore there is an algebraical relation between U and $\tan \frac{\pi u}{\omega}$, which is of degree κ in the latter and of degree m in the former. Hence *U is an algebraic function of* $\tan \frac{\pi u}{\omega}$ *and therefore also of* $e^{\frac{i\pi u}{\omega}}$.

Next, let κ be infinite, so that the original integer q is infinite. Then, as in the Note in § 152, the equation between U, V, W will cease to be algebraical unless each aggregate of values $u_\lambda + p\omega$, for each particular value of p and for the infinite sequence $\lambda = 1, 2, \ldots, \kappa$, can be arranged in a system or a set of systems, say σ in number, each of the form $f_\rho(u + p\omega, p_\rho)$ for an infinite series of values of p_ρ. Each of these implies a series of values $f_\rho(v + p'\omega, p_\rho')$ of the argument of V for the same series of values of p_ρ' as of p_ρ, and also a series of values $f_\rho(u + v + p''\omega, p_\rho'')$ of the argument of W for the same series of values of p_ρ''. By proceeding as in § 152, it follows that

$$f_\rho(u + p\omega, p_\rho) = u + p\omega + p_\rho\omega_\rho',$$

where ω_ρ' is an appropriate constant, the ratio of which to ω can be proved (as in § 106) to be not purely real, and p_ρ has a simply-infinite succession of values. The integer σ may be finite or it may be infinite.

When ω and all the constants ω' which thus arise are linearly equivalent to two quantities Ω_1 and Ω_2, so that the terms additive to u can be expressed in the form $s_1\Omega_1 + s_2\Omega_2$, then the aggregate of values u can be expressed in the form

$$u_\rho + p_1\Omega_1 + p_2\Omega_2,$$

for a simply-infinite series for p_1 and for p_2; and ρ has a series of values $1, 2, \ldots, \sigma$. This case is, in effect, the same as case (b).

When ω and all the constants ω' are not linearly equivalent to only two quantities, such as Ω_1 and Ω_2, we have a case which, in effect, is the same as case (c).

These two cases must therefore now be considered.

For case (b), either as originally obtained or as derived through part of case (a), each of the (doubly) infinite series of values of u can be expressed in the form

$$u_\lambda + p_1\Omega_1 + p_2\Omega_2,$$

for $\lambda = 1, 2, \ldots, \sigma$ and for integral values of p_1 and p_2. The integer σ may be finite or infinite; the original integer q is infinite.

First, let σ be finite. Then the values of the argument for W are of the type

$$u_\lambda + p_1\Omega_1 + p_2\Omega_2 + v_\mu + p_1'\Omega_1 + p_2'\Omega_2,$$

that is,

$$u_\lambda + v_\mu + p_1''\Omega_1 + p_2''\Omega_2,$$

for all combinations of λ and μ and for integral values of p_1'' and p_2''. There are thus σ^2 series of values, each series containing a doubly-infinite number of terms of this type.

For every argument there are m values of W; and the set of m values is the same for all the arguments in one and the same infinite series. Thus W has $m\sigma^2$ values for all the arguments in all the series, that is, for a given value of U and a given value of V; and it follows, as before, from the consideration of the algebraical relation, that m is finite.

The function U thus has m values for each value of the argument u; and for a given value of the function there are σ series of values of the argument, each series being of the form $u_\lambda + p_1\Omega_1 + p_2\Omega_2$.

Take a doubly-periodic function Θ having Ω_1 and Ω_2 for its periods, such* that for a given value of Θ the values of its arguments are of the foregoing form. Whatever be the expression of the function, it is of the order σ. Then U has m values for each value of Θ, and Θ has one value for each value of U; hence there is an algebraical equation between U and Θ, of

* All that is necessary for this purpose is to construct, by the use of Prop. XII, § 118, a function having, as its irreducible simple infinities, a series of points $a_1, a_2, \ldots, a_\sigma$—special values of $u_1, u_2, \ldots, u_\sigma$—in the parallelogram of periods, chosen so that no two of the σ points a coincide.

the first degree in the latter and of the mth degree in U: that is, U is an algebraical function of Θ. But, by Prop. XV. § 119, Θ can be expressed in the form

$$\frac{M + N\wp'(u)}{L},$$

where L, M, N are rational integral algebraical functions of $\wp(u)$, if Ω_1 and Ω_2 be the periods of $\wp(u)$; and $\wp'(u)$ is a two-valued algebraical function of $\wp(u)$, so that Θ is an algebraical function of $\wp(u)$. *Hence also U is an algebraical function of $\wp(u)$, the periods of $\wp(u)$ being properly chosen.*

This inference requires that σ, the order of Θ, be greater than 1. Because U has m values for an argument u, the symmetric function ΣU has one value for an argument u and it is therefore a uniform function. But each term of the sum has the same value for $u + p_1\Omega_1 + p_2\Omega_2$ as for u; and therefore this uniform function is doubly-periodic. The number of independent doubly-infinite series of values of u for a uniform doubly-periodic function is at least two: and therefore there must be at least two doubly-infinite series of values of u, so that $\sigma > 1$. Hence a function, that possesses an addition-theorem, cannot have only one doubly-infinite series of values for its argument.

If σ be infinite, there is an infinite series of values of u of the form $u_\lambda + p_1\Omega_1 + p_2\Omega_2$; an argument, similar to that in case (a), shews that this is, in effect, the same as case (c).

It is obvious that cases (ii), (iii) and (iv) of § 152 are now completely covered; case (v) of § 152 is covered by case (c) now to be discussed in § 154.

154. For case (c), we have the series of values u represented by a number of series of the form

$$u_\lambda + \sum_{r=1} m_r\omega_r,$$

where the quantities ω are not linearly equivalent to two quantities Ω_1 and Ω_2. The original integer q is infinite.

Then, by §§ 108, 110, it follows that integers m can be chosen in an unlimited variety of ways so that the modulus of

$$\sum_{r=1} m_r\omega_r$$

is infinitesimal, and therefore in the immediate vicinity of any point u_λ there is an infinitude of points at which the function resumes its value. Such a function would, as in previous instances, degenerate into a mere constant; and therefore the combination of values which gives rise to this case does not occur.

All the possible cases have been considered: and the truth of Weierstrass's theorem* that a function, which has an algebraical addition-theorem, is either an algebraical function of u, or of $e^{\frac{i\pi u}{\omega}}$ (where ω is suitably chosen), or of $\wp(u)$, where the periods of $\wp(u)$ are suitably chosen, is established; and it has incidentally been established—it is, indeed, essential to the derivation of the theorem—that *a function, which has an algebraical addition-theorem, has only a finite number of values for a given argument.*

It is easy to see that *the first derivative has only a finite number of values for a given argument;* for the elimination of U between the algebraical equations

$$G(U, u) = 0, \quad \frac{\partial G}{\partial U} U' + \frac{\partial G}{\partial u} = 0,$$

leads to an equation in U' of the same finite degree as G in U.

Further, it is now easy to see that *if the analytical function $\phi(u)$, which possesses an algebraical addition-theorem, be uniform, then it is a rational function either of u, or of $e^{\frac{i\pi u}{\omega}}$, or of $\wp(u)$ and $\wp'(u)$;* and that any uniform function, which is transcendental in the sense of § 47 and which possesses an algebraical addition-theorem, is either a simply-periodic function or a doubly-periodic function.

The following examples will illustrate some of the inferences in regard to the number of values of $\phi(u+v)$ arising from series of values for u and v.

Ex. 1. Let
$$U = u^{\frac{1}{2}} + (2u+1)^{\frac{1}{2}}.$$

Evidently m, the number of values of U for a value of u, is 4; and, as the rationalised form of the equation is

$$u^2 + 2u(1 - 3U^2) + (U^2 - 1)^2 = 0,$$

the value of p, being the number of values of u for a given value of U, is 2. Thus the equation in W should be, by § 151, of degree $(4 \,.\, 2^2 =)\, 16$.

This equation is
$$\prod_{r=1}^{8} \{3(W^2 - U^2 - V^2) + 1 - 2k_r\} = 0,$$

where k_r is any one of the eight values of

$$W(2W^2-1)^{\frac{1}{2}} + U(2U^2-1)^{\frac{1}{2}} + V(2V^2-1)^{\frac{1}{2}};$$

an equation, when rationalised, of the 16th degree in W.

Ex. 2. Let $U = \cos u$.

Evidently $m=1$; the values of u for a given value of U are contained in the double series $u+2\pi n$, $-u+2\pi n$, for all values of n from $-\infty$ to $+\infty$. The values of $u+v$ are

$u+2\pi n+v+2\pi m$, that is, $u+v+2\pi p$; $-u+2\pi n+v+2\pi m$, that is, $-u+v+2\pi p$;

$u+2\pi n-v+2\pi m$, that is, $u-v+2\pi p$; $-u+2\pi n-v+2\pi m$, that is, $-u-v+2\pi p$,

* The theorem has been used by Schwarz, *Ges. Werke*, t. ii, pp. 260—268, in determining all the families of plane isothermic curves which are algebraical curves, an 'isothermic' curve being of the form $u=c$, where u is a function satisfying the potential-equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

so that the number of series of values of $u+v$ is four, each series being simply-infinite. It might thus be expected that the equation between U, V, W would be of degree $(1.4=)\,4$ in W; but it happens that

$$\cos(u+v)=\cos(-u-v),$$

and so the degree of the equation in W is reduced to half its degree. The equation is

$$W^2-2WUV+U^2+V^2-1=0.$$

Ex. 3. Let $U=\operatorname{sn} u$.

Evidently $m=1$; and there are two doubly-infinite series of values of u determined by a given value of U, having the form $u+2m\omega+2m'\omega'$, $\omega-u+2m\omega+2m'\omega'$. Hence the values of $u+v$ are

$$\equiv \quad u+v\,(\text{mod. } 2\omega,\ 2\omega')\,;\ \equiv\omega-u+v\,(\text{mod. } 2\omega,\ 2\omega')\,;$$

$$\equiv\omega+u-v\,(\text{mod. } 2\omega,\ 2\omega')\,;\ \equiv\ -u-v\,(\text{mod. } 2\omega,\ 2\omega')\,;$$

four in number. The equation may therefore be expected to be of the fourth degree in W; it is

$$4(1-U^2)(1-V^2)(1-W^2)=(2-U^2-V^2-W^2+k^2U^2V^2W^2)^2.$$

155. But it must not be supposed that any algebraical equation between U, V, W, which is symmetrical in U and V, is one necessarily implying the representation of an algebraical addition-theorem. Without entering into a detailed investigation of the formal characteristics of the equations that are suitable, a latent test is given by implication in the following theorem, also due to Weierstrass:—

If an analytical function possess an algebraical addition-theorem, an algebraical equation involving the function and its first derivative with regard to its argument exists; and the coefficients in this equation do not involve the argument of the function.

The proposition might easily be derived by assuming the preceding proposition, and applying the known results relating to the algebraical dependence between those functions, the types of which are suited to the representation of the functions in question, and their derivatives; we shall, however, proceed more directly from the equation expressing the algebraical addition-theorem in the form

$$G(U,\ V,\ W)=0,$$

which may be regarded as a rationally irreducible equation.

Differentiating with regard to u, we have

$$\frac{\partial G}{\partial U}U'+\frac{\partial G}{\partial W}W'=0,$$

and similarly, with regard to v, we have

$$\frac{\partial G}{\partial V}V'+\frac{\partial G}{\partial W}W'=0,$$

from which it follows that

$$\frac{\partial G}{\partial U}U'-\frac{\partial G}{\partial V}V'=0.$$

This equation* will, in general, involve W; in order to obtain an equation free from W, we eliminate W between

$$G=0 \text{ and } \frac{\partial G}{\partial U}U'=\frac{\partial G}{\partial V}V',$$

the elimination being possible because both equations are of finite degree; and thus in any case we have an algebraical equation independent of W and involving U, U', V, V'.

Not more than one equation can arise by assigning various values to v, a quantity that is independent of u; for we should have either inconsistent equations or simultaneous equations which, being consistent, determine a limited number of values of U and U' for all values of u, that is, only a number of constants. Hence there can be only one equation, obtained by assigning varying values to v; and this single equation is the algebraical equation between the function and its first derivative, the coefficients being independent of the argument of the function.

Note. A test of suitability of an algebraical equation $G=0$ between three variables U, V, W to represent an addition-theorem is given by the condition that the elimination of W between

$$G=0 \text{ and } U'\frac{\partial G}{\partial U}=V'\frac{\partial G}{\partial V}$$

leads to only a single equation between U and U' for different values of V and V'.

Ex. Consider the equation

$$(2-U-V-W)^2-4(1-U)(1-V)(1-W)=0.$$

The deduced equation involving U' and V' is

$$(2VW-V-W+U)U'=(2UW-U-W+V)V',$$

so that

$$W=\frac{(V-U)(V'+U')}{(2V-1)U'-(2U-1)V'}.$$

The elimination of W is simple. We have

$$1-W=\frac{(V+U-1)(U'-V')}{(2V-1)U'-(2U-1)V'},$$

and

$$2-U-V-W=2\frac{(V+U-1)\{(1-V)U'-(1-U)V'\}}{(2V-1)U'-(2U-1)V'}.$$

Neglecting $4(V+U-1)=0$, which is an irrelevant equation, and multiplying by $(2V-1)U'-(2U-1)V'$, which is not zero unless the numerator also vanish, and this would make both U' and V' zero, we have

$$(V+U-1)\{(1-V)U'-(1-U)V'\}^2=(1-U)(1-V)(U'-V')\{2V-1)U'-(2U-1)V'\},$$

and therefore

$$V(U-V)(1-V)U'^2+U(V-U)(1-U)V'^2=0.$$

* It is permissible to adopt any subsidiary irrational or non-algebraical form as the equivalent of $G=0$, provided no special limitation to the subsidiary form be implicitly adopted. Thus, if W can be expressed explicitly in terms of U and V, this resoluble (but irrational) equivalent of the equation often leads rapidly to the equation between U and its derivative.

When the irrelevant factor $U-V$ is neglected, this equation gives

$$\frac{U'^2}{U(1-U)}=\frac{V'^2}{V(1-V)},$$

the equation required: and this, indeed, is the necessary form in which the equation involving U and U' arises in general, the variables being combined in associate pairs. Each side is evidently a constant, say $4a^2$; and then we have

$$U'^2=4a^2U(1-U).$$

Then the value of U is $\sin^2(au+\beta)$, the arbitrary additive constant of integration being β; by substitution in the original equation, β is easily proved to be zero.

156. Again, if the elimination between

$$G=0 \text{ and } \frac{\partial G}{\partial U}U'=\frac{\partial G}{\partial V}V'$$

be supposed to be performed by the ordinary algebraical process for finding the greatest common measure of G and $U'\frac{\partial G}{\partial U}-V'\frac{\partial G}{\partial V}$, regarded as functions of W, the final remainder is the eliminant which, equated to zero, is the differential equation involving U, U', V, V'; and the greatest common measure, equated to zero, gives the simplest equation in virtue of which the equations $G=0$ and $\frac{\partial G}{\partial U}U'=\frac{\partial G}{\partial V}V'$ subsist. It will be of the form

$$f(W, U, V, U', V')=0.$$

If the function have only one value for each value of the argument, so that it is a uniform function, this last equation can give only one value for W; for all the other magnitudes that occur in the equation are uniform functions of their respective arguments. Since it is linear in W, the equation can be expressed in the form

$$W=R(U, V, U', V'),$$

where R denotes a rational function. Hence*:—

A uniform analytical function $\phi(u)$, which possesses an algebraical addition-theorem, is such that $\phi(u+v)$ can be expressed rationally in terms of $\phi(u)$, $\phi'(u)$, $\phi(v)$ and $\phi'(v)$.

It need hardly be pointed out that this result is not inconsistent with the fact that the algebraical equation between $\phi(u+v)$, $\phi(u)$ and $\phi(v)$ does not, in general, express $\phi(u+v)$ as a rational function of $\phi(u)$ and $\phi(v)$. And it should be noticed that the rationality of the expression of $\phi(u+v)$ in terms of $\phi(u)$, $\phi(v)$, $\phi'(u)$, $\phi'(v)$ is characteristic of functions with an algebraical addition-theorem. Instances do occur of functions such that $\phi(u+v)$ can be expressed, not rationally, in terms of $\phi(u)$, $\phi(v)$, $\phi'(u)$, $\phi'(v)$; they do not possess an algebraical addition-theorem. Such an instance is furnished by $\zeta(u)$; the expression of $\zeta(u+v)$, given in Ex. 3 of § 131, can be modified so as to have the form indicated.

* The theorem is due to Weierstrass; see Schwarz, § 2, (l.c. in note to p. 297).

CHAPTER XIV.

Connection of Surfaces.

157. In proceeding to the discussion of multiform functions, it was stated (§ 100) that there are two methods of special importance, one of which is the development of Cauchy's general theory of functions of complex variables and the other of which is due to Riemann. The former has been explained in the immediately preceding chapters; we now pass to the consideration of Riemann's method. But, before actually entering upon it, there are some preliminary propositions on the connection of surfaces which must be established; as they do not find a place in treatises on geometry, an outline will be given here but only to that elementary extent which is necessary for our present purpose.

In the integration of meromorphic functions, it proved to be convenient to exclude the poles from the range of variation of the variable by means of infinitesimal closed simple curves, each of which was thereby constituted a limit of the region: the full boundary of the region was composed of the aggregate of these non-intersecting curves.

Similarly, in dealing with some special cases of multiform functions, it proved convenient to exclude the branch-points by means of infinitesimal curves or by loops. And, in the case of the fundamental lemma of § 16, the region over which integration extended was considered as one which possibly had several distinct curves as its complete boundary.

These are special examples of a general class of regions, at all points within the area of which the functions considered are monogenic, finite, and continuous and, as the case may be, uniform or multiform. But, important as are the classes of functions which have been considered, it is necessary to consider wider classes of multiform functions and to obtain the regions which are appropriate for the representation of the variation of the variable in each case. The most conspicuous examples of such new functions are the algebraic functions, adverted to in §§ 94—99; and it is chiefly in view of their value and of the value of functions dependent upon them, as well as of the kind of surface on which their variable can be simply represented, that we now proceed to establish some of the topological properties of surfaces in general.

158. A surface is said to be *connected* when, from any point of it to any other point of it, a continuous line can be drawn without passing out of the surface. Thus the surface of a circle, that of a plane ring such as arises in Lambert's Theorem, that of a sphere, that of an anchor-ring, are connected surfaces. Two non-intersecting spheres, not joined or bound together in any manner, are not a connected surface but are two different connected surfaces. It is often necessary to consider surfaces, which are constituted by an aggregate of several sheets; but, in order that the surface may be regarded as connected, there must be junctions between the sheets.

One of the simplest connected surfaces is such a plane area as is enclosed and completely bounded by the circumference of a circle. All lines drawn in it from one internal point to another can be deformed into one another; any simple closed line lying entirely within it can be deformed so as to be evanescent, without in either case passing over the circumference; and any simple line from one point of the circumference to another, when regarded as an impassable barrier, divides the surface into two portions. Such a surface is called* *simply connected.*

The kind of connected surface next in point of simplicity is such a plane area as is enclosed between and is completely bounded by the circumferences of two concentric circles. All lines in the surface from one point to another cannot necessarily be deformed into one another, e.g., the lines z_0az and z_0bz; a simple closed line cannot necessarily be deformed so as to be evanescent without crossing the boundary, e.g., the line az_0bza; and a simple line from a point in one part of the boundary to a point in another and different part of the boundary, such as a line AB, does not divide the surface into two portions but, set as an impassable barrier, it makes the surface simply connected.

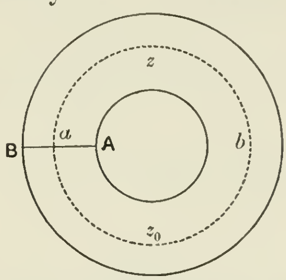

Fig. 35.

Again, on the surface of an anchor-ring, a closed line can be drawn in two essentially distinct ways, abc, $ab'c'$, such that neither can be deformed so as to be evanescent or so as to pass continuously into the other. If abc be made the only impassable barrier, a line such as $\alpha\beta\gamma$ cannot be deformed so as to be evanescent; if $ab'c'$ be made the only impassable barrier, the same holds of a line such as $\alpha\beta'\gamma'$. In order to make the surface simply connected, two impassable barriers, such as abc and $ab'c'$, must be set.

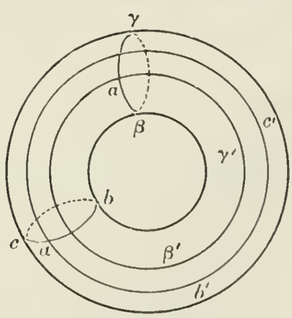

Fig. 36.

Surfaces, like the flat ring or the anchor-

* Sometimes the term *monadelphic* is used. The German equivalent is *einfach zusammenhängend.*

ring, are called* *multiply connected*; the establishment of barriers has made it possible, in each case, to modify the surface into one which is simply connected.

159. It proves to be convenient to arrange surfaces in classes according to the character of their connection; and these few illustrations suggest that the classification may be made to depend, either upon the resolution of the surface, by the establishment of barriers, into one that is simply connected, or upon the number of what may be called independent irreducible circuits. The former mode—that of dependence upon the establishment of barriers—will be adopted, thus following Riemann†; but whichever of the two modes be adopted (and they are not necessarily the only modes) subsequent demands require that the two be brought into relation with one another.

The most effective way of securing the impassability of a barrier is to suppose the surface actually cut along the line of the barrier. Such a section of a surface is either a cross-cut or a loop-cut.

If the section be made through the interior of the surface from one point

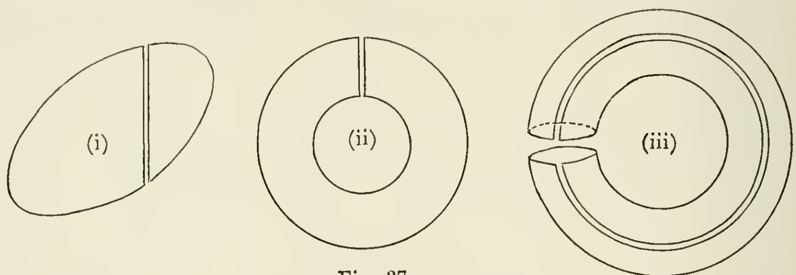

Fig. 37.

of the boundary to another point of the boundary, without intersecting itself or meeting the boundary save at its extremities, it is called a *cross-cut*‡. Every part of it, as it is made, is to be regarded as boundary during the formation of the remainder; and any cross-cut, once made, is to be regarded as boundary during the formation of any cross-cut subsequently made. Illustrations are given in Fig. 37.

The definition and explanation imply that the surface has a boundary. Some surfaces, such as a complete sphere and a complete anchor-ring, do not possess a boundary; but, as will be seen later (§§ 163, 168) from the discussion of the evanescence of circuits, it is desirable to assign some boundary in order to avoid merely artificial difficulties as to the numerical

* Sometimes the term *polyadelphie* is used. The German equivalent is *mehrfach zusammenhängend*.

† "Grundlagen für eine allgemeine Theorie der Functionen einer veränderlichen complexen Grösse," Riemann's *Gesammelte Werke*, pp. 9—12; "Theorie der Abel'schen Functionen," ib., pp. 84—89. When reference to either of these memoirs is made, it will be by a citation of the page or pages in the volume of Riemann's Collected Works.

‡ This is the equivalent used for the German word *Querschnitt;* French writers use *Section*, and Italian writers use *Trasversale* or *Taglio trasversale*.

expression of the connection. This assignment usually is made by taking for the boundary of a surface, which otherwise has no boundary, an infinitesimal closed curve, practically a point; thus in the figure of the anchor-ring (Fig. 36) the point a is taken as a boundary, and each of the two cross-cuts begins and ends in a.

If the section be made through the interior of the surface from a point not on the boundary and, without meeting the boundary or crossing itself, return to the initial point, (so that it has the form of a simple curve lying

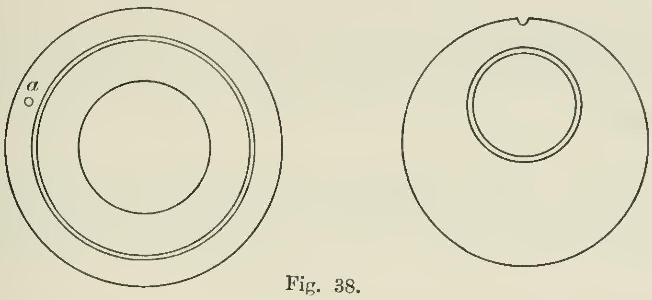

Fig. 38.

entirely in the surface), it is called* a *loop-cut.* Thus a piece can be cut out of a bounded spherical surface by a loop-cut (Fig. 38); but it does not necessarily give a separate piece when made in the surface of an anchor-ring.

It is evident that both a cross-cut and a loop-cut furnish a double boundary-edge to the whole aggregate of surface, whether consisting of two pieces or of only one piece after the section.

Moreover, these sections represent the impassable barriers of the preliminary explanations; and no specified form was assigned to those barriers. It is thus possible, within certain limits, to deform a cross-cut or a loop-cut continuously into a closely contiguous and equivalent position. If, for instance, two barriers initially coincide over any finite length, one or other can be slightly deformed so that finally they intersect only in a point; the same modification can therefore be made in the sections.

The definitions of simple connection and of multiple connection will now† be as follows:—

A surface is simply connected, if it be resolved into two distinct pieces by every cross-cut; but if there be any cross-cut, which does not resolve it into distinct pieces, the surface is multiply connected.

160. Some fundamental propositions, relating to the connection of surfaces, may now be derived.

* This is the equivalent used for the German word *Rückkehrschnitt;* French writers use the word *Rétrosection.*

† Other definitions will be required, if the classification of surfaces be made to depend on methods other than resolution by sections.

I. *Each of the two distinct pieces, into which a simply connected surface S is resolved by a cross-cut, is itself simply connected.*

If either of the pieces, made by a cross-cut ab, be not simply connected, then some cross-cut cd must be possible which will not resolve that piece into distinct portions.

If neither c nor d lie on ab, then the obliteration of the cut ab will restore the original surface S, which now is not resolved by the cut cd into distinct pieces.

If one of the extremities of cd, say c, lie on ab, then the obliteration of the portion cb will change the two pieces into a single piece which is the original surface S; and S now has a cross-cut acd, which does not resolve it into distinct pieces.

If both the extremities lie on ab, then the obliteration of that part of ab which lies between c and d will change the two pieces into one; this is the original surface S, now with a cross-cut $acdb$, which does not resolve it into distinct pieces.

These are all the possible cases should either of the distinct pieces of S not be simply connected; each of them leads to a contradiction of the simple connection of S; therefore the hypothesis on which each is based is untenable, that is, the distinct pieces of S in all the cases are simply connected.

COROLLARY 1. *A simply connected surface is resolved by n cross-cuts into $n+1$ distinct pieces, each simply connected; and an aggregate of m simply connected surfaces is resolved by n cross-cuts into $n+m$ distinct pieces each simply connected.*

COROLLARY 2. *A surface that is resolved into two distinct simply connected pieces by a cross-cut is simply connected before the resolution.*

COROLLARY 3. *If a multiply connected surface be resolved into two different pieces by a cross-cut, both of these pieces cannot be simply connected.*

We now come to a theorem* of great importance:—

II. *If a resolution of a surface by m cross-cuts into n distinct simply connected pieces be possible, and also a different resolution of the same surface by μ cross-cuts into ν distinct simply connected pieces, then $m-n=\mu-\nu$.*

Let the aggregate of the n pieces be denoted by S and the aggregate of the ν pieces by Σ: and consider the effect on the original surface of a united system of $m+\mu$ simultaneous cross-cuts made up of the two systems of the m and of the μ cross-cuts respectively. The operation of this system can be carried out in two ways: (i) by effecting the system of μ cross-cuts on S and

* The following proof of this proposition is substantially due to Neumann, p. 157. Another proof is given by Riemann, pp. 10, 11, and is amplified by Durège, *Elemente der Theorie der Functionen*, pp. 183—190; and another by Lippich, see Durège, pp. 190—197.

(ii) by effecting the system of m cross-cuts on Σ: with the same result on the original surface.

After the explanation of § 159, we may justifiably assume that the lines of the two systems of cross-cuts meet only in points, if at all: let δ be the number of points of intersection of these lines. Whenever the direction of a cross-cut meets a boundary line, the cross-cut terminates; and if the direction continue beyond that boundary line, that produced part must be regarded as a new cross-cut.

Hence the new system of μ cross-cuts applied to S is effectively equivalent to $\mu + \delta$ new cross-cuts. Before these cuts were made, S was composed of n simply connected pieces; hence, after they are applied, the new arrangement of the original surface is made up of $n + (\mu + \delta)$ simply connected pieces.

Similarly, the new system of m cross-cuts applied to Σ will give an arrangement of the original surface made up of $\nu + (m + \delta)$ simply connected pieces. These two arrangements are the same: and therefore

$$n + \mu + \delta = \nu + m + \delta,$$

so that

$$m - n = \mu - \nu.$$

It thus appears that, if by any system of q cross-cuts a multiply connected surface be resolved into a number p of pieces distinct from one another and all simply connected, the integer $q - p$ is independent of the particular system of the cross-cuts and of their configuration. The integer $q - p$ is therefore essentially associated with the character of the multiple connection of the surface: and its invariance for a given surface enables us to arrange surfaces according to the value of the integer.

No classification among the multiply connected surfaces has yet been made: they have merely been defined as surfaces in which cross-cuts can be made that do not resolve the surface into distinct pieces.

It is natural to arrange them in classes according to the number of cross-cuts which are necessary to resolve the surface into one of simple connection or a number of pieces each of simple connection.

For a simply connected surface, no such cross-cut is necessary: then $q = 0$, $p = 1$, and in general $q - p = -1$. We shall say that the *connectivity** is unity. Examples are furnished by the area of a plane circle, and by a spherical surface with one hole†.

A surface is called doubly-connected when, by one appropriate cross-cut, the surface is changed into a single surface of simple connection: then $q = 1$, $p = 1$ for this particular resolution, and therefore in general, $q - p = 0$. We

* Sometimes *order of connection*, sometimes *adelphic order*; the German word, that is used, is *Grundzahl*.

† The hole is made to give the surface a boundary (§ 163).

shall say that the connectivity is 2. Examples are furnished by a plane ring and by a spherical surface with two holes.

A surface is called triply-connected when, by two appropriate cross-cuts, the surface is changed into a single surface of simple connection: then $q = 2$, $p = 1$ for this particular resolution and therefore, in general, $q - p = 1$. We shall say that the connectivity is 3. Examples are furnished by the surface of an anchor-ring with one hole in it*, and by the surfaces† in Figure 39, the surface in (2) not being in one plane but one part beneath another.

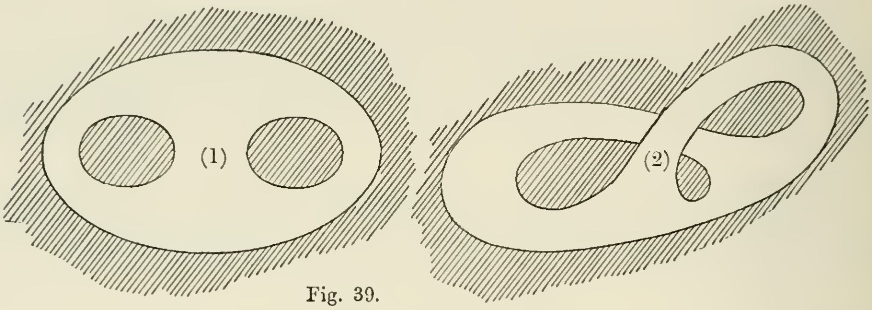

Fig. 39.

And, in general, a surface will be said to be N-ply connected or its connectivity will be denoted by N, if, by $N - 1$ appropriate cross-cuts, it can be changed into a single surface that is simply connected‡. For this particular resolution $q = N - 1$, $p = 1$: and therefore in general

$$q - p = N - 2,$$

or

$$N = q - p + 2.$$

Let a cross-cut l be drawn in a surface of connectivity N. There are two cases to be considered, according as it does not or does divide the surface into distinct pieces.

First, let the surface be only one piece after l is drawn: and let its connectivity then be N'. If in the original surface q cross-cuts (one of which can, after the preceding proposition, be taken to be l) be drawn dividing the surface into p simply connected pieces, then

$$N = q - p + 2.$$

To obtain these p simply connected pieces from the surface after the cross-cut l, it is evidently sufficient to make the $q - 1$ original cross-cuts other than l; that is, the modified surface is such that by $q - 1$ cross-cuts it is resolved into p simply connected pieces, and therefore

$$N' = (q - 1) - p + 2.$$

Hence $N' = N - 1$, or the connectivity of the surface is diminished by unity.

* The hole is made to give the surface a boundary (§ 163).

† Riemann, p. 89.

‡ A few writers estimate the connectivity of such a surface as $N - 1$, the same as the number of cross-cuts which can change it into a single surface of the simplest rank of connectivity: the estimate in the text seems preferable.

Secondly, let the surface be two pieces after l is drawn, of connectivities N_1 and N_2 respectively. Let the appropriate $N_1 - 1$ cross-cuts in the former, and the appropriate $N_2 - 1$ in the latter, be drawn so as to make each a simply connected piece. Then, together, there are two simply connected pieces.

To obtain these two pieces from the original surface, it will suffice to make in it the cross-cut l, the $N_1 - 1$ cross-cuts, and the $N_2 - 1$ cross-cuts, that is, $1 + (N_1 - 1) + (N_2 - 1)$ or $N_1 + N_2 - 1$ cross-cuts in all. Since these, when made in the surface of connectivity N, give two pieces, we have

$$N = (N_1 + N_2 - 1) - 2 + 2,$$

and therefore

$$N_1 + N_2 = N + 1.$$

If one of the pieces be simply connected, the connectivity of the other is N; so that, if a simply connected piece of surface be cut off a multiply connected surface, the connectivity of the remainder is unchanged. Hence:

III. *If a cross-cut be made in a surface of connectivity N and if it do not divide it into separate pieces, the connectivity of the modified surface is $N - 1$; but if it divide the surface into two separate pieces of connectivities N_1 and N_2, then $N_1 + N_2 = N + 1$.*

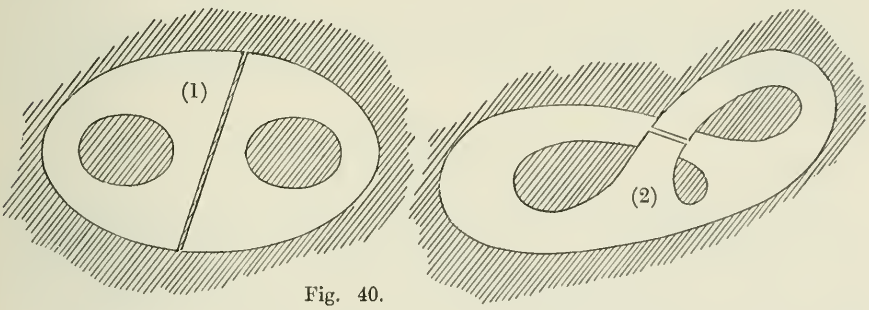

Fig. 40.

Illustrations are shewn, in Fig. 40, of the effect of cross-cuts on the two surfaces in Fig. 39.

IV. In the same way it may be proved that, *if s cross-cuts be made in a surface of connectivity N and divide it into $r + 1$ separate pieces (where $r \leqslant s$) of connectivities $N_1, N_2, \ldots, N_{r+1}$ respectively, then*

$$N_1 + N_2 + \ldots + N_{r+1} = N + 2r - s,$$

a more general result including both of the foregoing cases.

Thus far we have been considering only cross-cuts: it is now necessary to consider loop-cuts, so far as they affect the connectivity of a surface in which they are made.

A loop-cut is changed into a cross-cut, if from A any point of it a cross-cut be made to any point C in a boundary-curve of the original surface, for $CAbdA$ (Fig. 41) is then evidently a cross-cut of the original surface; and CA is a cross-cut of the surface, which is the modification of the original surface after the loop-cut has been made. Since, by definition, a loop-cut does not meet the boundary, the cross-cut CA does not divide the modified surface into distinct pieces; hence, according as the effect of the loop-cut is, or is not, that of making distinct pieces, so will the effect of the whole cross-cut be, or not be, that of making distinct pieces.

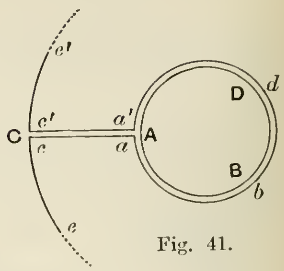

Fig. 41.

161. Let a loop-cut be drawn in a surface of connectivity N; as before for a cross-cut, there are two cases for consideration, according as the loop-cut does or does not divide the surface into distinct pieces.

First, let it divide the surface into two distinct pieces, say of connectivities N_1 and N_2 respectively. Change the loop-cut into a cross-cut of the original surface by drawing a cross-cut in either of the pieces, say the second, from a point in the course of the loop-cut to some point of the original boundary. This cross-cut, as a section of that piece, does not divide it into distinct pieces: and therefore the connectivity is now $N_2' (= N_2 - 1)$. The effect of the whole section, which is a single cross-cut, of the original surface is to divide it into two pieces, the connectivities of which are N_1 and N_2': hence, by § 160, III.,

$$N_1 + N_2' = N + 1,$$

and therefore

$$N_1 + N_2 = N + 2.$$

If the piece cut out be simply connected, say $N_1 = 1$, then the connectivity of the remainder is $N + 1$. But such a removal of a simply connected piece by a loop-cut is the same as making a hole in a continuous part of the surface: and therefore *the effect of making a simple hole in a continuous part of a surface is to increase by unity the connectivity of the surface.*

If the piece cut out be doubly connected, say $N_1 = 2$, then the connectivity of the remainder is N, the same as the connectivity of the original surface. Such a portion would be obtained by cutting out a piece with a hole in it which, so far as concerns the original surface, would be the same as merely enlarging the hole—an operation that naturally would not affect the connectivity.

Secondly, let the loop-cut not divide the surface into two distinct pieces: and let N' be the connectivity of the modified surface. In this modified surface make a cross-cut k from any point of the loop-cut to a point of the boundary: this does not divide it into distinct pieces and therefore the connectivity after this last modification is $N' - 1$. But the surface thus finally modified is derived from the original surface by the single cross-cut, constituted by the combination of k with the loop-cut: this single cross-cut does not divide the surface into distinct pieces and therefore the connectivity after the modification is $N-1$. Hence

$$N'-1=N-1,$$

that is, $N'=N$, or *the connectivity of a surface is not affected by a loop-cut which does not divide the surface into distinct pieces.*

Both of these results are included in the following theorem:—

V. *If after any number of loop-cuts made in a surface of connectivity N, there be $r+1$ distinct pieces of surface, of connectivities $N_1, N_2, \ldots, N_{r+1}$, then*

$$N_1+N_2+\ldots\ldots+N_{r+1}=N+2r.$$

Let the number of loop-cuts be s. Each of them can be changed into a cross-cut of the original surface, by drawing in some one of the pieces, as may be convenient, a cross-cut from a point of the loop-cut to a point of a boundary; this new cross-cut does not divide the piece in which it is drawn into distinct pieces. If k such cross-cuts (where k may be zero) be drawn in the piece of connectivity N_m, the connectivity becomes N_m', where

$$N_m'=N_m-k;$$

hence $$\sum_{m=1}^{r+1} N_m'=\sum_{m=1}^{r+1} N_m-\Sigma k=\sum_{m=1}^{r+1} N_m-s.$$

We now have s cross-cuts dividing the surface of connectivity N into $r+1$ distinct pieces, of connectivities $N_1', N_2', \ldots, N_r', N_{r+1}'$; and therefore, by § 160, IV.,

$$N_1'+\ldots+N_r'+N_{r+1}'=N+2r-s,$$

so that $$N_1+N_2+\ldots+N_{r+1}=N+2r.$$

This result could have been obtained also by combination and repetition of the two results obtained for a single loop-cut.

Thus a spherical surface with one hole in it is simply connected: when $n-1$ other different holes* are made in it, the edges of the holes being outside one another, the connectivity of the surface is increased by $n-1$, that is, it becomes n. Hence *a spherical surface with n holes in it is n-ply connected.*

162. Occasionally, it is necessary to consider the effect of a slit made in the surface.

If the slit have neither of its extremities on a boundary (and therefore no point on a boundary) it can be regarded as the limiting form of a loop-cut which makes a hole in the surface. Such a slit therefore (§ 161) increases the connectivity by unity.

* These are holes in the surface, not holes bored through the volume of the sphere; one of the latter would give two holes in the surface.

If the slit have one extremity (but no other point) on a boundary, it can be regarded as the limiting form of a cross-cut, which returns on itself as in the figure, and cuts off a single simply connected piece. Such a slit therefore (§ 160, III.) leaves the connectivity unaltered.

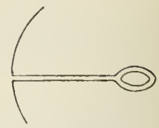

Fig. 42.

If the slit have both extremities on boundaries, it ceases to be merely a slit: it is a cross-cut the effect of which on the connectivity has been obtained. We do not regard such sections as slits.

163. In the preceding investigations relative to cross-cuts and loop-cuts, reference has continually been made to the boundary of the surface considered.

The *boundary* of a surface consists of a line returning to itself, or of a system of lines each returning to itself. Each part of such a boundary-line as it is drawn is considered a part of the boundary, and thus a boundary-line cannot cut itself and pass beyond its earlier position, for a boundary cannot be crossed: each boundary-line must therefore be a simple curve*.

Most surfaces have boundaries: an exception arises in the case of closed surfaces whatever be their connectivity. It was stated (§ 159) that a boundary is assigned to such a surface by drawing an infinitesimal simple curve in it or, what is the same thing, by making a small hole. The advantage of this can be seen from the simple example of a spherical surface.

When a small hole is made in any surface the connectivity is increased by unity: the connectivity of the spherical surface after the hole is made is unity, and therefore the connectivity of the complete spherical surface must be taken to be zero.

The mere fact that the connectivity is less than unity, being that of the simplest connected surfaces with which we have to deal, is not in itself of importance. But let us return for a moment to the suggested method of determining the connectivity by means of the evanescence of circuits without crossing the boundary. When the surface is the complete spherical surface (Fig. 43), there are two essentially distinct ways of making a circuit C evanescent, first, by making it collapse into the point a, secondly by making it expand over the equator and then collapse into the point b. One of the two is superfluous: it introduces an element of doubt as to the mode of evanescence unless that mode be specified—a specification which in itself is tantamount to an assignment of

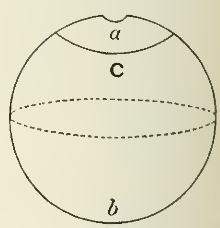

Fig. 43.

* Also a line not returning to itself may be a boundary; it can be regarded as the limit of a simple curve when the area becomes infinitesimal.

boundary. And in the case of multiply connected surfaces the absence of boundary, as above, leads to an artificial reduction of the connectivity by unity, arising not from the greater simplicity of the surface but from the possibility of carrying out in two ways the operation of reducing any circuit to given circuits, which is most effective when only one way is permissible. We shall therefore assume a boundary assigned to such closed surfaces as in the first instance are destitute of boundary.

164. The relations between the number of boundaries and the connectivity of a surface are given by the following propositions.

I. *The boundary of a simply connected surface consists of a single line.*

When a boundary consists of separate lines, then a cross-cut can be made from a point of one to a point of another. By proceeding from P, a point on one side of the cross-cut, along the boundary $ac...c'a'$ we can by a line lying wholly in the surface reach a point Q on the other side of the cross-cut: hence the parts of the surface on opposite sides of the cross-cut are connected. The surface is therefore not resolved into distinct pieces by the cross-cut.

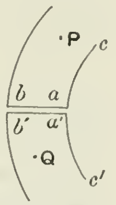

Fig. 44.

A simply connected surface is resolved into distinct pieces by each cross-cut made in it: such a cross-cut as the foregoing is therefore not possible, that is, there are not separate lines which make up its boundary. It has a boundary: the boundary therefore consists of a single line.

II. *A cross-cut either increases by unity or diminishes by unity the number of distinct boundary-lines of a multiply connected surface.*

A cross-cut is made in one of three ways: either from a point a of one boundary-line A to a point b of another boundary-line B; or from a point a of a boundary-line to another point a' of the same boundary-line; or from a point of a boundary-line to a point in the cut itself.

If made in the first way, a combination of one edge of the cut, the remainder of the original boundary A, the other edge of the cut and the remainder of the original boundary B taken in succession, form a single piece of boundary; this replaces the two boundary-lines A and B which existed distinct from one another before the cross-cut was made. Hence the number of lines is diminished by unity. An example is furnished by a plane ring (ii., Fig. 37, p. 314).

If made in the second way, the combination of one edge of the cut with the piece of the boundary on one side of it makes one boundary-line, and the combination of the other edge of the cut with the other piece of the boundary makes another boundary-line. Two boundary-lines, after the cut is made, replace a single boundary-line, which existed before it was made: hence the number of lines is increased by unity. Examples are furnished by the cut surfaces in Fig. 40, p. 319.

If made in the third way, the cross-cut may be considered as constituted by a loop-cut and a cut joining the loop-cut to the boundary. The boundary-lines may now be considered as constituted (Fig. 41, p. 320) by the closed curve ABD and the closed boundary $abda'c'e'...eca$; that is, there are now two boundary-lines instead of the single boundary-line $ce...e'c'c$ in the uncut surface. Hence the number of distinct boundary-lines is increased by unity.

COROLLARY. *A loop-cut increases the number of distinct boundary-lines by two.*

This result follows at once from the last discussion.

III. *The number of distinct boundary-lines of a surface of connectivity N is $N-2k$, where k is a positive integer that may be zero.*

Let m be the number of distinct boundary-lines; and let $N-1$ appropriate cross-cuts be drawn, changing the surface into a simply connected surface. Each of these cross-cuts increases by unity or diminishes by unity the number of boundary-lines; let these units of increase or of decrease be denoted by $\epsilon_1, \epsilon_2, \ldots, \epsilon_{N-1}$. Each of the quantities ϵ is ± 1; let k of them be positive, and $N-1-k$ negative. The total number of boundary-lines is therefore

$$m+k-(N-1-k).$$

The surface now is a single simply connected surface, and there is therefore only one boundary-line; hence

$$m+k-(N-1-k)=1,$$

so that

$$m=N-2k;$$

and evidently k is an integer that may be zero.

COROLLARY 1. *A closed surface with a single boundary-line* is of odd connectivity.*

For example, the surface of an anchor-ring, when bounded, is of connectivity 3; the surface, obtained by boring two holes through the volume of a solid sphere, is, when bounded, of connectivity 5.

If the connectivity of a closed surface with a single boundary be $2p+1$, the surface is often said† to be of *class* p (§ 178, p. 349.)

COROLLARY 2. *If the number of distinct boundary lines of a surface of connectivity N be N, any loop-cut divides the surface into two distinct pieces.*

After the loop-cut is made, the number of distinct boundary-lines is $N+2$; the connectivity of the whole of the cut surface is therefore not less

* See § 159.

† The German word is *Geschlecht;* French writers use the word *genre*, and Italians *genere*.

than $N+2$. It has been proved that a loop-cut, which does not divide the surface into distinct pieces, does not affect the connectivity; hence as the connectivity has been increased, the loop-cut must divide the surface into two distinct pieces. It is easy, by the result of § 161, to see that, after the loop-cut is made, the sum of connectivities of the two pieces is $N+2$, so that the connectivity of the whole of the cut surface is equal to $N+2$.

Note. Throughout these propositions, a tacit assumption has been made, which is important for this particular proposition when the surface is the means of representing the variable. The assumption is that *the surface is bifacial and not unifacial*; it has existed implicitly throughout all the geometrical representations of variability: it found explicit expression in § 4 when the plane was brought into relation with the sphere: and a cut in a surface has been counted a single cut, occurring in one face, though it would have to be counted as two cuts, one on each side, were the surface unifacial.

The propositions are not necessarily valid, when applied to unifacial surfaces. Consider a surface made out of a long rectangular slip of paper, which is twisted once (or any odd number of times) and then has its ends fastened together. This surface is of double connectivity, because one section can be made across it which does not divide it into separate pieces; it has only a *single* boundary-line, so that Prop. III. just proved does not apply. The surface is unifacial; and it is possible, without meeting the boundary, to pass continuously in the surface from a point P to another point Q which could be reached merely by passing through the material at P.

We therefore do not retain unifacial surfaces for consideration.

165. The following proposition, substantially due to Lhuilier*, may be taken in illustration of the general theory.

If a closed surface of connectivity $2N+1$ (or of class N) be divided by circuits into any number of simply connected portions, each in the form of a curvilinear polygon, and if F be the number of polygons, E be the number of edges and S the number of angular points, then

$$2N = 2 + E - F - S.$$

Let the edges E be arranged in systems, a system being such that any line in it can be reached by passage along some other line or lines of the system; let k be the number of such systems†. To resolve the surface into a number of simply connected pieces composed of the F polygons, the cross-cuts will be made along the edges; and therefore, unless a boundary be assigned

* Gergonne, *Ann. de Math.*, t. iii, (1813), pp. 181—186; see also Möbius, *Ges. Werke*, t. ii, p. 468. A *circuit* is defined in § 166.

† The value of k is 1 for the proposition and is greater than 1 for the Corollary.

to the surface in each system of lines, the first cut for any system will be a loop-cut. We therefore take k points, one in each system as a boundary; the first will be taken as the natural boundary of the surface, and the remaining $k-1$, being the limiting forms of $k-1$ infinitesimal loop-cuts, increase the connectivity of the surface by $k-1$, that is, the connectivity now is $2N+k$.

The result of the cross-cuts is to leave F simply connected pieces: hence Q, the number of cross-cuts, is given by

$$Q = 2N + k + F - 2.$$

At every angular point on the uncut surface, three or more polygons are contiguous. Let S_m be the number of angular points, where m polygons are contiguous; then

$$S = S_3 + S_4 + S_5 + \dots$$

Again, the number of edges meeting at each of the S_3 points is three, at each of the S_4 points is four, at each of the S_5 points is five, and so on; hence, in taking the sum $3S_3 + 4S_4 + 5S_5 + \dots$, each edge has been counted twice, once for each extremity. Therefore

$$2E = 3S_3 + 4S_4 + 5S_5 + \dots$$

Consider the composition of the extremities of the cross-cuts; the number of the extremities is $2Q$, twice the number of cross-cuts.

Each of the k points furnishes two extremities; for each such point is a boundary on which the initial cross-cut for each of the systems must begin and must end. These points therefore furnish $2k$ extremities.

The remaining extremities occur in connection with the angular points. In making a cut, the direction passes from a boundary along an edge, past the point along another edge and so on, until a boundary is reached; so that on the first occasion when a cross-cut passes through a point, it is made along two of the edges meeting at the point. Every other cross-cut passing through that point must begin or end there, so that each of the S_3 points will furnish one extremity (corresponding to the remaining one cross-cut through the point), each of the S_4 points will furnish two extremities (corresponding to the remaining two cross-cuts through the point), and so on. The total number of extremities thus provided is

$$S_3 + 2S_4 + 3S_5 + \dots$$

Hence $$2Q = 2k + S_3 + 2S_4 + 3S_5 + \dots$$

$$= 2k + 2E - 2S,$$

or $$Q = k + E - S,$$

which combined with $$Q = 2N + k + F - 2,$$

leads to the relation $$2N = 2 + E - F - S.$$

The simplest case is that of a sphere, when Euler's relation $F+S=E+2$ is obtained. The case next in simplicity is that of an anchor-ring, for which the relation is $F+S=E$.

COROLLARY. *If the result of making the cross-cuts along the various edges be to give the F polygons, not simply connected areas but areas of connectivities* $N_1+1, N_2+1, \ldots, N_F+1$ *respectively, then the connectivity of the original surface is given by*

$$2N=2+E-F-S+\sum_{r=1}^{F} N_r.$$

166. The method of determining the connectivity of a surface by means of a system of cross-cuts, which resolve it into one or more simply connected pieces, will now be brought into relation with the other method, suggested in § 159, of determining the connectivity by means of irreducible circuits.

A closed line drawn on the surface is called a *circuit.*

A circuit, which can be reduced to a point by continuous deformation without crossing the boundary, is called *reducible*; a circuit, which cannot be so reduced, is called *irreducible.*

An irreducible circuit is either (i) *simple,* when it cannot without crossing the boundary be deformed continuously into repetitions of one or more circuits; or (ii) *multiple,* when it can without crossing the boundary be deformed continuously into repetitions of a single circuit; or (iii) *compound,* when it can without crossing the boundary be deformed continuously into combinations of different circuits, that may be simple or multiple. The distinction between simple circuits and compound circuits, that involve no multiple circuits in their combination, depends upon conventions adopted for each particular case.

A circuit is said to be *reconcileable* with the system of circuits into a combination of which it can be continuously deformed.

If a system of circuits be reconcileable with a reducible circuit, the system is said to be reducible.

As there are two directions, one positive and the other negative, in which a circuit can be described, and as there are possibilities of repetitions and of compositions of circuits, it is clear that circuits can be represented by linear algebraical expressions involving real quantities and having merely numerical coefficients.

Thus a reducible circuit can be denoted by 0.

If a simple irreducible circuit, positively described, be denoted by a, the same circuit, negatively described, can be denoted by $-a$.

The multiple circuit, which is composed of m positive repetitions of the simple irreducible circuit a, would be denoted by ma; but if the m repetitions were negative, the multiple circuit would be denoted by $-ma$.

A compound circuit, reconcileable with a system of simple irreducible circuits $a_1, a_2, \ldots, a_n$ would be denoted by $m_1a_1 + m_2a_2 + \ldots + m_na_n$, where $m_1, m_2, \ldots, m_n$ are positive or negative integers, being the net number of positive or negative descriptions of the respective simple irreducible circuits.

The condition of the reducibility of a system of circuits $a_1, a_2, \ldots, a_n$, each one of which is simple and irreducible, is that integers $m_1, m_2, \ldots, m_n$ should exist such that

$$m_1a_1 + m_2a_2 + \ldots + m_na_n = 0,$$

the sign of equality in this equation, as in other equations, implying that continuous deformation without crossing the boundary can change into one another the circuits, denoted by the symbols on either side of the sign.

The representation of any compound circuit in terms of a system of independent irreducible circuits is unique: if there were two different expressions, they could be equated in the foregoing sense and this would imply the existence of a relation

$$p_1a_1 + p_2a_2 + \ldots + p_na_n = 0,$$

which is excluded by the fact that the system is irreducible.

Further, equations can be combined linearly, provided that the coefficients of the combinations be merely numerical.

167. In order, then, to be in a position to estimate circuits on a multiply connected surface, it is necessary that an irreducible system of irreducible simple circuits should be known, such a system being considered complete when every other circuit on the surface is reconcileable with the system.

Such a system is not necessarily unique; and it must be proved that, *if more than one complete system be obtainable, any circuit can be reconciled with each system.*

First, *the number of simple irreducible circuits in any complete system must be the same for the same surface.*

Let $a_1, \ldots, a_p$; and $b_1, \ldots, b_n$; be two complete systems. Because $a_1, \ldots, a_p$ constitute a complete system, every circuit of the system of circuits b is reconcileable with it; that is, integers m_{ij} exist, such that

$$b_r = m_{1r}a_1 + m_{2r}a_2 + \ldots + m_{pr}a_p,$$

for $r = 1, 2, \ldots, n$. If n were $> p$, then by combining linearly each equation after the first p equations with those p equations, and eliminating $a_1, \ldots, a_p$ from the set of $p+1$ equations, we could derive $n-p$ relations of the form

$$M_1b_1 + M_2b_2 + \ldots + M_nb_n = 0,$$

where the coefficients M, being determinants the constituents of which are integers, would be integers. The system of circuits b is irreducible, and there are therefore no such relations; hence n is not greater than p.

Similarly, by considering the reconciliation of each circuit a with the irreducible system of circuits b, it follows that p is not greater than n.

Hence p and n are equal to one another. And, because each system is a complete system, there are integers A and B such that

$$\left.\begin{array}{l} a_r = A_{r1}b_1 + A_{r2}b_2 + \dots + A_{rn}b_n \quad (r = 1, \dots, n) \\ b_s = B_{s1}a_1 + B_{s2}a_2 + \dots + B_{sn}a_n \quad (s = 1, \dots, n) \end{array}\right\}.$$

The determinant of the integers A is equal to ± 1; likewise the determinant of the integers B.

Secondly, *let x be a circuit reconcileable with the system of circuits a: it is reconcileable with any other complete system of circuits.*

Since x is reconcileable with the system a, integers $m_1, \dots, m_n$ can be found such that

$$x = m_1 a_1 + \dots + m_n a_n.$$

Any other complete system of n circuits b is such that the circuits a can be expressed in the form

$$a_r = A_{r1}b_1 + \dots + A_{rn}b_n, \quad (r = 1, \dots, n),$$

where the coefficients A are integers; and therefore

$$\begin{aligned} x &= b_1 \sum_{r=1}^{n} m_r A_{r1} + b_2 \sum_{r=1}^{n} m_r A_{r2} + \dots + b_n \sum_{r=1}^{n} m_r A_{rn} \\ &= q_1 b_1 + q_2 b_2 + \dots + q_n b_n, \end{aligned}$$

where the coefficients q are integers, that is, x is reconcileable with the complete system of circuits b.

168. It thus appears that for the construction of any circuit on a surface, it is sufficient to know some one complete system of simple irreducible circuits. A complete system is supposed to contain the smallest possible number of simple circuits: any one which is reconcileable with the rest is omitted, so that the circuits of a system may be considered as independent. Such a system is indicated by the following theorems:—

I. *No irreducible simple circuit can be drawn on a simply connected surface*.*

If possible, let an irreducible circuit C be drawn in a simply connected surface with a boundary B. Make a loop-cut along C, and change it into a cross-cut by making a cross-cut A from some point of C to a point of B; this cross-cut divides the surface into two simply connected pieces, one of which is bounded by B, the two edges of A, and one edge of the cut along C, and the other of which is bounded entirely by the cut along C.

The latter surface is smaller than the original surface; it is simply connected and has a single boundary. If an irreducible simple circuit can be drawn on it, we proceed as before, and again obtain a still smaller simply connected surface. In this way, we ultimately obtain an infinitesimal

* All surfaces considered are supposed to be bounded.

element; for every cut divides the surface, in which it is made, into distinct pieces. Irreducible circuits cannot be drawn in this element; and therefore its boundary is reducible. This boundary is a circuit in a larger portion of the surface: the circuit is reducible so that, in that larger portion no irreducible circuit is possible and therefore its boundary is reducible. This boundary is a circuit in a still larger portion, and the circuit is reducible: so that in this still larger portion no irreducible circuit is possible and once more the boundary is reducible.

Proceeding in this way, we find that no irreducible simple circuit is possible in the original surface.

COROLLARY. *No irreducible circuit can be drawn on a simply connected surface.*

II. *A complete system of irreducible simple circuits for a surface of connectivity N contains $N-1$ simple circuits, so that every other circuit on the surface is reconcileable with that system.*

Let the surface be resolved by cross-cuts into a single simply connected surface: $N-1$ cross-cuts will be necessary. Let CD be any one of them: and let a and b be two points on the opposite edges of the cross-cut. Then since the surface is simply connected, a line can be drawn in the surface from a to b without passing out of the surface or without meeting a part of the boundary, that is, without meeting any other cross-cut. The cross-cut CD ends either in another cross-cut or in a boundary; the line $ae \ldots fb$ surrounds that other cross-cut or that boundary as the case may be: hence, if the cut CD be obliterated, the line $ae \ldots fba$ is irreducible on the surface in which the other $N-2$ cross-cuts are made. But it meets none of those cross-cuts; hence, when they are all obliterated so as to restore the unresolved surface of connectivity N, it is an irreducible circuit. It is evidently not a repeated circuit; hence it is an irreducible simple circuit. Hence *the line of an irreducible simple circuit on an unresolved surface is given by a line passing from a point on one edge of a cross-cut in the resolved surface to a point on the opposite edge.*

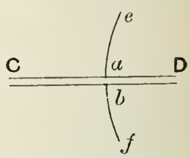

Fig. 45.

Since there are $N-1$ cross-cuts, it follows that $N-1$ irreducible simple circuits can thus be obtained: one being derived in the foregoing manner from each of the cross-cuts, which are necessary to render the surface simply connected. It is easy to see that each of the irreducible circuits on an unresolved surface is, by the cross-cuts, rendered impossible as a circuit on the resolved surface.

But every other irreducible circuit C is reconcileable with the $N-1$ circuits, thus obtained. If there be one not reconcileable with these $N-1$ circuits, then, when all the cross-cuts are made, the circuit C is not rendered impossible, if it be not reconcileable with those which are rendered impossible by the cross-cuts: that is, there is on the resolved surface an irreducible circuit. But the resolved surface is simply connected, and therefore no irreducible circuit can be drawn on it: hence the hypothesis as to C, which leads to this result, is not tenable.

Thus every other circuit is reconcileable with the system of $N-1$ circuits: and therefore *the system is complete**.

This method of derivation of the circuits at once indicates how far a system is arbitrary. Each system of cross-cuts leads to a complete system of irreducible simple circuits, and vice versa; as the one system is not unique, so the other system is not unique.

For the general question, Jordan's memoir, *Des contours tracés sur les surfaces*, Liouville, 2[me] Sér., t. xi., (1866), pp. 110—130, may be consulted.

Ex. 1. On a doubly connected surface, one irreducible simple circuit can be drawn. It is easily obtained by first resolving the surface into one that is simply connected—a single cross-cut CD is effective for this purpose—and then by drawing a curve aeb in the

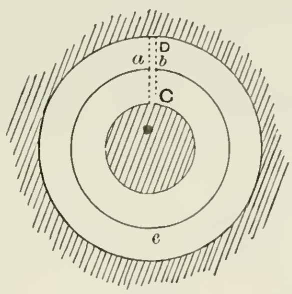

Fig. 46, (i).

surface from one edge of the cross-cut to the other. All other irreducible circuits on the unresolved surface are reconcileable with the circuit $aeba$.

Ex. 2. On a triply-connected surface, two independent irreducible circuits can be

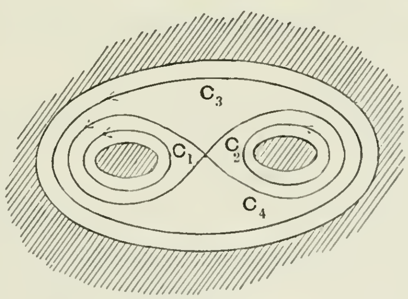

Fig. 46, (ii).

* If the number of independent irreducible simple circuits be adopted as a basis for the definition of the connectivity of a surface, the result of the proposition would be taken as the definition: and the resolution of the surface into one, which is simply connected, would then be obtained by developing the preceding theory in the reverse order.

drawn. Thus in the figure C_1 and C_2 will form a complete system. The circuits C_3 and C_4 are also irreducible : they can evidently be deformed into C_1 and C_2 and reducible circuits by continuous deformation : in the algebraical notation adopted, we have

$$C_3 = C_1 + C_2, \quad C_4 = C_1 - C_2.$$

Ex. 3. Another example of a triply connected surface is given in Fig. 47. Two irreducible simple circuits are C_1 and C_2. Another irreducible circuit is C_3; this can be

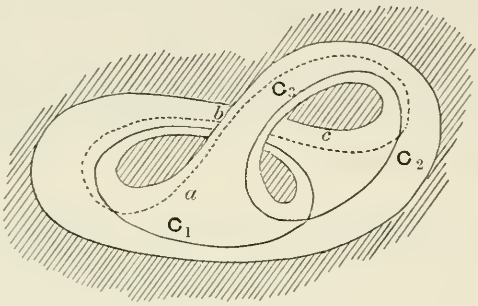

Fig. 47.

reconciled with C_1 and C_2 by drawing the point a into coincidence with the intersection of C_1 and C_2, and the point c into coincidence with the same point.

Ex. 4. As a last example, consider the surface of a solid sphere with n holes bored through it. The connectivity is $2n+1$: hence $2n$ independent irreducible simple circuits

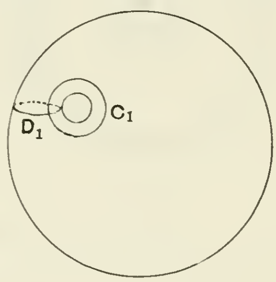

Fig. 48.

can be drawn on the surface. The simplest complete system is obtained by taking $2n$ curves: made up of a set of n, each round one hole, and another set of n, each through one hole.

A resolution of this surface is given by taking cross-cuts, one round each hole (making the circuits through the holes no longer possible) and one through each hole (making the circuits round the holes no longer possible).

The simplest case is that for which $n=1$: the surface is equivalent to the anchor-ring.

169. Surfaces are at present being considered in view of their use as a means of representing the value of a complex variable. The foregoing investigations imply that surfaces can be classed according to their connectivity; and thus, having regard to their designed use, the question arises as to whether all surfaces of the same connectivity are equivalent to one another, so as to be transformable into one another.

Moreover, a surface can be physically deformed and still remain suitable for representation of the variable, provided certain conditions are satisfied. We thus consider geometrical transformation as well as physical deformation; but we are dealing only with the general results and not with the mathematical relations of stretching and bending, which are discussed in treatises on Analytical Geometry*.

It is evident that continuity is necessary for both: discontinuity would imply discontinuity in the representation of the variable. Points that are contiguous (that is, separated only by small distances measured in the surface) must remain contiguous†: and one point in the unchanged surface must correspond to only one point in the changed surface. Hence *in the continuous deformation of a surface there may be stretching and there may be bending; but there must be no tearing and there must be no joining.*

For instance, a single untwisted ribbon, if cut, comes to be simply connected. If a twist through 180° be then given to one end and that end be then joined to the other, we shall have a once-twisted ribbon, which is a surface with only one face and only one edge; it cannot be looked upon as an equivalent of the former surface.

A spherical surface with a single hole can have the hole stretched and the surface flattened, so as to be the same as a bounded portion of a plane: the two surfaces are equivalent to one another. Again, in the spherical surface, let a large indentation be made: let both the outer and the inner surfaces be made spherical; and let the mouth of the indentation be contracted into the form of a long, narrow hole along a part of a great circle. When each point of the inner surface is geometrically moved so that it occupies the position of its reflexion in the diametral plane of the hole, the final form§ of the whole surface is that of a two-sheeted surface with a junction along a line: it is a spherical winding-surface, and is equivalent to the simply connected spherical surface.

170. It is sufficient, for the purpose of representation, that the two surfaces should have a point-to-point transformation: it is not necessary that physical deformation, without tears or joins, should be actually possible. Thus a ribbon with an even number of twists would be as effective as a limited portion of a cylinder, or (what is the same thing) an untwisted ribbon: but it is not possible to deform the one into the other physically‡.

It is easy to see that either deformation or transformation of the kind considered *will change a bifacial surface into a bifacial surface;* that *it will not alter the connectivity,* for it will not change irreducible circuits into

* See, for instance, Frost's *Solid Geometry*, (3rd ed.), pp. 342—352.

† Distances between points must be measured along the surface, not through space; the distance between two points is a length which one point would traverse before reaching the position of the other, the motion of the point being restricted to take place in the surface. Examples will arise later, in Riemann's surfaces, in which points that are contiguous in space are separated by finite distances on the surface.

§ Clifford, *Coll. Math. Papers*, p. 250.

‡ The difference between the two cases is that, in physical deformation, the surfaces are the surfaces of continuous matter and are impenetrable; while, in geometrical transformation, the surfaces may be regarded as penetrable without interference with the continuity.

reducible circuits, and the number of independent irreducible circuits determines the connectivity: and that *it will not alter the number of boundary curves*, for a boundary will be changed into a boundary. These are necessary relations between the two forms of the surface: it is not difficult to see that they are sufficient for correspondence. For if, on each of two bifacial surfaces with the same number of boundaries and of the same connectivity, a complete system of simple irreducible circuits be drawn, then, when the members of the systems are made to correspond in pairs, the full transformation can be effected by continuous deformation of those corresponding irreducible circuits. It therefore follows that:—

The necessary and sufficient conditions, that two bifacial surfaces may be equivalent to one another for the representation of a variable, are that the two surfaces should be of the same connectivity and should have the same number of boundaries.

As already indicated, this equivalence is a geometrical equivalence: deformation may be (but is not of necessity) physically possible.

Similarly, the presence of one or of several knots in a surface makes no essential difference in the use of the surface for representing a variable. Thus a long cylindrical surface is changed into an anchor-ring when its ends are joined together; but the changed surface would be equally effective for purposes of representation if a knot were tied in the cylindrical surface before the ends are joined.

But it need hardly be pointed out that though surfaces, thus twisted or knotted, are equivalent for the purpose indicated, they are not equivalent for all topological enumerations.

Seeing that bifacial surfaces, with the same connectivity and the same number of boundaries, are equivalent to one another, it is natural to adopt, as the surface of reference, some simple surface with those characteristics; thus for a surface of connectivity $2p+1$ with a single boundary, the surface of a solid sphere, bounded by a point and pierced through with p holes, could be adopted.

Klein calls* such a surface of reference a *Normal Surface.*

It has been seen that a bounded spherical surface and a bounded simply connected part of a plane are equivalent—they are, moreover, physically deformable into one another.

An untwisted closed ribbon is equivalent to a bounded piece of a plane with one hole in it—they are deformable into one another: but if the ribbon, previous to being closed, have undergone an even number of twists each through 180°, they are still equivalent but are not physically deformable into one another. Each of the bifacial surfaces is doubly connected (for a single cross-cut renders each simply connected) and each of them

* *Ueber Riemann's Theorie der algebraischen Functionen und ihrer Integrale*, (Leipzig, Teubner, 1882), p. 26.

has two boundaries. If however the ribbon, previous to being closed, have undergone an odd number of twists each through 180°, the surface thus obtained is not equivalent to the single-holed portion of the plane ; it is unifacial and has only one boundary.

A spherical surface pierced in $n+1$ holes is equivalent to a bounded portion of the plane with n holes ; each is of connectivity $n+1$ and has $n+1$ boundaries. The spherical surface can be deformed into the plane surface by stretching one of its holes into the form of the outside boundary of the plane surface.

Ex. Prove that the surface of a bounded anchor-ring can be physically deformed into the surface in Fig. 47, p. 332.

For continuation and fuller development of the subjects of the present chapter, the following references, in addition to those which have been given, will be found useful :

Klein, *Math. Ann.*, t. vii, (1874), pp. 548—557; ib., t. ix, (1876), pp. 476—482.

Lippich, *Math. Ann.*, t. vii, (1874), pp. 212—229 ; *Wiener Sitzungsb.*, t. lxix, (ii), (1874), pp. 91—99.

Durège, *Wiener Sitzungsb.*, t. lxix, (ii), (1874), pp. 115—120 ; and section 9 of his treatise, quoted on p. 316, note.

Neumann, chapter vii of his treatise, quoted on p. 5, note.

Dyck, *Math. Ann.*, t. xxxii, (1888), pp. 457—512, ib., t. xxxvii, (1890), pp. 273—316; at the beginning of the first part of this investigation, a valuable series of references is given.

Dingeldey, *Topologische Studien*, (Leipzig, Teubner, 1890).

CHAPTER XV.

Riemann's Surfaces.

171. The method of representing a variable by assigning to it a position in a plane or on a sphere is effective when properties of uniform functions of that variable are discussed. But when multiform functions, or integrals of uniform functions occur, the method is effective only when certain parts of the plane are excluded, due account being subsequently taken of the effect of such exclusions; and this process, the extension of Cauchy's method, was adopted in Chapter IX.

There is another method, referred to in § 100 as due to Riemann, of an entirely different character. In Riemann's representation, the region, in which the variable z exists, no longer consists of a single plane but of a number of planes; they are distinct from one another in geometrical conception, yet, in order to preserve a representation in which the value of the variable is obvious on inspection, the planes are infinitesimally close to one another. The number of planes, often called *sheets*, is the same as the number of distinct values (or branches) of the function w for a general argument z and, unless otherwise stated, will be assumed finite; each sheet is associated with one branch of the function, and changes from one branch of the function to another are effected by making the z-variable change from one sheet to another, so that, to secure the possibility of change of sheet, it is necessary to have means of passage from one sheet to another. The aggregate of all the sheets is a surface, often called a *Riemann's Surface.*

For example, consider the function

$$w=z^{\frac{1}{3}}+(z-1)^{-\frac{1}{3}},$$

the cube roots being independent of one another. It is evidently a nine-valued function; the number of sheets in the appropriate Riemann's surface is therefore nine.

The branch-points are $z=0$, $z=1$, $z=\infty$. Let ω and α denote a cube-root of unity, independently of one another; then the values of $z^{\frac{1}{3}}$ can be represented in the form

$z^{\frac{1}{3}}$, $\omega z^{\frac{1}{3}}$, $\omega^2 z^{\frac{1}{3}}$; and the values of $(z-1)^{-\frac{1}{3}}$ can be represented in the form $(z-1)^{-\frac{1}{3}}$, $a^2(z-1)^{-\frac{1}{3}}$, $a(z-1)^{-\frac{1}{3}}$. The nine values of w can be symbolically expressed as follows :—

w_1	1	1
w_2	ω	1
w_3	ω^2	1

w_4	1	a^2
w_5	ω	a^2
w_6	ω^2	a^2

w_7	1	a
w_8	ω	a
w_9	ω^2	a

where the symbols opposite to w give the coefficients of $z^{\frac{1}{3}}$ and of $(z-1)^{-\frac{1}{3}}$ respectively.

Now when z describes a small simple circuit positively round the origin, the groups in cyclical order are w_1, w_2, w_3; w_4, w_5, w_6; w_7, w_8, w_9. And therefore, in the immediate vicinity of the origin, there must be means of passage to enable the z-point to make the corresponding changes from sheet to sheet. Taking a section of the whole surface near the origin so as to indicate the passages and regarding the right-hand sides as the part from which the z-variable moves when it describes a circuit positively, the passages must be in character as indicated in Fig. 49. And it is evident that the further description of small simple circuits round the origin will, with these passages, lead to the proper values: thus w_5, which after the single description is the value of w_4, becomes w_6 after another description and it is evident that a point in the w_5 sheet passes into the w_6 sheet.

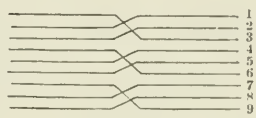

Fig. 49.

When z describes a small simple circuit positively round the point 1, the groups in cyclical order are w_1, w_4, w_7; w_2, w_5, w_8; w_3, w_6, w_9: and therefore, in the immediate vicinity of the point 1, there must be means of passage to render possible the corresponding changes of z from sheet to sheet. Taking a section as before near the point 1 and with similar convention as to the positive direction of the z-path, the passages must be in character as indicated in Fig. 50.

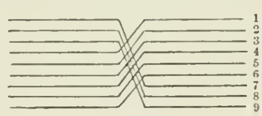

Fig. 50.

Similarly for infinitely large values of z.

If then the sheets can be so joined as to give these possibilities of passage and also give combinations of them corresponding to combinations of the simple paths indicated, then there will be a surface to any point of which will correspond one and only one value of w: and when the value of w is given for a point z in an ordinary plane of variation, then that value of w will determine the sheet of the surface in which the point z is to be taken. A surface will then have been constructed such that the function w, which is multiform for the single-plane representation of the variable, is uniform for variations in the many-sheeted surface.

Again, for the simple example arising from the two-valued function, defined by the equation

$$w=\{(z-a)(z-b)(z-c)\}^{-\frac{1}{2}},$$

the branch-points are a, b, c, ∞; and a small simple circuit round any one of these four points interchanges the two values. The Riemann's surface is two-sheeted and there must be means of passage between the two sheets in the vicinity of a, that of b, that of c and at the infinite part of the plane.

These examples are sufficient to indicate the main problem. It is the construction of a surface in which the independent variable can move so that, for variations of z in that surface, the multiformity of the function is changed to uniformity. From the nature of the case, the character of the surface will depend on the character of the function: and thus, though all the functions are uniform within their appropriate surfaces, these surfaces are widely various. Evidently for uniform functions of z the appropriate surface on the above method is the single plane already adopted.

172. The simplest classes of functions for which a Riemann's surface is useful are (i) those called (§ 94) *algebraic* functions, that is, multiform functions of the independent variable defined by an algebraical equation of the form

$$f(w, z) = 0,$$

which is of finite degree, say n, in w; and (ii) those usually called *Abelian* functions, which arise through integrals connected with algebraic functions.

Of such an algebraic function there are, in general, n distinct values; but for the special values of z, that are the branch-points, two or more of the values coincide. The appropriate Riemann's surface is composed of n sheets; one branch, and only one branch, of w is associated with a sheet. The variable z, in its relation to the function, is determined not merely by its modulus and argument but also by its sheet; that is, in the language of the earlier method, we take account of the path by which z acquires a value. The particular sheet in which z lies determines the particular branch of the function. Variations of z, which occur within a sheet and do not coincide with points lying in regions of passage between the sheets, lead to variations in the value of the branch of w associated with the sheet; a return to an initial value of z, by a path that nowhere lies within a region of passage, leaves the z-point in the same sheet as at first and so leads to the initial branch (and to the initial value of the branch) of w. But a return to an initial value of z by a path, which, in the former method of representation, would enclose a branch-point, implies a change of the branch of the function according to the definite order prescribed by the branch-point. Hence the final value of the variable z on the Riemann's surface must lie in a sheet that is different from that of the initial (and algebraically equal) value; and therefore the sheets must be so connected that, in the immediate vicinity of branch-points, there are means of passage from one sheet to another, securing the proper interchanges of the branches of the function as defined by the equation.

173. The first necessity is therefore the consideration of the mode in which the sheets of a Riemann's surface are joined: the mode is indicated by the theorem that *sheets of a Riemann's surface are joined along lines.*

The junction might be made either at a point, as with two spheres in contact, or by a common portion of a surface, as with one prism lying on another, or along lines; but whatever the character of the junction be, it must be such that a single passage across it (thereby implying entrance to the junction and exit from it) must change the sheet of the variable.

If the junction were at a point, then the z-variable could change from one sheet into another sheet, only if its path passed through that point: any other closed path would leave the z-variable in its original sheet. A small closed curve, infinitesimally near the point and enclosing it and no other branch-point, is one which ought to transfer the variable to another sheet because it encloses a branch-point: and this is impossible with a point-junction when the path does not pass through the point. Hence a junction *at a point only* is insufficient to provide the proper means of passage from sheet to sheet.

If the junction were effected by a common portion of surface, then a passage through it (implying an entrance into that portion and an exit from it) ought to change the sheet. But, in such a case, closed contours can be constructed which make such a passage without enclosing the branch-point a: thus the junction would cause a change of sheet for certain circuits the description of which ought to leave the z-variable in the original sheet. Hence a junction by a *continuous area of surface* does not provide proper means of passage from sheet to sheet.

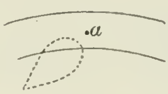

Fig. 51.

The only possible junction which remains is a line. The objection in the last case does not apply to a closed contour which does not contain the branch-point; for the line cuts the curve twice and there are therefore two crossings; the second of them makes the variable return to the sheet which the first crossing compelled it to leave.

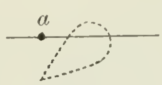

Fig. 52.

Hence the junction between any two sheets takes place along a line.

Such a line is called* a *branch-line*. The branch-points of a multiform function lie on the branch-lines, after the foregoing explanations; and a branch-line can be crossed by the variable only if the variable change its sheet at crossing, in the sequence prescribed by the branch-point of the function which lies on the line. Also, the sequence is reversed when the branch-line is crossed in the reversed direction.

Thus, if two sheets of a surface be connected along a branch-line, a point which crosses the line from the first sheet must pass into the second and a point which crosses the line from the second sheet must pass into the first.

Again, if, along a common direction of branch-line, the first sheet of a surface be connected with the second, the second with the third, and the third with

* Sometimes *cross-line*, sometimes *branch-section*. The German title is *Verzweigungschnitt;* the French is *ligne de passage;* see also the note on the equivalents of branch-point, p. 15.

the first, a point which crosses the line from the first sheet in one direction must pass into the second sheet, but if it cross the line in the other direction it must pass into the third sheet.

A branch-point does not necessarily affect all the branches of a function: when it affects only some of them, the corresponding property of the Riemann's surface is in evidence as follows. Let $z=a$ determine a branch-point affecting, say, only r branches. Take n points a, one in each of the sheets; and through them draw n lines *cab*, having the same geometrical position in the respective sheets. Then in the vicinity of the point a in each of the n sheets, associated with the r affected branches, there must be means of passage from each one to all the rest of them; and the lines *cab* can conceivably be the branch-lines with a properly established sequence. The point a does not affect the other $n-r$ branches: there is therefore no necessity for means of passage in the vicinity of a among the remaining $n-r$ sheets. In each of these remaining sheets, the point a and the line *cab* belong to their respective sheets alone: for them, the point a is not a branch-point and the line *cab* is not a branch-line.

174. Several essential properties of the branch-lines are immediate inferences from these conditions.

I. *A free end of a branch-line in a surface is a branch-point.*

Let a simple circuit be drawn round the free end so small as to enclose no branch-point (except the free end, if it be a branch-point). The circuit meets the branch-line once, and the sheet is changed because the branch-line is crossed; hence the circuit includes a branch-point which therefore can be only the free end of the line.

Note. A branch-line may terminate in the boundary of the surface, and then the extremity need not be a branch-point.

II. *When a branch-line extends beyond a branch-point lying in its course, the sequence of interchange is not the same on the two sides of the point.*

If the sequence of interchange be the same on the two sides of the branch-point, a small circuit round the point would first cross one part of the branch-line and therefore involve a change of sheet and then, in its course, would cross the other part of the branch-line in the other direction which, on the supposition of unaltered sequence, would cause a return to the initial sheet. In that case, a circuit round the branch-point would fail to secure the proper change of sheet. Hence the sequence of interchange caused by the branch-line cannot be the same on the two sides of the point.

III. *If two branch-lines with different sequences of interchange have a common extremity, that point is either a branch-point or an extremity of at least one other branch-line.*

If the point be not a branch-point, then a simple curve enclosing it, taken so small as to include no branch-point, must leave the variable in its initial sheet. Let A be such a point, AB and AC be two branch-lines having A for a common extremity; let the sequence be as in the figure, taken for a simple case; and suppose that the variable initially is in the rth sheet. A passage across AB makes the variable pass into the sth sheet. If there be no branch-line between AB and AC having an extremity at A, and if neither n nor m be s, then the passage across AC makes no change in the sheet of the variable and, therefore, in order to restore r before AB, at least one branch-line must lie in the angle between AC and AB, estimated in the positive trigonometrical sense.

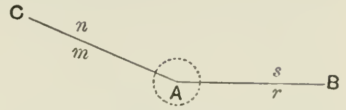

Fig. 53.

If either n or m, say n, be s, then after passage across AC, the point is in the mth sheet; then, since the sequences are not the same, m is not r and there must be some branch-line between AC and AB to make the point return to the rth sheet on the completion of the circuit.

If then the point A be not a branch-point, there must be at least one other branch-line having its extremity at A. This proves the proposition.

Corollary 1. *If both of two branch-lines extend beyond a point of intersection, which is not a branch-point, no sheet of the surface has both of them for branch-lines.*

Corollary 2. *If a change of sequence occur at any point of a branch-line, then either that point is a branch-point or it lies also on some other branch-line.*

Corollary 3. *No part of a branch-line with only one branch-point on it can be a closed curve.*

It is evidently superfluous to have a branch-line without any branch-point on it.

175. On the basis of these properties, we can obtain a system of branch-lines satisfying the requisite conditions which are:—

(i) the proper sequences of change from sheet to sheet must be secured by a description of a simple circuit round a branch-point: if this be satisfied for each of the branch-points, it will evidently be satisfied for any combination of simple circuits, that is, for any path whatever enclosing one or more branch-points.

(ii) the sheet, in which the variable re-assumes its initial value after describing a circuit that encloses no branch-point, must be the initial sheet.

In the z-plane of Cauchy's method, let lines be drawn from any point I, not a branch-point in the first instance, to each of the branch-points, as in fig. 19, p. 156, so that the joining lines do not meet except at I: and suppose the n-sheeted Riemann's surface to have branch-lines coinciding geometrically with these lines, as in § 173, and having the sequence of interchange for passage across each the same as the order in the cycle of functional values for a small circuit round the branch-point at its free end. No line (or part of a line) can be a closed curve; the lines need not be straight, but they will be supposed drawn as direct as possible to the points in angular succession.

The first of the above requisite conditions is satisfied by the establishment of the sequence of interchange.

To consider the second of the conditions, it is convenient to divide circuits into two kinds, (α) those which exclude I, (β) those which include I, no one of either kind (for our present purpose) including a branch-point.

A closed circuit, excluding I and all the branch-points, must intersect a branch-line an even number of times, if it intersect the line in real points. Let the figure (fig. 54) represent such a case: then the crossings at A and B counteract one another and so the part between A and B may without effect be transferred across IB_3 so as not to cut the branch-line at all. Similarly for the points C and D: and a similar transference of the part now between C and D may be made across the branch-line without effect: that is, the circuit can, without effect, be changed so as not to cut the branch-line IB_3 at all. A similar change can be made for each of the branch-lines: and so the circuit can, without effect, be changed into one which meets no branch-line and therefore, on its completion, leaves the sheet unchanged.

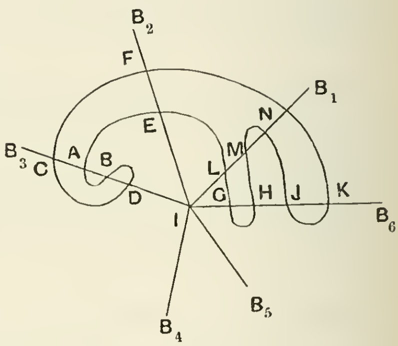

Fig. 54.

A closed circuit, including I but no branch-point, must meet each branch-line an odd number of times. A change similar in character to that in the previous case may be made for each branch-line: and without affecting the result, the circuit can be changed so that it meets each branch-line only once. Now the effect produced by a branch-line on the function is the same as the description of a simple loop round the branch-point which with I determines the branch-line: and therefore the effect of the circuit at present contemplated is, after the transformation which does not affect the result, the same as that of a circuit, in the previously adopted mode of representation, enclosing all the branch-points. But, by Cor. III. of § 90, the effect of a circuit which encloses all the branch-points (including $z = \infty$, if it be a branch-point) is to restore the value of the function which it had at the beginning of the circuit: and therefore in the present case the effect is to make the point return to the sheet in which it lay initially.

It follows therefore that, for both kinds of a closed circuit containing no branch-point, the effect is to make the z-variable return to its initial sheet on resuming its initial value at the close of the circuit.

Next, let the point I be a branch-point; and let it be joined by lines, as direct* as possible, to each of the other branch-points in angular succession. These lines will be regarded as the branch-lines; and the sequence of interchange for passage across any one is made that of the interchange prescribed by the branch-point at its free extremity.

The proper sequence of change is secured for a description of a simple closed circuit round each of the branch-points other than I. Let a small circuit be described round I; it meets each of the branch-lines once and therefore its effect is the same as, in the language of the earlier method of representing variation of z, that of a circuit enclosing all the branch-points except I. Such a circuit, when taken on the Neumann's sphere, as in Cor. III., § 90 and Ex. 2, § 104, may be regarded in two ways, according as one or other of the portions, into which it divides the area of the sphere, is regarded as the included area; in one way, it is a circuit enclosing all the branch-points except I, in the other it is a circuit enclosing I alone and no other branch-point. Without making any modification in the final value of w, it can (by § 90) be deformed, either into a succession of loops round all the branch-points save one, or into a loop round that one; the effect of these two deformations is therefore the same. Hence the effect of the small closed circuit round I meeting all the branch-lines is the same as, in the other mode of representation, that of a small curve round I enclosing no other branch-point; and therefore the adopted set of branch-lines secures the proper sequence of change of value for description of a circuit round I.

The first of the two necessary conditions is therefore satisfied by the present arrangement of branch-lines.

The proof, that the second of the two necessary conditions is also satisfied by the present arrangement of branch-lines, is similar to that in the preceding case, save that only the first kind of circuit of the earlier proof is possible.

It thus appears that a system of branch-lines can be obtained which secures the proper changes of sheet for a multiform function: and therefore Riemann's surfaces can be constructed for such a function, the essential property being that over its appropriate surface an otherwise multiform function of the variable is a uniform function.

* The reason for this will appear in §§ 183, 184.

The multipartite character of the function has its influence preserved by the character of the surface to which the function is referred: the surface, consisting of a number of sheets joined to one another, may be a multiply connected surface.

In thus proving the general existence of appropriate surfaces, there has remained a large arbitrary element in their actual construction: moreover, in particular cases, there are methods of obtaining varied configurations of branch-lines. Thus the assignment of the n branches to the n sheets has been left unspecified, and is therefore so far arbitrary: the point I, if not a branch-point, is arbitrarily chosen and so there is a certain arbitrariness of position in the branch-lines. Naturally, what is desired is the simplest appropriate surface: the particularisation of the preceding arbitrary qualities is used to derive a canonical form of the surface.

176. The discussion of one or two simple cases will help to illustrate the mode of junction between the sheets, made by branch-lines.

The simplest case of all is that in which the surface has only a single sheet: it does not require discussion.

The case next in simplicity is that in which the surface is two-sheeted: the function is therefore two-valued and is consequently defined by a quadratic equation of the form

$$Lu^2 + 2Mu + N = 0,$$

where L and M are uniform functions of z. When a new variable w is introduced, defined by $Lu + M = w$, so that values of w and of u correspond uniquely, the equation is

$$w^2 = M^2 - LN = P(z).$$

It is evident that every branch-point of u is a branch-point of w, and vice versa; hence the Riemann's surface is the same for the two equations. Now any root of $P(z)$ of odd degree is a branch-point of w. If then

$$P(z) = Q^2(z) R(z),$$

where $R(z)$ is a product of only simple factors, every factor of $R(z)$ leads to a branch-point. If the degree of $R(z)$ be even, the number of branch-points for finite values of the variable is even and $z = \infty$ is not a branch-point; if the degree of $R(z)$ be odd, the number of branch-points for finite values of the variable is odd and $z = \infty$ is a branch-point: in either case, the number of branch-points is even.

There are only two values of w, and the Riemann's surface is two-sheeted: crossing a branch-line therefore merely causes a change of sheet. The free ends of branch-lines are branch-points; a small circuit round any branch-point causes an interchange of the branches w, and a circuit round any two branch-points restores the initial value of w at the end and therefore leaves the variable in the same sheet as at the beginning. These are the essential requirements in the present case; all of them are satisfied by taking *each* *branch-line as a line connecting two (and only two) of the branch-points.* The ends of all the branch-lines are free: and their number, in this method, is one-half that of the (even) number of branch-points. A small circuit round a branch-point meets a branch-line once and causes a change of sheet; a circuit round two (and not more than two) branch-points causes either no crossing of branch-line or an even number of crossings and therefore restores the variable to the initial sheet.

A branch-line is, in this case, usually drawn in the form of a straight line when the surface is plane: but this form is not essential and all that is desirable is to prevent intersections of the branch-lines.

Note. Junction between the sheets along a branch-line is easily secured. The two sheets to be joined are cut along the branch-line. One edge of the cut in the upper sheet, say its right edge looking along the section, is joined to the left edge of the cut in the lower sheet; and the left edge in the upper sheet is joined to the right edge in the lower.

A few simple examples will illustrate these remarks as to the sheets: illustrations of closed circuits will arise later, in the consideration of integrals of multiform functions.

Ex. 1. Let
$$w^2 = A(z-a)(z-b),$$
so that a and b are the only branch-points. The surface is two-sheeted: the line ab may be made the branch-line. In Fig. 55 only part of the upper sheet is shewn*, as likewise only part of the lower sheet. Continuous lines imply what is visible; and dotted lines what is invisible, on the supposition that the sheets are opaque.

The circuit, closed in the surface and passing round O, is made up of OJK in the upper sheet: the point crosses the branch-line and then passes into the lower sheet, where it describes the dotted line KLH: it then meets and crosses the branch-line at H, passes into the upper sheet and in that sheet returns to O. Similarly of the line ABC, the part AB lies in the lower sheet, the part BC in the upper: of the line DG the part DE lies in the upper sheet, the part EFG in the lower, the piece FG of this part being there visible beyond the boundary of the retained portion of the upper surface.

Ex. 2. Let
$$\lambda w^2 = z^3 - a^3.$$
The branch-points (Fig. 56) are $A\ (=a)$, $B\ (=\alpha a)$, $C\ (=\alpha a^2)$, where α is a primitive cube root of unity, and $z=\infty$. The branch-lines can be made by BC, $A\infty$; and the two-sheeted surface will be a surface over which w is uniform. Only a part of each sheet is shewn in the figure; a section also is made at M across the surface, cutting the branch-line $A\infty$.

Ex. 3. Let
$$w^m = z^n,$$
where n and m are prime to each other. The branch-points are $z=0$ and $z=\infty$; and the branch-line extends from 0 to ∞. There are m sheets; if we associate them in order with the branches w_s, where
$$w_s = r^{\frac{n}{m}} e^{\frac{(n\theta+2s\pi)i}{m}}$$
for $s=1, 2, \ldots, m$, then the first sheet is connected with the second forwards, the second with the third forwards, and so on; the mth being connected with the first forwards.

* The form of the three figures in the plate opposite p. 346 is suggested by Holzmüller, *Einführung in die Theorie der isogonalen Verwandschaften und der conformen Abbildungen*, (Leipzig, Teubner, 1882), in which several illustrations are given.

The surface is sometimes also called a *winding-surface;* and a branch-point such as $z=0$ on the surface, where a number m of sheets pass into one another in succession, is also called a *winding-point* of order $m-1$ (see p. 15, note). An illustration of the surface for $m=3$ is given in Fig. 57, the branch-line being cut so as to shew the branching: what is visible is indicated by continuous lines; what is in the second sheet, but is invisible, is indicated by the thickly dotted line; what is in the third sheet, but is invisible, is indicated by the thinly dotted line.

Ex. 4. Consider a three-sheeted surface having four branch-points at a, b, c, d; and let each point interchange two branches, say, w_2, w_3 at a; w_1, w_3 at b; w_2, w_3 at c; w_1, w_2

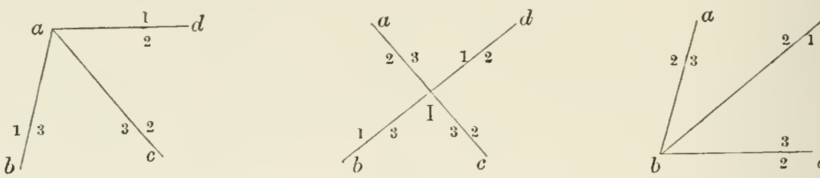

Fig. 58.

at d; the points being as in Fig. 58. It is easy to verify that these branch-points satisfy the condition that a circuit, enclosing them all, restores the initial value of w.

The branching of the sheets may be made as in the figure, the integers on the two sides of the line indicating the sheets that are to be joined along the line.

A canonical form for such a surface can be derived from the more general case given later (in §§ 186—189).

Ex. 5. Shew that, if the equation

$$f(w, z)=0$$

be of degree n in w and be irreducible, all the n sheets of the surface are connected, that is, it is possible by an appropriate path to pass from any sheet to any other sheet.

177. It is not necessary to limit the surface representing the variable to a set of planes; and, indeed, as with uniform functions, there is a convenience in using the sphere for the purpose.

We take n spheres, each of diameter unity, touching the Riemann's plane surface at a point A; each sphere is regarded as the stereographic projection of a plane sheet, with regard to the other extremity A' of the spherical diameter through A. Then, the sequence of these spherical sheets being the same as the sequence of the plane sheets, branch-points in the plane surface project into branch-points on the spherical surface: branch-lines between the plane sheets project into branch-lines between the spherical sheets and are terminated by corresponding points; and if a branch-line extend in the plane surface to $z=\infty$, the corresponding branch-line in the spherical surface is terminated at A'.

A surface will thus be obtained consisting of n spherical sheets; like the plane Riemann's surface, it is one over which the n-valued function is a uniform function of the position of the variable point.

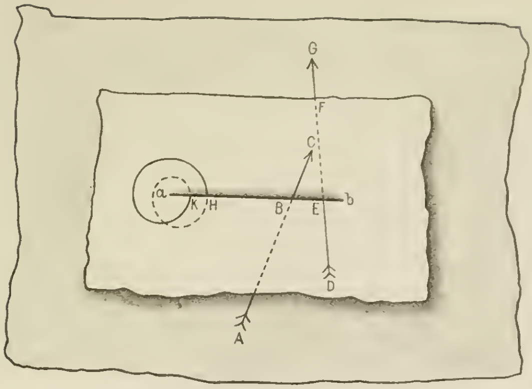

Fig. 55

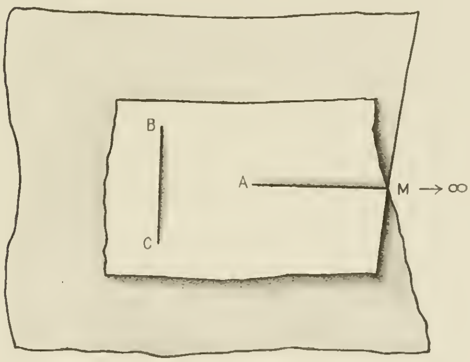

Fig. 56.

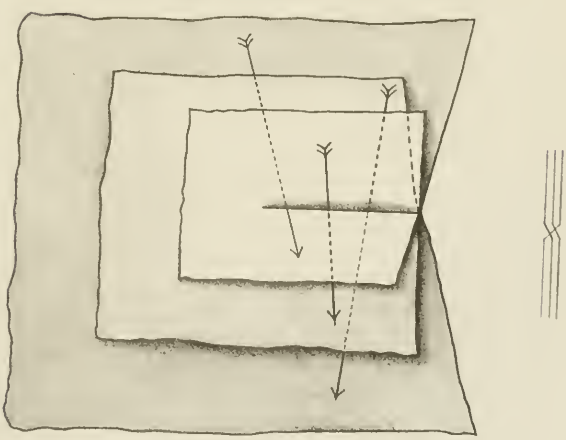

Fig. 57.

To face p. 346.

But also *the connectivity of the n-sheeted spherical surface is the same as that of the n-sheeted plane surface with which it is associated.*

In fact, the plane surface can be mechanically changed into the spherical surface without tearing, or repairing, or any change except bending and compression: all that needs to be done is that the n plane sheets shall be bent, without making any change in their sequence, each into a spherical form, and that the boundaries at infinity (if any) in the plane-sheet shall be compressed into an infinitesimal point, being the South pole of the corresponding spherical sheet or sheets. Any junctions between the plane sheets extending to infinity are junctions terminated at the South pole. As the plane surface has a boundary, which, if at infinity on one of the sheets, is therefore not a branch-line for that sheet, so the spherical surface has a boundary which, if at the South pole, cannot be the extremity of a branch-line.

178. We proceed to obtain the connectivity of a Riemann's surface: it is determined by the following theorem:—

Let the total number of branch-points in a Riemann's n-sheeted surface be r; and let the number of branches of the function interchanging at the first point be m_1, *the number interchanging at the second be* m_2, *and so on. Then the connectivity of the surface is*

$$\Omega - 2n + 3,$$

where Ω *denotes* $m_1 + m_2 + \dots + m_r - r$.

Take* the surface in the bounded spherical form, the connectivity N of which is the same as that of the plane surface: and let the boundary be a small hole A in the outer sheet. By means of cross-cuts and loop-cuts, the surface can be resolved into a number of distinct simply connected pieces.

First, make a slice bodily through the sphere, the edge in the outside sheet meeting A and the direction of the slice through A being chosen so that none of the branch-points lie in any of the pieces cut off. Then n parts, one from each sheet and each simply connected, are taken away. The remainder of the surface has a cup-like form; let the connectivity of this remainder be M.

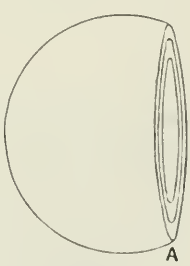

Fig. 59.

This slice has implied a number of cuts.

The cut made in the outside sheet is a cross-cut, because it begins and ends in the boundary A. It divides the surface into two distinct pieces, one being the portion of the outside sheet cut off, and this piece is simply connected;

* The proof is founded on Neumann's, pp. 168—172.

hence, by Prop. III. of § 160, the remainder has its connectivity still represented by N.

The cuts in all the other sheets, caused by the slice, are all loop-cuts, because they do not anywhere meet the boundary. There are $n-1$ loop-cuts, and each cuts off a simply connected piece; and the remaining surface is of connectivity M. Hence, by Prop. V. of § 161,

$$M + n - 1 = N + 2(n-1),$$

and therefore

$$M = N + n - 1.$$

In this remainder, of connectivity M, make $r-1$ cuts, each of which begins in the rim and returns to the rim, and is to be made through the n sheets together; and choose the directions of these cuts so that each of the r resulting portions of the surface contains one (and only one) of the branch-points.

Consider the portion of the surface which contains the branch-point where m_1 sheets of the surface are connected. The m_1 connected sheets constitute a piece of a winding-surface round the winding-point of order $m_1 - 1$; the remaining sheets are unaffected by the winding-point, and therefore the parts of them are $n - m_1$ distinct simply connected pieces. The piece of winding-surface is simply connected; because a circuit, that does not contain the winding-point, is reducible without passing over the winding-point, and a circuit, that does contain the winding-point, is reducible to the winding-point, so that no irreducible circuit can be drawn. Hence the portion of the surface under consideration consists of $n - m_1 + 1$ distinct simply connected pieces.

Similarly for the other portions. Hence the total number of distinct simply connected pieces is

$$\begin{aligned} &\sum_{q=1}^{r} (n - m_q + 1) \\ &= nr - \sum_{q=1}^{r} m_q + r \\ &= nr - \Omega. \end{aligned}$$

But in the portion of connectivity M each of the $r-1$ cuts causes, in each of the sheets, a cut passing from the boundary and returning to the boundary, that is, a cross-cut. Hence there are n cross-cuts from each of the $r-1$ cuts, and therefore $n(r-1)$ cross-cuts altogether, made in the portion of surface of connectivity M.

The effect of these $n(r-1)$ cross-cuts is to resolve the portion of connectivity M into $nr - \Omega$ distinct simply connected pieces; hence, by § 160,

$$M = n(r-1) - (nr - \Omega) + 2,$$

and therefore

$$N = M - (n-1) = \Omega - 2n + 3,$$

the connectivity of the Riemann's surface.

The quantity Ω, having the value $\sum_{q=1}^{r}(m_q-1)$, may be called the *ramification* of the surface, as indicating the aggregate sum of the orders of the different branch-points.

Note. The surface just considered is a closed surface to which a point has been assigned for boundary; hence, by Cor. I., Prop. III., § 164, its connectivity is an odd integer. Let it be denoted by $2p+1$; then

$$2p = \Omega - 2n + 2,$$

and $2p$ is the number of cross-cuts which change the Riemann's surface into one that is simply connected.

The integer p is often called (Cor. I., Prop. III., § 164) the *class* of the Riemann's surface; and *the equation*

$$f(w, z) = 0$$

is said to be of class p, *when* p *is the class of the associated Riemann's surface.*

Ex. 1. When the equation is

$$w^2 = \lambda(z-a)(z-b),$$

we have a two-sheeted surface, $n=2$. There are two branch-points, $z=a$ and $z=b$; but $z=\infty$ is not a branch-point; so that $r=2$. At each of the branch-points the two values are interchanged, so that $m_1=2$, $m_2=2$; thus $\Omega=2$. Hence the connectivity $=2-4+3=1$, that is, the surface is simply connected.

The surface can be deformed, as in the example in § 169, into a sphere.

Ex. 2. When the equation is

$$\begin{aligned} w^2 &= 4z^3 - g_2 z - g_3 \\ &= 4(z-e_1)(z-e_2)(z-e_3), \end{aligned}$$

we have $n=2$. There are four branch-points, viz., e_1, e_2, e_3, ∞, so that $r=4$; and at each of them the two values of w are interchanged, so that $m_s=2$ (for $s=1, 2, 3, 4$), and therefore $\Omega=8-4=4$. Hence the connectivity is $4-4+3$, that is, 3; and the value of p is unity.

Similarly, the surface associated with the equation

$$w^2 = U(z),$$

where $U(z)$ is a rational, integral, algebraical function of degree $2m-1$ or of degree $2m$, is of connectivity $2m+1$; so that $p=m$. The equation

$$w^2 = (1-z^2)(1-k^2z^2)$$

is of class $p=1$. The case next in importance is that of the algebraical equation leading to the hyperelliptic functions, when U is either a quintic or a sextic; and then $p=2$.

Ex. 3. Obtain the connectivity of the Riemann's surface associated with the equation

$$w^3 + z^3 - 3awz = 1,$$

where a is a constant, (i) when a is zero, (ii) when a is different from zero.

Ex. 4. Shew that, if the surface associated with the equation

$$f(w, z) = 0,$$

have μ boundary-lines instead of one, and if the equation have the same branch-points as in the foregoing proposition, the connectivity is $\Omega - 2n + \mu + 2$.

179. The consideration of irreducible circuits on the surface at once reveals the multiple connection of the surface, the numerical measure of which has been obtained. In a Riemann's surface, a simple closed *circuit cannot be deformed over a branch-point.* Let A be a branch-point, and let $AE...$ be the branch-line having a free end at A. Take a curve $...CED...$ crossing the branch-line at E and passing into a sheet different from that which contains the portion CE; and, if possible, let a slight deformation of the curve be made so as to transfer the portion CE across the branch-point A. In the deformed position, the curve $...C'E'D'...$ does not meet the branch-line; there is, consequently, no change of sheet in its course near A and therefore $E'D'...$, which is the continuation of $...C'E'$, cannot be regarded as the deformed position of ED. The two paths are essentially distinct; and thus the original path cannot be deformed over the branch-point.

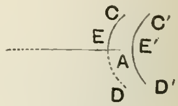

Fig. 60.

It therefore follows that continuous deformation of a circuit over a branch-point on a Riemann's surface is a geometrical impossibility.

Ex. Trace the variation of the curve CED, as the point E moves up to A and then returns along the other side of the branch-line.

Hence a circuit containing two or more of the branch-points is irreducible; but a circuit containing all the branch-points is equivalent to a circuit that contains none of them, and it is therefore reducible.

If a circuit contain only one branch-point, it can be continuously deformed so as to coincide with the point on each sheet and therefore, being deformable into a point, it is a reducible circuit. An illustration has already occurred in the case of a portion of winding-surface containing a single winding-point (p. 348); all circuits drawn on it are reducible.

It follows from the preceding results that the Riemann's surface associated with a multiform function is generally one of multiple connection; we shall find it convenient to know how it can be resolved, by means of cross-cuts, into a simply connected surface. The representative surface will be supposed a closed surface with a single boundary; its connectivity, necessarily odd, being $2p + 1$, the number of cross-cuts necessary to resolve the surface into one that is simply connected is $2p$; when these cuts have been made, the simply connected surface then obtained will have its boundary composed of a single closed curve.

One or two simple examples of resolution of special Riemann's surfaces will be useful in leading up to the general explanation; in the examples it will be shewn how, in conformity with § 168, the resolving cross-cuts render irreducible circuits impossible.

Ex. 1. Let the equation be

$$w^2 = A(z-a)(z-b)(z-c)(z-d),$$

where a, b, c, d are four distinct points, all of finite modulus. The surface is two-sheeted; each of the points a, b, c, d is a branch-point where the two values of w interchange; and so the surface, assumed to have a single boundary, is triply connected, the value of p being unity. The branch-lines are two, each connecting a pair of branch-points; let them be ab and cd.

Two cross-cuts are necessary and sufficient to resolve the surface into one that is simply connected. We first make a cross-cut, beginning at the boundary B, (say it is in the upper sheet), continuing in that sheet and returning to B, so that its course encloses the branch-line ab (but not cd) and meets no branch-line. It is a cross-cut, and not a loop-cut, for it begins and ends in the boundary; it is evidently a cut in the upper sheet alone, and does not divide the surface into distinct portions; and, once made, it is to be regarded as boundary for the partially cut surface.

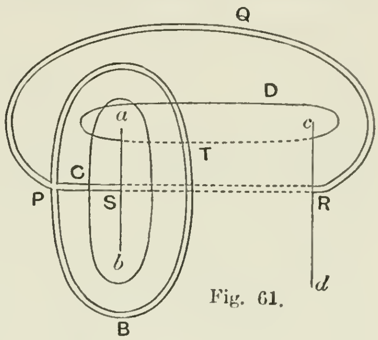

Fig. 61.

The surface in its present condition is connected: and therefore it is possible to pass from one edge to the other of the cut just made. Let P be a point on it; a curve that passes from one edge to the other is indicated by the line PQR in the upper sheet, RS in the lower, and SP in the upper. Along this line make a cut, beginning at P and returning to P; it is a cross-cut, partly in the upper sheet and partly in the lower, and it does not divide the surface into distinct portions.

Two cross-cuts in the triply connected surface have now been made; neither of them, as made, divides the surface into distinct portions, and each of them when made reduces the connectivity by one unit; hence the surface is now simply connected. It is easy to see that the boundary consists of a single line not intersecting itself; for beginning at P, we have the outer edge of PBT, then the inner edge of $PQRSP$, then the inner edge of PTB, and then the outer edge of $PSRQP$, returning to P.

The required resolution has been effected.

Before the surface was resolved, a number of irreducible circuits could be drawn; a complete system of irreducible circuits is composed of two, by § 168. Such a system may be taken in various ways; let it be composed of a simple curve C lying in the upper sheet and containing the points a and b, and a simple curve D, lying partly in the upper and partly in the lower sheet and containing the points a and c; each of these curves is irreducible, because it encloses two branch-points. Every other irreducible circuit is reconcileable with these two; the actual reconciliation in particular cases is effected most simply when the surface is taken in a spherical form.

The irreducible circuit C on the unresolved surface is impossible on the resolved surface owing to the cross-cut $SPQRS$; and the irreducible circuit D on the unresolved surface is impossible on the resolved surface owing to the cross-cut PTB. It is easy to verify that no irreducible circuit can be drawn on the resolved surface.

In practice, it is conveniently effective to select a complete system of irreducible simple circuits and then to make the cross-cuts so that each of them renders one circuit of the system impossible on the resolved surface.

Ex. 2. If the equation be

$$w^2 = 4z^3 - g_2 z - g_3$$
$$= 4(z - e_1)(z - e_2)(z - e_3),$$

the branch-points are e_1, e_2, e_3 and ∞. When the two-sheeted surface is spherical, and the branch-lines are taken to be (i) a line joining e_1, e_2; and (ii) a line joining e_3 to the South pole, the discussion of the surface is similar in detail to that in the preceding example.

Ex. 3. Let the equation be

$$w^2 = Az(1-z)(\kappa - z)(\lambda - z)(\mu - z),$$

and for simplicity suppose that κ, λ, μ are real quantities subject to the inequalities

$$1 < \kappa < \lambda < \mu < \infty.$$

The associated surface is two-sheeted and has a boundary assigned to it; assuming that its sheets are planes, we shall take some point in the finite part of the upper sheet, not being a branch-point, as the boundary. There are six branch-points, viz., 0, 1, κ, λ, μ, ∞ at each of which the two values of w interchange; and so the connectivity of the surface is 5 and its class, p, is 2. The branch-lines can be taken as three, this being the simplest arrangement; let them be the lines joining 0, 1; κ, λ; μ, ∞.

Four cross-cuts are necessary to resolve the surface into one that is simply connected and has a single boundary. They may be obtained as follows.

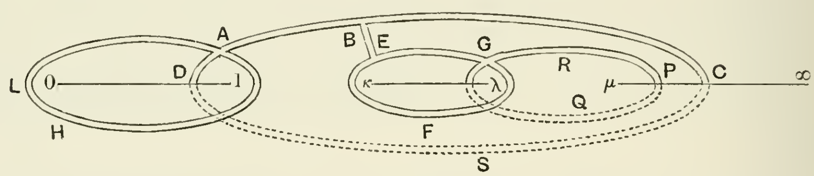

Fig. 62.

Beginning at the boundary L, let a cut LHA be made entirely in the upper sheet along a line which, when complete, encloses the points 0 and 1 but no other branch-points; let the cut return to L. This is a cross-cut and it does not divide the surface into distinct pieces; hence, after it is made, the connectivity of the modified surface is 4, and there are two boundary lines, being the two edges of the cut LHA.

Beginning at a point A in LHA, make a cut along ABC in the upper sheet until it meets the branch-line $\mu\infty$, then in the lower sheet along CSD until it meets the branch-line 01, and then in the upper sheet from D returning to the initial point A. This is a cross-cut and it does not divide the surface into distinct pieces; hence, after it is made, the connectivity of the modified surface is 3, and it is easy to see that there is only one boundary edge, similar to the single boundary in *Ex.* 1 when the surface in that example has been completely resolved.

Make a loop-cut EFG along a line, enclosing the points κ and λ but no other branch-points; and change it into a cross-cut by making a cut from E to some point B of the boundary. This cross-cut can be regarded as $BEFGE$, ending at a point in its own earlier course. As it does not divide the surface into distinct pieces, the connectivity is reduced to 2; and there are two boundary lines.

Beginning at a point G make another cross-cut $GQPRG$, as in the figure, enclosing the two branch-points λ and μ and lying partly in the upper sheet and partly in the lower. It does not divide the surface into distinct pieces: the connectivity is reduced to unity and there is a single boundary line.

Four cross-cuts have been made; and the surface has been resolved into one that is simply connected.

It is easy to verify:

(i) that neither in the upper sheet, nor in the lower sheet, nor partly in the upper sheet and partly in the lower, can an irreducible circuit be drawn in the resolved surface; and

(ii) that, owing to the cross-cuts, the simplest irreducible circuits in the unresolved surface—viz. those which enclose $0, 1$; $1, \kappa$; κ, λ; λ, μ; respectively—are rendered impossible in the resolved surface.

The equation in the present example, and the Riemann's surface associated with it, lead to the theory of hyperelliptic functions*.

180. The last example suggests a method of resolving any two-sheeted surface into a surface that is simply connected.

The number of its branch-points is necessarily even, say $2p+2$. The branch-lines can be made to join these points in pairs, so that there will be $p+1$ of them. To determine the connectivity (§ 178), we have $n=2$ and, since two values are interchanged at every branch-point, $\Omega = 2p+2$; so that the connectivity is $2p+1$. Then $2p$ cross-cuts are necessary for the required resolution of the surface.

We make cuts round p of the branch-lines, that is, round all of them but one; each cut is made to enclose two branch-points, and each lies entirely in the upper sheet. These are cuts corresponding to the cuts LHA and EFG in fig. 62; and, as there, the cut round the first branch-line begins and ends in the boundary, so that it is a cross-cut. All the remaining cuts are loop-cuts at present. The system of p cuts we denote by $a_1, a_2, \dots, a_p$.

We make other p cuts, one passing from the inner edge of each of the p cuts a already made to the branch-line which it surrounds, then in the lower sheet to the $(p+1)$th branch-line, and then in the upper sheet returning to the point of the outer edge of the cut a at which it began. This system of cuts corresponds to the cuts $ADSCBA$ and $GQPRG$ in fig. 62. Each of them can be taken so as to meet no one of the cuts a except the one in which it begins and ends; and they can be taken so as not to meet one another. This system of p cuts we denote by $b_1, b_2, \dots, b_p$, where b_r is the cut which begins and ends in a_r. All these cuts are cross-cuts, because they begin and end in boundary-lines.

Lastly, we make other $p-1$ cuts from a_r to b_{r-1}, for $r = 2, 3, \dots, p$, all in

* One of the most direct discussions of the theory from this point of view is given by Prym, *Neue Theorie der ultraelliptischen Functionen*, (Berlin, Mayer and Müller, 2nd ed., 1885).

the upper sheet; no one of them, except at its initial and its final points, meets any of the cuts already made. This system of $p-1$ cuts we denote by $c_2, c_3, \ldots, c_p$.

Because b_{r-1} is a cross-cut, the cross-cut c_r changes a_r (hitherto a loop-cut) into a cross-cut when c_r and a_r are combined into a single cut.

It is evident that no one of these cuts divides the surface into distinct pieces; and thus we have a system of $2p$ cross-cuts resolving the two-sheeted surface of connectivity $2p+1$ into a surface that is simply connected. The cross-cuts in order* are

$$a_1, b_1, c_2 \text{ and } a_2, b_2, c_3 \text{ and } a_3, b_3, \ldots, c_p \text{ and } a_p, b_p.$$

181. This resolution of a general two-sheeted surface suggests† Riemann's general resolution of a surface with any (finite) number of sheets.

As before, we assume that the surface is closed and has a single boundary and that its class is p, so that $2p$ cross-cuts are necessary for its resolution into one that is simply connected.

Make a cut in the surface such as not to divide it into distinct pieces; and let it begin and end in the boundary. It is a cross-cut, say a_1; it changes the number of boundary-lines to 2 and it reduces the connectivity of the cut surface to $2p$.

Since the surface is connected, we can pass in the surface along a continuous line from one edge of the cut a_1 to the opposite edge. Along this line make a cut b_1: it is a cross-cut, because it begins and ends in the boundary. It passes from one edge of a_1 to the other, that is, from one boundary-line to another. Hence, as in Prop. II. of § 164, it does not divide the surface into distinct pieces; it changes the number of boundaries to 1 and it reduces the connectivity to $2p-1$.

The problem is now the same as at first, except that now only $2p-2$ cross-cuts are necessary for the required resolution. We make a loop-cut a_2, not resolving the surface into distinct pieces, and a cross-cut c_1 from a point of a_2 to a point on the boundary at b_1; then c_1 and a_2, taken together, constitute a cross-cut that does not resolve the surface into distinct pieces. It therefore reduces the connectivity to $2p-2$ and leaves two pieces of boundary.

The surface being connected, we can pass in the surface along a continuous line from one edge of a_2 to the opposite edge. Along this line we make a cut b_2, evidently a cross-cut, passing, like b_1 in the earlier case, from one boundary-line to the other. Hence it does not divide the surface into

* See Neumann, pp. 178—182; Prym, *Zur Theorie der Functionen in einer zweiblättrigen Fläche*, (1866).

† Riemann, *Ges. Werke*, pp. 122, 123; Neumann, pp. 182—185.

distinct pieces; it changes the number of boundaries to 1 and it reduces the connectivity to $2p-3$.

Proceeding in p stages, each of two cross-cuts, we ultimately obtain a simply connected surface with a single boundary; and the general effect on the original unresolved surface is to have a system of cross-cuts somewhat of the form

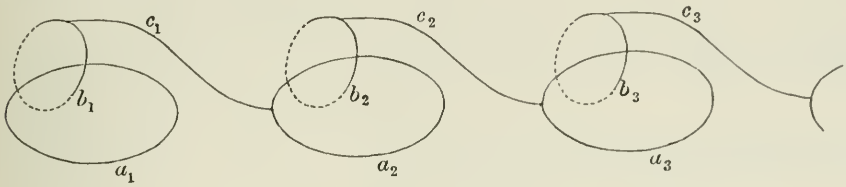

Fig. 63.

The foregoing resolution is called the *canonical resolution* of a Riemann's surface.

Ex. 1. Construct the Riemann's surface for the equation

$$w^3+z^3-3awz=1,$$

both for $a=0$ and for a different from zero; and resolve it by cross-cuts into a simply connected surface with a single boundary, shewing a complete system of irreducible simple circuits on the unresolved surface.

Ex. 2. Shew that the Riemann's surface for the equation

$$w^3=\frac{(z-a)(z-b)}{(z-c)(z-d)}$$

is of class $p=2$; indicate the possible systems of branch-lines, and, for each system, resolve the surface by cross-cuts into a simply connected surface with a single boundary.

(Burnside.)

182. Among algebraical equations with their associated Riemann's surfaces, two general cases of great importance and comparative simplicity distinguish themselves. The first is that in which the surface is two-sheeted; round each branch-point the two branches interchange. The second is that in which, while the surface has a finite number of sheets greater than two, all the branch-points are of the first order, that is, are such that round each of them only two branches of the function interchange. The former has already been considered, in so far as concerns the surface; we now proceed to the consideration of the latter.

The equation is $$f(w, z)=0,$$

of degree n in w; and, for our present purpose, it is convenient to regard $f=0$ as an equation corresponding to a generalised plane curve of degree n, so that no term in f is of dimensions higher than n.

The total number of branch-points has been proved, in § 98, to be

$$n(n-1)-2\delta-2\kappa,$$

where δ is the number of points which are the generalisation of double points on the curve with non-coincident tangents and κ is the number of double points on the curve with coincident tangents. Round each of these branch-points, two branches of w interchange and only two, so that all the numbers m_q of § 178 are equal to 2; hence the ramification Ω is

$$2\{n(n-1)-2\delta-2\kappa\}-\{n(n-1)-2\delta-2\kappa\},$$

that is, $$\Omega=n(n-1)-2\delta-2\kappa.$$

The connectivity of the surface is therefore

$$n(n-1)-2\delta-2\kappa-2n+3;$$

and therefore the class p of the surface is

$$\tfrac{1}{2}(n-1)(n-2)-\delta-\kappa.$$

Now this integer is known* as the *deficiency* of the curve; and therefore it appears that *the deficiency of the curve is the same as the class of the Riemann surface associated with its equation, and also is the same as the class of its equation.*

Moreover, the number of branch-points of the original equation is Ω, that is,

$$\begin{aligned}&=2p+2n-2\\&=2\{p+(n-1)\}.\end{aligned}$$

Note. The equality of these numbers, representing the deficiency and the class, is one among many reasons that lead to the close association of algebraic functions (and of functions dependent on them) with the theory of plane algebraic curves, in the investigations of Nöther, Brill, Clebsch and others, referred to in §§ 191, 242.

183. With a view to the construction of a canonical form of Riemann's surface of class p for the equation under consideration, it is necessary to consider in some detail the relations between the branches of the functions as they are affected by the branch-points.

The effect produced on any value of the function by the description of a small circuit, enclosing one branch-point (and only one), is known. But when the small circuit is part of a loop, the effect on the value of the function with which the loop begins to be described depends upon the form of the loop; and various results (e.g. Ex. 1, § 104) are obtained by taking different loops. In the first form (§ 175) in which the branch-lines were established as junctions between sheets, what was done was the equivalent

* Salmon's *Higher Plane Curves*, §§ 44, 83; Clebsch's *Vorlesungen über Geometrie*, (edited by Lindemann), t. i, pp. 351—429, the German word used instead of deficiency being *Geschlecht*. The name 'deficiency' was introduced by Cayley in 1865: see *Proc. Lond. Math. Soc.*, vol. i., "On the transformation of plane curves."

of drawing a number of straight loops, which had one extremity common to all and the other free, and of assigning the law of junction according to the law of interchange determined by the description of the loop. As, however, there is no necessary limitation to the forms of branch-lines, we may draw them in other forms, always, of course, having branch-points at their free extremities; and according to the variation in the form of the branch-line, (that is, according to the variation in the form of the corresponding loop or, in other words, according to the deformation of the loop over other branch-points from some form of reference), there will be variation in the law of junction along the branch-lines.

There is thus a large amount of arbitrary character in the forms of the branch-lines, and consequently in the laws of junction along the branch-lines, of the sheets of a Riemann's surface. Moreover, the assignment of the n branches of the function to the n sheets is arbitrary. Hence a considerable amount of arbitrary variation in the configuration of a Riemann's surface is possible within the limits imposed by the invariance of its connectivity. The canonical form will be established by making these arbitrary elements definite.

184. After the preceding explanation and always under the hypothesis that the branch-points are simple, we shall revert temporarily to the use of loops and shall ultimately combine them into branch-lines.

When, with an ordinary point as origin, we construct a loop round a branch-point, two and only two of the values of the function are affected by that particular loop; they are interchanged by it; but a different form of loop, from the same origin round the same branch-point, might affect some other pair of values of the function.

To indicate the law of interchange, a symbol will be convenient. If the two values interchanged by a given loop be w_i and w_m, the loop will be denoted by im; and i and m will be called the numbers of the symbol of that loop.

For the initial configuration of the loops, we shall (as in § 175) take an ordinary point O: we shall make loops beginning at O, forming them in the sequence of angular succession of the branch-points round O and drawing the double linear part of the loop as direct as possible from O to its branch-point: and, in this configuration, we shall take the law of interchange by a loop to be the law of interchange by the branch-point in the loop.

In any other configuration, the symbol of a loop round any branch-point depends upon its form, that is, depends upon the deformation over other branch-points which the loop has suffered in passing from its initial form. The effect of such deformation must first be obtained: it is determined by the following lemma:—

When one loop is deformed over another, the symbol of the deformed loop is unaltered, if neither of its numbers or if both of its numbers occur in the symbol of the unmoved loop; but if, before deformation, the symbols have one number common, the new symbol of the deformed loop is obtained from the old symbol by substituting, for the common number, the other number in the symbol of the unmoved loop.

The sufficient test, to which all such changes must be subject, is that the effect on the values of the function at any point of a contour enclosing both branch-points is the same at that point for all deformations into two loops. Moreover, a complete circuit of all the loops is the same as a contour enclosing all the branch-points; it therefore (Cor. III. § 90) restores the initial value with which the circuit began to be described.

Obviously there are three cases.

First, when the symbols have no number common: let them be mn, rs. The branch-point in the loop rs does not affect w_m or w_n: it is thus effectively not a branch-point for either of the values w_m and w_n; and therefore (§ 91) the loop mn can be deformed across the point, that is, it can be deformed across the loop mn.

Secondly, when the symbols are the same: the symbol of the deformed loop must be unaltered, in order that the contour embracing only the two branch-points may, as it should, restore after its complete description each of the values affected.

Thirdly, when the symbols have one number common: let O be any point and let the loops be OA, OB in any given position such as (i), Fig. 64, with symbols mr, nr respectively. Then OB may be deformed over OA as in (ii), or OA over OB as in (iii).

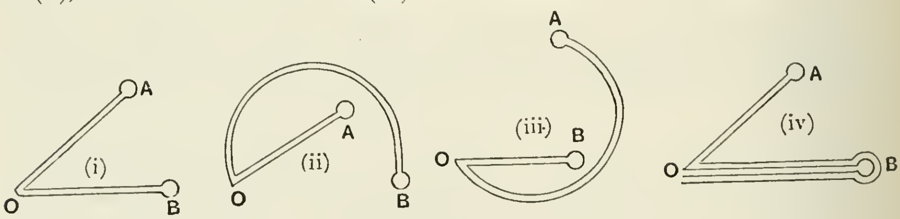

Fig. 64.

The effect at O of a closed circuit, including the points A and B and described positively beginning at O, is, in (i) which is the initial configuration, to change w_m into w_r, w_r into w_n, w_n into w_m; this effect on the values at O, unaltered, must govern the deformation of the loops.

The two alternative deformations (ii) and (iii) will be considered separately.

When, as in (ii), OB is deformed over OA, then OA is unmoved and therefore unaltered: it is still mr. Now, beginning at O with w_m, the loop

OA changes w_m into w_r: the whole circuit changes w_m into w_r, so that OB must now leave w_r unaltered. Again, beginning with w_n, it is unaltered by OA, and the whole circuit changes w_n into w_m: hence OB must change w_n into w_m, that is, the symbol of OB must be mn. And, this being so, an initial w_r at O is changed by the whole circuit into w_n, as it should be. Hence the new symbol mn of the deformed loop OB in (ii) is obtained from the old symbol by substituting, for the common number r, the other number m in the symbol of the unmoved loop OA.

We may proceed similarly for the deformation in (iii); or the new symbol may be obtained as follows. The loop OA in (iii) may be deformed to the form in (iv) without crossing any branch-point and therefore without changing its symbol. When this form of the loop is described in the positive direction, w_n initially at O is changed into w_r after the first loop OB, for this loop has the position of OB in (i), then it is changed into w_m after the loop OA, for this loop has the position of OA in (i), and then w_m is unchanged after the second (and inner) loop OB. Thus w_n is changed into w_m, so that the symbol is mn, a symbol which is easily proved to give the proper results with an initial value w_m or w_r for the whole contour. This change is as stated in the theorem, which is therefore proved.

Ex. If the deformation from (i) to (ii) be called superior, and that from (i) to (iii) inferior, then x successive superior deformations give the same loop-configuration, in symbols and relative order for positive description, as $6-x$ successive inferior deformations.

COROLLARY. *A loop can be passed unchanged over two loops that have the same symbol.*

Let the common symbol of the unmoved loops be mn. If neither number of the deformed loop be m or n, passage over each of the loops mn makes no difference, after the lemma; likewise, if its symbol be mn. If only one of its numbers, say n, be in mn, its symbol is nr, where r is different from m. When the loop nr is deformed over the first loop mn, its new symbol is mr; when this loop mr is deformed over the second loop mn, its new symbol is nr, that is, the final symbol is the same as the initial symbol, or the loop is unchanged.

185. The initial configuration of the loops is used by Clebsch and Gordan to establish their simple cycles and thence to deduce the periodicity of the Abelian integrals connected with the equation $f(w, z)=0$, without reference to the Riemann's surface; and this method of treating the functions that arise through the equation, always supposed to have merely simple branch-points, has been used by Casorati* and Lüroth†.

We can pass from any value of w at the initial point O to any other

* *Annali di Matematica*, 2da Ser., t. iii, (1870), pp. 1—27.

† *Abh. d. K. bay. Akad.* t. xvi, i Abth., (1887), pp. 199—241.

value by a suitable series of loops; because, were it possible to interchange the values of only some of the branches, an equation could be constructed which had those branches for its roots. The fundamental equation could then be resolved into this equation and an equation having the rest of the branches for its roots: that is, the fundamental equation would cease to be irreducible.

We begin then with any loop, say one connecting w_1 with w_2. There will be a loop, connecting the value w_3 with either w_1 or w_2; there will be a loop, connecting the value w_4 with either w_1, w_2, or w_3; and so on, until we select a loop, connecting the last value w_n with one of the other values. Such a set of loops, $n-1$ in number, is called *fundamental.*

A passage round the set will not at the end restore the branch with which the description began. When we begin with any value, any other value can be obtained after the description of properly chosen loops of the set.

Any other loop, when combined with a set of fundamental loops, gives a system the description of suitably chosen loops of which restores some initial value; only two values can be restored by the description of loops of the combined system. Thus if the loops in order be 12, 13, 14, ..., $1n$ and a loop qr be combined with them, the value w_q is changed into w_1 by $1q$, into w_r by $1r$, into w_q by qr; and similarly for w_r. Such a combination of n loops is called a *simple cycle.*

The total number of branch-points, and therefore of loops, is (§ 182)

$$2\{p+(n-1)\};$$

and therefore the total number of simple cycles is $2p+n-1$. But these simple cycles are not independent of one another.

In the description of any cycle, the loops vary in their operation according to the initial value of w: and, for two different initial values of w, no loop is operative in the same way. For otherwise all the preceding and all the succeeding loops would operate in the same way and would lead, on reversal, to the same initial value of w. Hence a loop of a given cycle can be operative in only two descriptions, once when it changes, say, w_i into w_j, and the other when it changes w_j into w_i.

Now consider the circuit made up of all the loops. When w_1 is taken as the initial value, it is restored at the end: and in the description only a certain number of loops have been operative: the cycle made up of these loops can be resolved into the operative parts of simple cycles, that is, into simple cycles: hence one relation among the simple cycles is given by the consideration of the operative loops when the whole system of the loops is described with an initial value.

Similarly when any other initial value is taken; so that apparently there are n relations, one arising from each initial value. These n relations are not independent: for a simultaneous combination of the operations of all the loops in all the circuits leads to an identically null effect (but no smaller combination would be effective), for each loop is operative twice (and only twice) with opposite effects, shewing that one and only one of the relations is derivable from the remainder. Hence there are $n-1$ independent relations and therefore* the number of independent simple cycles is $2p$.

186. We now proceed to obtain a typical form of the Riemann's surface by deforming the initial configuration of the loops into a typical configuration†. The final arrangement of the loops is indicated by the two theorems:—

I. *The loops can be made in pairs in which all loop-symbols are of the form* $(m, m+1)$, *for* $m = 1, 2, \ldots, n-1$. (With this configuration, w_1 can be changed by a loop only into w_2, w_2 by a loop only into w_3, and so on in succession, each change being effected by an even number of loops.) This theorem is due to Lüroth.

II. *The loops can be made so that there is only one pair* 12, *only one pair* 23, …, *only one pair* $(n-2, n-1)$, *and the remaining* $p+1$ *pairs are* $(n-1, n)$. This theorem is due to Clebsch.

187. We proceed to prove Lüroth's theorem, assuming that the loops have the initial configuration of § 184.

Take any loop 12, say OA: beginning it with w_1, describe loops positively and in succession; then as the value w_1 is restored sooner or later, for it must be restored by the circuit of all the loops, let it be restored first by a loop OB, the symbol of OB necessarily containing the number 1. Between OA and OB there may be loops whose symbols contain 1 but which have been inoperative. Let each of these in turn be deformed so as to pass back over all the loops between its initial position and OA; and then finally over OA. Before passing over OA its symbol must contain 1, for there is no loop over which it has passed that, having 1 in its symbol, could make it drop 1 in the passage; but it cannot contain 2, for, if it did, the effect of OA and the deformed loop would be to restore 1, an effect that would have been caused in the original position, contrary to the hypothesis that OB is the first loop that restores 1. Hence after it has passed over OA its symbol no longer contains 1.

* Clebsch und Gordan, *Theorie der Abel'schen Functionen*, p. 85.

† The investigation is based upon the following memoirs:—

Lüroth, "Note über Verzweigungsschnitte und Querschnitte in einer Riemann'schen Fläche," *Math. Ann.*, t. iv, (1871), pp. 181—184; "Ueber die kanonischen Perioden der Abel'schen Integrale," *Abh. d. K. bay. Akad.*, t. xv, ii Abth., (1885), pp. 329—366.

Clebsch, "Zur Theorie der Riemann'schen Flächen," *Math. Ann.*, t. vi, (1873), pp. 216—230.

Clifford, "On the canonical form and dissection of a Riemann's Surface," *Lond. Math. Soc. Proc.*, vol. viii, (1877), pp. 292—304.

Next, pass OB over the loops between its initial position and OA but not over OA: its symbol must be 12 in the deformed position since w_1 is restored by the loop OB. Then OA and the deformed loop OB are each 12; hence each of the loops, between the new position and the old position of OB, can be passed over OA and the new loop OB without any change in its symbol. There are therefore, behind OA, a series of loops that do not affect w_1. Thus the loops are

(a) loops behind OA not affecting w_1, (b) OA, OB each 12,
(c) other loops beyond the initial position of OB.

Begin now with w_2 at the loop OB and again describe loops positively and in succession: then w_2 must be restored sooner or later. It may be only after OA is described, so that there has been a complete circuit of all the loops; or it may first be by an intermediate loop, say OC.

For the former case, when OA is the first loop by which w_2 is restored, we deform as follows. Deform all loops affecting w_1, which lie between OB and OA, in the positive direction from OB back over other loops and over OB. The symbol of each just before its deformation contains 1 but not 2, and therefore after its deformation it does not contain 1. Moreover just after OB is described, w_1 is the value, and just before OA is described, w_1 is the value; hence the intermediate loops, which have affected w_1, must be even in number. Let OG be the first after OB which affects w_1, and let the symbol of OG be $1r$. Then beginning OG with w_1, the value w_1 must be restored by a complete circuit of all the loops, that is, it must be restored by OB; and therefore the value must be w_1 when beginning OA, or w_1 must be restored before OA. Let OH be the first loop after OG to restore w_1; then, by proceeding as above, we can deform all the loops between OG and OH over OG, with the result that no such deformed loop affects w_1 and that OG and OH are both $1r$. Hence all the loops affecting w_1 can be arranged in pairs having the same symbol.

Since OG and OH are a pair with the same symbol, every loop between OB and OG can be passed unchanged over OG and OH together. When this is done, pass OG over OB so that it becomes $2r$, and then OH over OB so that it also is $2r$. Thus these deformed loops OG, OH are a pair $2r$; and therefore OA can, without change, be deformed over both so as to be next to OB. Let this be done with all the pairs; then, finally, we have

(a) loops not affecting w_1, (b) a pair with the symbol 12,
(c) pairs affecting w_2 and not w_1, (d) loops not affecting w_1.

We thus have a pair 12 and loops not affecting w_1, so that such a change has been effected as to make all the loops affecting w_1 possess the symbol 12.

For the second case, when OC is the first loop to restore w_2, the value with which the loop OB whose symbol is 12 began to be described, we treat the loops between OB and OC in a manner similar to that adopted in the former case for loops between OA and OB; so that, remembering that now w_2 instead of the former w_1 is the value dealt with in the recurrence, we can deform these loops into

(*a*) loops behind OB which change w_1 but not w_2,

(*b*) OB and OC, the symbol of each of which is 12.

Now OB was next to OA; hence the set (*a*) are now next to OA. Each of them when passed over OA drops the number 1 from its symbol and so the whole system now consists of

(*a*) loops behind OA not affecting w_1, (*b*) OA, OB, OC each of which is 12, (*c*) other loops.

Begin again with the value w_1 before OA. Before OC the value is w_1; and the whole circuit of the loops must restore w_1, which must therefore occur before OA. Let OD be the first loop by which w_1 is restored. Then treating the loops between OC and OD, as formerly those between the initial positions of OA and OB were treated, we shall have

(*a*) loops behind OA not affecting w_1, (*b*) OA, OB each being 12, (*c*) loops between OB and OC not affecting w_1, (*d*) OC, OD each being 12, (*e*) other loops.

Except that fewer loops affecting w_1 have to be reckoned with, the configuration is now in the same condition as at the end of the first stage. Proceeding therefore as before, we can arrange that all the loops affecting w_1 occur in pairs with the symbol 12. Moreover, each of the loops in the set (*c*) can be passed unchanged over OA and OB; so that, finally, we have

(*a*) pairs of loops with the symbol 12, (*b'*) loops not affecting w_1.

We keep (*a*) in pairs, so that any desired deformation of loops in (*b'*) over them can be made without causing any change; and we treat the set (*b'*) in the same manner as before, with the result that the set (*b'*) is replaced by

(*b*) pairs of loops with the symbol 23, (*c'*) loops not affecting w_1 or w_2.

And so on, with the ultimate result that *the loops can be made in pairs in which each symbol is of the form* $(m, m+1)$ *for* $m = 1, \dots, n-1$.

188. We now come to Clebsch's Theorem that the loops thus made can be so deformed that there is only one pair 12, only one pair 23, and so on, until the last symbol $(n-1, n)$, which is the common symbol of $p+1$ pairs.

This can be easily proved after the establishment of the lemma that, *if there be two pairs* 12 *and one pair* 23, *the loops can be deformed into one pair* 12 *and two pairs* 23.

The actual deformation leading to the lemma is shewn in the accompanying scheme: the deformations implied by the continuous lines are those of a loop from the left to the right of the respective lines, and those implied by the dotted lines are those of a loop from the right to the left of the respective lines. It is interesting to draw figures, representing the loops in the various configurations.

12	12	12	12	23	23
12	12	12	23	13	23
12	12	23	13	13	23
12	12	13	13	23	23
12	23	12	13	23	23
12	23	23	12	23	23
12	12	23	23	23	23

By the continued use of this lemma we can change all but one of the pairs 12 into pairs 23, all but one of the pairs 23 into pairs 34, and so on, the final configuration being that there are one pair 12, one pair 23, ... and $p+1$ pairs $(n-1, n)$. Thus Clebsch's theorem is proved.

189. We now proceed to the construction of the Riemann's surface.

Each loop is associated with a branch-point, and the order of interchange for passage round the branch-point, by means of the loop, is given by the numbers in the symbol of the loop.

Hence, in the configuration which has been obtained, there are two branch-points 12: we therefore connect them (as in § 176) by a line, not necessarily along the direction of the two loops 12 but necessarily such that it can, without passing over any branch-point, be deformed into the lines of the two loops; and we make this the branch-line between the first and the second sheets. There are two branch-points 23: we connect them by a line not meeting the former branch-line, and we make it the branch-line between the second and the third sheets. And so on, until we come to the last two sheets. There are $2p+2$ branch-points $n-1, n$: we connect these in pairs (as in § 176) by $p+1$ lines, not meeting one another or any of the former lines, and we make them the $p+1$ branch-lines between the last two sheets.

It thus appears that, *when the winding-points of a Riemann's surface with n sheets of connectivity $2p+1$ are all simple, the surface can be taken in such a form that there is a single branch-line between consecutive sheets except for the last two sheets: and between the last two sheets there are $p+1$ branch-lines.* This form of Riemann's surface may be regarded as the canonical form for a surface, all the branch-points of which are simple.

Further, let AB be a branch-line such as 12. Let two points P and Q be taken in the first sheet on opposite sides of AB, so that PQ in space is infinitesimal; and let P' be the point in the second sheet determined by the same value of z as P, so that $P'Q$ in the sheet is infinitesimal. Then the value w_1 at P is changed by a loop round A (or round B) into a value at Q differing only infinitesimally from w_2, which is the value at P': that is, the change in the function from Q to P' is infinitesimal. Hence *the value of the function is continuous across a line of passage from one sheet to another.*

190. The class of the foregoing surface is p; and it was remarked, in § 170, that a convenient surface of reference of the same class is that of a solid sphere with p holes bored through it. It is, therefore, proper to indicate the geometrical deformation of a Riemann's surface of this canonical form into a p-holed sphere.

The Riemann's surface consists of n sheets connected chainwise each with a single branch-line to the sheet on either side of it, except that the first is connected only with the second and that the last two have $p+1$ branch-lines. We may also consider the whole surface as spherical and the sequence of the sheets from the inside outwards: and the outmost sheet can be considered as bounded.

Let the branch-line between the first and the second sheets be made to lie along part of a great circle. Let the first sheet of the Riemann's surface be reflected in the plane of this great circle: the line becomes a long narrow hole along the great circle, and the reflected sheet becomes a large indentation in the second sheet. Reversing the process of § 169, we can change the new form of the second sheet, so that it is spherical again: it is now the inmost of the $n-1$ sheets of the surface, the connectivity and the ramification of which are unaltered by the operation.

Let this process be applied to each surviving inner sheet in succession. Then, after $n-2$ operations, there will be left a two-sheeted surface; the outer sheet is bounded and the two sheets are joined by $p+1$ branch-lines; so that the connectivity is still $2p+1$. Let these branch-lines be made to lie along a great circle: and let the inner surface be reflected in the plane of this circle. Then, after the reflexion, each of the branch-lines becomes a long narrow hole along the great circle; and there are two spherical surfaces which pass continuously into one another at these holes, the outer of the surfaces being bounded. By stretching one of the holes and flattening the two surfaces, the new form is that of a bifacial flat surface: each of the p holes then becomes a hole through the body bounded by that surface; the stretched hole gives the extreme geometrical limits of the extension of the surface, and the original boundary of the outer surface becomes a boundary hole existing in only one face. The body can now be distended until it takes the form of a sphere, and the final form is that of the surface of a solid sphere with p holes bored through it and having a single boundary.

This is the normal surface of reference (§ 170) of connectivity $2p+1$.

As a last ground of comparison between the Riemann's surface in its canonical form and the surface of the bored sphere, we may consider the system of cross-cuts necessary to transform each of them into a simply connected surface.

We begin with the spherical surface. The simplest irreducible circuits are of two classes, (i) those which go round a hole, (ii) those which go through a hole; the cross-cuts, $2p$ in number, which make the surface simply connected, must be such as to prevent these irreducible circuits.

Round each of the holes we make a cut a, the first of them beginning and ending in the boundary: these cuts prevent circuits through the holes. Through each hole we make a cut b, beginning and ending at a point in the corresponding cut a: we then make from the first b a cut c_1 to the second a, from the second b a cut c_2 to the third a, and so on. The surface is then simply connected: a_1 is a cross-cut, b_1 is a cross-cut, $c_1 + a_2$ is a cross-cut, b_2 is a cross-cut, $c_2 + a_3$ is a cross-cut, and so on. The total number is evidently $2p$, the number proper for the reduction; and it is easy to verify that there is a single boundary.

To compare this dissection with the resolution of a Riemann's surface by cross-cuts, say of a two-sheeted surface (the n-sheeted surface was transformed into a two-sheeted surface), it must be borne in mind that only p of the $p+1$ branch-lines were changed into holes and the remaining one, which, after the partial deformation, was a hole of the Riemann's surface, was stretched out so as to give the boundary.

It thus appears that the direction of a cut a round a hole in the normal surface of reference is a cut round a branch-line in one sheet, that is, it is a cut a as in the resolution (§ 180) of the Riemann's surface into one that is simply connected.

Again, a cut b is a cut from a point in the boundary across a cut a and through the hole back to the initial point; hence, in the Riemann's surface, it is a cut from some one assigned branch-line across a cut a_r, meeting the branch-line surrounded by a_r, passing into the second sheet and, without meeting any other cut or branch-line in that surface, returning to the initial point on the assigned branch-line. It is a cut b as in the resolution of the Riemann's surface.

Lastly, a cut c is made from a cut b to a cut a. It is the same as in the resolution of the Riemann's surface, and the purpose of each of these cuts is to change each of the loop-cuts a (after the first) into cross-cuts.

A simple illustration arises in the case of a two-sheeted Riemann's surface, of class $p=2$. The various forms are:

(i) the surface of a two-holed sphere, with the directions of cross-cuts that resolve it into a simply connected surface; as in (i), Fig. 65, B, K being at opposite edges of the cut c_1 where it meets a_2: H, C at opposite edges where it meets b_1: and so on;

(ii) the spherical surface, resolved into a simply connected surface, bent, stretched, and flattened out; as in (ii), Fig. 65;

(iii) the plane Riemann's surface, resolved by the cross-cuts; as in Fig. 63, p. 355.

Numerous illustrations of transformations of Riemann's surfaces are given by Hofmann, *Methodik der stetigen Deformation von zweiblättrigen Riemann'schen Flächen*, (Halle a. S., Nebert, 1888).

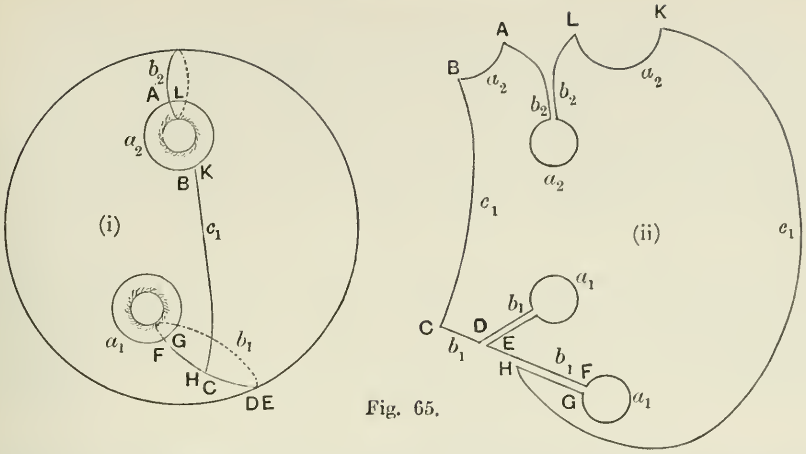

Fig. 65.

191. We have seen that a bifacial surface with a single boundary can be deformed, at least geometrically, into any other bifacial surface with a single boundary, provided the two surfaces have the same connectivity; and the result is otherwise independent of the constitution of the surface, in regard to sheets and to form or position of branch-lines. Further, in all the geometrical deformations adopted, the characteristic property is the uniform correspondence of points on the surfaces.

Now with every Riemann's surface, in its initial form, an algebraical equation $f(w, z) = 0$ is associated; but when deformations of the surface are made, the relations that establish uniform correspondence between different forms, practically by means of conformal representation, are often of a transcendental character (Chap. XX.). Hence, when two surfaces are thus equivalent to one another, and when points on the surfaces are determined solely by the variables in the respective algebraical equations, no relations other than algebraical being taken into consideration, the uniform correspondence of points can only be secured by assigning a new condition that there be uniform transformation between the variables w and z of one surface and the variables w' and z' of the other surface. And, when this condition is satisfied, the equations are such that the deficiencies of the two (generalised) curves represented by the equations are the same, because they are equal to the common connectivity. It may therefore be expected that, when the variables in an equation are subjected to uniform transformation, the class of the equation is unaltered; or in other words that the deficiency of a curve is an invariant for uniform transformation.

This inference is correct: the actual proof is more directly connected with geometry and the theory of Abelian functions, and must be sought elsewhere*. The result is of importance in justifying the adoption of a simple normal surface of the same class as a surface of reference.

* Clebsch's *Vorlesungen über Geometrie*, t. i, p. 459, where other references are given; Salmon's *Higher Plane Curves*, pp. 93, 319; Clebsch und Gordan, *Theorie der Abel'schen Functionen*, Section 3; Brill, *Math. Ann.*, t. vi, pp. 33—65.

CHAPTER XVI.

Algebraic Functions and their Integrals.

192. In the preceding chapter sufficient indications have been given as to the character of the Riemann's surface on which the n-branched function w, determined by the equation

$$f(w, z) = 0,$$

can be represented as a uniform function of the position of the variable. It is unnecessary to consider algebraically multiform functions of position on the surface, for such multiformity would merely lead to another surface of the same kind, on which the algebraically multiform functions would be uniform functions of position; transcendentally multiform functions of position will arise later, through the integrals of algebraic functions. It therefore remains, at the present stage, only to consider the most general uniform function of position on the Riemann's Surface.

On the other hand, it is evident that a Riemann's Surface of any number of sheets can be constructed, with arbitrary branch-points and assigned sequence of junction; the elements of the surface being subject merely to general laws, which give a necessary relation between the number of sheets, the ramification and the connectivity, and which require the restoration of any value of the function after the description of some properly chosen irreducible circuit. The essential elements of the arbitrary surface, and the merely general laws indicated, are independent of any previous knowledge of an algebraical equation associated with the surface; and a question arises whether, when a Riemann's surface is given, an associated algebraical equation necessarily exists.

Two distinct subjects of investigation, therefore, arise. The first is the most general uniform function of position on a surface associated with a given algebraical equation, and its integral; the second is the discussion of the existence of functions of position on a surface that is given independently of an algebraical equation. Both of them lead, as a matter of fact, to the theory of transcendental (that is, non-algebraical) functions of the most general type, commonly called Abelian transcendents. But the first is, naturally, the more direct, in that the algebraical equation is initially given: whereas, in the second, the prime necessity is the establishment of the so-called Existence-Theorem—that such functions, algebraical and transcendental, exist.

193. Taking the subjects of investigation in the assigned order, we suppose the fundamental equation to be irreducible, and algebraical as regards both the dependent and the independent variable; the general form is therefore

$$w^n G_0(z) + w^{n-1} G_1(z) + \dots + w G_{n-1}(z) + G_n(z) = 0,$$

the coefficients $G_0(z)$, $G_1(z)$, ..., $G_n(z)$ being rational, integral, algebraical functions.

The infinities of w are, by § 95, the zeros of $G_0(z)$ and, possibly, $z=\infty$. But, for our present purpose, no special interest attaches to the infinity of a function, as such; we therefore take $wG_0(z)$ as a new dependent variable, and the equation then is

$$f(w, z) = w^n + w^{n-1} g_1(z) + \dots + w g_{n-1}(z) + g_n(z) = 0,$$

in which the functions $g(z)$ are rational, integral, algebraical functions of z.

The distribution of the branches for a value of z which is an ordinary point, and the determination of the branch-points together with the cyclical grouping of the branches round a branch-point, may be supposed known. When the corresponding n-sheeted Riemann's surface (say of connectivity $2p+1$) is constructed, then w is a uniform function of position on the surface.

Now not merely w, but every rational function of w and z, is a uniform function of position on the surface; and its branch-points (though not necessarily its infinities) are the same as that of the function w.

Conversely, *every uniform function of position on the Riemann's surface, having accidental singularities and infinities only of finite order, is an algebraical rational function of w and z.* The proof* of this proposition, to which we now proceed, leads to the canonical expression for the most general uniform function of position on the surface, an expression which is used in Abel's Theorem in transcendental integrals.

Let w' denote the general uniform function, and let w_1', w_2', ..., w_n' denote the branches of this function for the points on the n sheets determined by

* The proof adopted follows Prym, *Crelle*, t. lxxxiii, (1877), pp. 251—261; see also Klein, *Ueber Riemann's Theorie der algebraischen Functionen und ihrer Integrale*, p. 57.

the algebraical magnitude z; and let $w_1, w_2, \ldots, w_n$ be the corresponding branches of w for the magnitude z. Then the quantity

$$w_1^s w_1' + w_2^s w_2' + \ldots + w_n^s w_n',$$

where s is any positive integer, is a symmetric function of the possible values of $w^s w'$; it has the same value in whatever sheet z may lie and by whatever path z may have attained its position in that sheet; the said quantity is therefore a uniform function of z. Moreover, all its singularities are accidental in character, by the initial hypothesis as to w' and the known properties of w; they are finite in number; and therefore the uniform function of z is algebraical. Let it be denoted by $h_s(z)$, which is an integral function only when the singularities are for infinite values of z; then

$$w_1^s w_1' + w_2^s w_2' + \ldots + w_n^s w_n' = h_s(z),$$

an equation which is valid for any positive integer s, there being of course the suitable changes among the rational integral algebraical functions $h(z)$ for changes in s. It is unnecessary to take $s \geqslant n$, when the equations for the values $0, 1, \ldots, n-1$ of s are retained: for the equations corresponding to values of $s \geqslant n$ can be derived, from the n equations that are retained, by using the fundamental equation determining w.

Solving the equations

$$\begin{aligned} w_1' + w_2' + \ldots + w_n' &= h_0(z), \\ w_1 w_1' + w_2 w_2' + \ldots + w_n w_n' &= h_1(z), \\ \ldots\ldots\ldots &\vdots \ldots\ldots\ldots \\ w_1^{n-1} w_1' + \ldots + w_n^{n-1} w_n' &= h_{n-1}(z), \end{aligned}$$

to determine w_1', we have

$$w_1' \begin{vmatrix} 1, & 1, & \ldots, & 1 \\ w_1, & w_2, & \ldots, & w_n \\ w_1^2, & w_2^2, & \ldots, & w_n^2 \\ \ldots & \ldots & \ldots & \ldots \\ w_1^{n-1}, & w_2^{n-1}, & \ldots, & w_n^{n-1} \end{vmatrix} = \begin{vmatrix} h_0(z), & 1, & \ldots, & 1 \\ h_1(z), & w_2, & \ldots, & w_n \\ h_2(z), & w_2^2, & \ldots, & w_n^2 \\ \ldots & \ldots & \ldots & \ldots \\ h_{n-1}(z), & w_2^{n-1}, & \ldots, & w_n^{n-1} \end{vmatrix}.$$

The right-hand side is evidently divisible by the product of the differences of $w_2, w_3, \ldots, w_n$; and this product is a factor of the coefficient of w_1'. Then, if

$$(w - w_2)(w - w_3) \ldots (w - w_n) = \sum_{r=1}^{n} k_r w^{n-r},$$

where k_1 is unity, we have, on removing the common factor,

$$w_1' = \frac{k_n h_0(z) + k_{n-1} h_1(z) + \ldots + k_2 h_{n-2}(z) + h_{n-1}(z)}{(w_1 - w_2)(w_1 - w_3) \ldots (w_1 - w_n)}.$$

But $$f(w, z) = (w - w_1)(w - w_2) \dots (w - w_n),$$
so that
$$k_2 = w_1 + g_1(z),$$
$$k_3 = w_1^2 + w_1 g_1(z) + g_2(z),$$
$$\dots\dots\dots \vdots \dots\dots\dots$$
$$k_n = w_1^{n-1} + w_1^{n-2} g_1(z) + \dots + g_{n-1}(z).$$
When these expressions for k are substituted in the numerator of the expression for w_1', it takes the form of a rational integral algebraical function of w of degree $n-1$ and of z, say
$$h_0(z) w_1^{n-1} + H_1(z) w_1^{n-2} + \dots + H_{n-2}(z) w_1 + H_{n-1}(z).$$
The denominator is evidently $\partial f/\partial w_1$, when w is replaced by w_1 after differentiation, so that we now have
$$w_1' = \frac{h_0(z) w_1^{n-1} + \dots + H_{n-1}(z)}{\partial f/\partial w_1}.$$
The corresponding form holds for each of the branches of w': and therefore we have
$$w' = \frac{h_0(z) w^{n-1} + H_1(z) w^{n-2} + \dots + H_{n-1}(z)}{\partial f/\partial w}$$
$$= \frac{h_0(z) w^{n-1} + H_1(z) w^{n-2} + \dots + H_{n-1}(z)}{n w^{n-1} + (n-1) w^{n-2} g_1(z) + \dots + g_{n-1}(z)},$$
so that w' is a rational, algebraical, function of w and z. The proposition is therefore proved.

By eliminating w between $f(w, z) = 0$ and the equation which expresses w' in terms of w and z, or by the use of § 99, it follows that w' satisfies an algebraical equation
$$F(w', z) = 0,$$
where F is of order n in w'; the equations $f(w, z) = 0$ and $F(w', z) = 0$ have the same Riemann's surface associated with them*.

194. It thus appears that there are uniform functions of position on the Riemann's surface just as there are uniform functions of position in a plane. The preceding proposition is limited to the case in which the infinities, whether at branch-points or not, are merely accidental; had the function possessed essential singularities, the general argument would still be valid, but the forms of the uniform functions $h(z)$ would no longer be algebraical. In fact, taking account of the difference in the form of the surface on which the independent variable is represented, we can extend to multiform functions, which are uniform on a Riemann's surface, those propositions for uniform functions which relate to expansion near an ordinary point or a singularity or, by using the substitution of § 93, a branch singularity, those which relate to continuation of functions, and so on;

* See § 191. Functions related to one another, as w and w' are, are called *gleichverzweigt*, Riemann, p. 93.

and their validity is not limited, as in Cor. VI., § 90, to a portion of the surface in which there are no branch-points.

Thus we have the theorem that *a uniform algebraical function of position on the Riemann's surface has as many zeros as it has infinities.*

This theorem may be proved as follows.

The function is a rational algebraical function of w and z. If it be also integral, let it be $w' = U(w, z)$, where U is integral.

Then the number of the zeros of w' on the surface is the number of simultaneous roots common to the two equations $U(w, z) = 0$, $f(w, z) = 0$. If u_λ and f_μ denote the aggregates of the terms of highest dimensions in these equations—say of dimensions λ and μ respectively—then $\lambda\mu$ is the number of common roots, that is, the number of zeros of w'.

The number of points, where w' assumes a value A, is the number of simultaneous roots common to the equations $U(w, z) = A$, $f(w, z) = 0$, that is, it is $\lambda\mu$ as before. Hence there are as many points where w' assumes a given value as there are zeros of w'; and therefore the number of the infinities is the same as the number of zeros. The number of infinities can also be obtained by considering them as simultaneous roots common to $u_\lambda = 0$, $f_\mu = 0$.

If the function be not integral, it can (§ 193) be expressed in the form $w' = \dfrac{U(w, z)}{V(w, z)}$, where U and V are integral, rational algebraical functions. The zeros of w' are the zeros of U and the infinities of V, the numbers of which, by what precedes, are respectively the same as the infinities of U and the zeros of V. The latter are the infinities of w'; and therefore w' has as many zeros as it has infinities.

Note. When the numerator and the denominator of a uniform fractional function of z have a common zero, we divide both of them by their greatest common measure; and the point is no longer a common zero of their new forms. But when the numerator $U(w, z)$ and the denominator $V(w, z)$ of a uniform function of position on a Riemann's surface have a common zero, so that there are simultaneous values of w and z for which both vanish, U and V do not necessarily possess a rational common factor; and then the common zero cannot be removed.

It is not difficult to shew that this possibility does not affect the preceding theorem.

195. In the case of uniform functions it was seen that, as soon as their integrals were considered, deviations from uniformity entered. Special investigations indicated the character of the deviations and the limitations to their extent. Incidentally, special classes of functions were introduced, such as many-valued functions, the values differing by multiples of a constant; and thence, by inversion, simply-periodic functions were deduced.

So, too, when multiform functions defined by an algebraical equation are considered, it is necessary to take into special account the deviations from uniformity of value on the Riemann's surface which may be introduced by processes of integration. It is, of course, in connexion with the branch-points that difficulties arise; but, as the present method of representing the variation of the variable is distinct from that adopted in the case of uniform functions, it is desirable to indicate how we deal with, not merely branch-points, but also singularities of functions when the integrals of such functions are under consideration. In order to render the ideas familiar and to avoid prolixity in the explanations relating to general integrals, we shall, after one or two propositions, discuss again some of the instances given in Chapter IX., taking the opportunity of stating general results as occasion may arise.

One or two propositions already proved must be restated: the difference from the earlier forms is solely in the mode of statement, and therefore the reasoning which led to their establishment need not be repeated.

I. *The path of integration between any two points on a Riemann's surface can, without affecting the value of the integral, be deformed in any possible continuous manner that does not make the path pass over any discontinuity of the subject of integration.*

This proposition is established in § 100.

II. *A simple closed curve on a Riemann's surface, which is a path of integration, can, without affecting the value of the integral, be deformed in any possible continuous manner that does not make the curve pass over any discontinuity of the subject of integration.*

Since the curve on the surface is closed, the initial and the final points are the same; the initial branch of the function is therefore restored after the description of the curve. This proposition is established in Corollary II., § 100.

III. *If the path of integration be a curve between two points on different sheets, determined by the same algebraical value of z, the curve is not a closed curve; it must be regarded as a path between the two points;* its deformation is subject to Proposition I.

No restatement, from Chapter IX., of the value of an integral, along a path which encloses a branch-point, is necessary. The method of dealing with the point when that value is infinite will be the same as the method of dealing with other infinities of the function.

196. We have already obtained some instances of multiple-valued functions, in the few particular integrals in Chapter IX.; the differences in the values of the functions, arising as integrals, consist solely of multiples of constants. The way in which these constants enter in Riemann's method is as follows.

When the surface is simply connected, there is no substantial difference from the previous theory for uniform functions; we therefore proceed to the consideration of multiply connected surfaces.

On a general surface, of any connectivity, take any two points z_0 and z. As the surface is one of multiple connection, there will be at least two essentially distinct paths between z_0 and z, that is, paths which cannot be reduced to one another; one of these paths can be deformed so as to be made equivalent to a combination of the other with some irreducible circuit. Let z_1 denote the extremity of the first path, and let z_2 denote the same point when regarded as the extremity of the second; then the difference of the two paths is an irreducible circuit passing from z_1 to z_2. When this circuit is made impossible by a cross-cut C passing through the point z, then z_1 and z_2 may be regarded as points on the opposite edges of the cross-cut: and the irreducible circuit on the unresolved surface becomes a path on the partially resolved surface passing from one edge of the cross-cut to the other.

When the surface is resolved by means of the proper number of cross-cuts into a simply connected surface, there is still a path in the surface from z_1 to z_2 on opposite edges of the cross-cut C: and all paths between z_1 and z_2 in the resolved surface are reconcileable with one another. One such path will be taken as the canonical path from z_1 to z_2; it evidently does not meet any of the cross-cuts, so that we consider only those paths which do not intersect any cross-cut.

If then Z be the function of position on the surface to be integrated, the value of the integral for the first path from z_0 to z_1 is

$$\int_{z_0}^{z_1} Zdz;$$

and for the second path it is $\int_{z_0}^{z_2} Zdz,$

or, by the assigned deformation of the second path, it is

$$\int_{z_0}^{z_1} Zdz + \int_{z_1}^{z_2} Zdz,$$

the second integral being taken along the canonical path from z_1 to z_2 in the surface, that is, along the irreducible circuit of canonical form, which would be possible in the otherwise resolved surface were the cross-cut C obliterated.

The difference of the values of the integral is evidently

$$\int_{z_1}^{z_2} Zdz,$$

which is therefore the change made in the value of the integral $\int_{z_0}^{z} Zdz$, when the upper limit passes from one edge of the cross-cut to the other; let it be denoted by I. As the curve is, in general, an irreducible circuit, this integral I may not, in general, be supposed zero.

We can arbitrarily assign the positive and the negative edges of some one cross-cut, say A. The edges of a cross-cut B that meets A are defined to be positive and negative as follows: when a point moves from one edge of B to the other, by describing the positive edge of A in a direction that is to the right of the negative edge of A, the edge of B on which the point initially lies is called its *positive* edge, and the edge of B on which the point finally lies is called its *negative* edge. And so on with the cross-cuts in succession.

The lower limit of the integral determining the modulus for a cross-cut is taken to lie on the negative edge, and the upper on the positive edge.

Regarding a point ζ on the cross-cut as defining two points z_1 and z_2 on opposite edges which geometrically are coincident, we now prove that *for all points on the cross-cut which can be reached from ζ without passing over any other cross-cut, when the surface is resolved into one that is simply connected, the integral I is a constant.* For, if ζ' be such a point, defining z_1' and z_2' on opposite edges, then $z_1 z_2 z_2' z_1' z_1$ is a circuit on the simply connected surface, which can be made evanescent; and it will be assumed that no infinities of Z lie in the surface within the circuit, an assumption which will be taken into account in §§ 197, 199. Therefore the integral of Z, taken round the circuit, is zero. Hence

$$\int_{z_1}^{z_2} Zdz + \int_{z_2}^{z_2'} Zdz + \int_{z_2'}^{z_1'} Zdz + \int_{z_1'}^{z_1} Zdz = 0,$$

that is,

$$\int_{z_1}^{z_2} Zdz - \int_{z_1'}^{z_2'} Zdz = \int_{z_1}^{z_1'} Zdz - \int_{z_2}^{z_2'} Zdz.$$

Along the direction of the cross-cut, the function Z is uniform: and therefore Zdz is the same for each element of the two edges, so long as the cross-cut is not met by any other. Hence the sums of the elements on the two edges are the same for all points on the cross-cut that can be reached from ζ without meeting a new cross-cut. The two integrals on the right-hand side of the foregoing equation are equal to one another, and therefore also those on the left-hand side, that is,

$$\int_{z_1}^{z_2} Zdz = \int_{z_1'}^{z_2'} Zdz,$$

which shews that *the integral I is constant for different points on a portion of cross-cut that is not met by any other cross-cut.*

If however the cross-cut be met by another cross-cut C', two cases arise according as C' has only one extremity, or has both extremities, on C.

First, let C' have only one extremity O on C. By what precedes, the integral is constant along OP, and it is constant along OQ; but we cannot infer that it is the same constant for the two parts. The preceding proof fails in this case; the distance $z_2 z_2'$ in the resolved surface is not infinitesimal, and therefore there is no element Zdz for $z_2 z_2'$ to be the same as the element for $z_1 z_1'$. Let I_2 be the constant for OP, I_1 that for QO; and let QP be the negative edge. Then

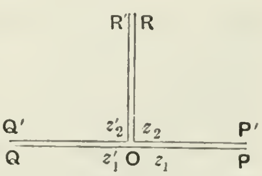

Fig. 66.

$$I_2 = \int_{z_1}^{z_2} Zdz, \quad I_1 = \int_{z_1'}^{z_2'} Zdz.$$

Let I' be the constant value for the cross-cut OR, and let OR be the negative edge; then

$$I' = \int_{z_2}^{z_2'} Zdz.$$

In the completely resolved surface, a possible path from z_2 to z_2' is z_2 to z_1, z_1 to z_1', z_1' to z_2'; it therefore is the canonical path, so that

$$I' = \int_{z_2}^{z_1} Zdz + \int_{z_1}^{z_1'} Zdz + \int_{z_1'}^{z_2'} Zdz$$
$$= -I_2 + I_1 + \int_{z_1}^{z_1'} Zdz.$$

But $\int_{z_1}^{z_1'} Zdz$ is an integral of a uniform finite function along an infinitesimal arc z_1Oz_1', and it is zero in the limit when we take z_1 and z_1' as coincident. Thus

$$I' = I_1 - I_2,$$

or *the constant for the cross-cut OR is the excess of the constant for the part of PQ at the positive edge of OR over the constant for the part of PQ at the negative edge.*

Secondly, let C' have both extremities on C, close to one another so that they may be brought together as in the figure: it is effectively the case of the directions of two cross-cuts intersecting one another, say at O. Let I_1, I_2, I_3, I_4 be the constants for the portions QO, OP, OR, SO of the cross-cuts respectively, and let z_3z_2 be the positive edge of QOP; then z_4z_3 is the positive edge of SOR. Then if $\Theta(z)$ denote the value of the integral $\int_{z_0}^{z} Zdz$ at O, which is definite because the surface is simply connected and no discontinuities of Z lie within the paths of integration, we have

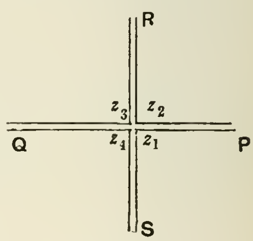

Fig. 67.

$$I_1 = \int_{z_4}^{z_3} Zdz = \Theta(z_3) - \Theta(z_4),$$
$$I_2 = \int_{z_1}^{z_2} Zdz = \Theta(z_2) - \Theta(z_1);$$

and $$I_3 = \int_{z_2}^{z_3} Zdz = \Theta(z_3) - \Theta(z_2), \quad I_4 = \int_{z_1}^{z_4} Zdz = \Theta(z_4) - \Theta(z_1);$$

so that $$I_1 - I_2 = I_3 - I_4,$$

or *the excess of the constant for the portion of a cross-cut on the positive edge, over the constant for the portion on the negative edge, of another cross-cut is equal to the excess, similarly estimated, for that other cross-cut.*

Ex. Consider the constants for the various portions of the cross-cuts in the canonical resolution (§§ 180, 181) of a Riemann's surface. Let the constants for the two portions of a_r be A_r, A_r'; and the constants for the two portions of b_r be B_r, B_r'; and let the constant for c_r be C_r.

Then, at the junction of c_r and a_{r+1}, we have

$$C_r = A_{r+1} \sim A_{r+1}';$$

at the junction of c_r and b_r, we have

$$C_r = B_r \sim B_r',$$

and, at the crossing of a_r and b_r, we have

$$A_r \sim A_r' = B_r \sim B_r'.$$

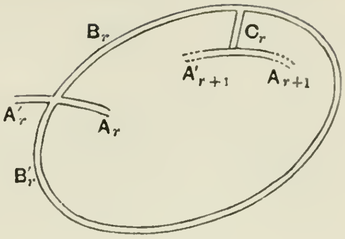

Fig. 68.

Now, because b_1 is the only cross-cut which meets a_1, we have $A_1 = A_1'$; hence $B_1 = B_1'$, and therefore $C_1 = 0$. Hence $A_2 = A_2'$; therefore $B_2 = B_2'$, and therefore also $C_2 = 0$. And so on.

Hence *the constant for each of the portions of a cross-cut a is the same; the constant for each of the portions of a cross-cut b is the same; and the constant for each cross-cut c is zero.* A single constant may thus be associated with each cross-cut a, and a single constant with each cross-cut b, in connexion with the integral of a given uniform function of position on the Riemann's surface. It has not been proved—and it is not necessarily the fact—that any one of these constants is different from zero; but it is sufficiently evident that, if all the constants be zero, the integral is a uniform function of position on the surface, that is, a rational algebraical function of w and z.

197. Hence the values of the integral at points on opposite edges of a cross-cut differ by a constant.

Suppose now that the cross-cut is obliterated: the two paths to the point z will be the same as in the case just considered and will furnish the same values respectively, say U and $U + I$. But the irreducible circuit which contributes the value I can be described any number of times; and therefore, taking account solely of this irreducible circuit and of the cross-cuts which render other circuits impossible on the resolved surface, the general value of the integral at the point z is

$$U + kI,$$

where k is an integer and U is the value for some prescribed path.

The constant I is called* a *modulus of periodicity.*

It is important that every modulus of periodicity should be finite; the path which determines the modulus can therefore pass through a point c where $Z = \infty$, or be deformed across it without change in the modulus, only if the limit of $(z - c) Z$ be a uniform zero at the point. If, however, the limit of $(z - c) Z$ at the point be a constant, implying a logarithmic infinity for the integral, or if it be an infinity of finite order (the order not being necessarily an integer), implying an algebraical infinity for the integral, we surround the point c by a simple small curve and exclude the internal area from the range of variation of the independent variable†. This exclusion is secured by making a small loop-cut in the surface round the point; it increases by unity the connectivity of the surface on which the variable is represented.

* Sometimes the *modulus for the cross-cut.*

† This is the reason for the assumption made on p. 375.

When the limit of $(z-c)Z$ is a uniform zero at c, no such exclusion is necessary: the order of the infinity for Z is easily seen to be a proper fraction and the point to be a branch-point.

Similarly, if the limit of zZ for $z=\infty$ be not zero and the path which determines a modulus can be deformed so as to become infinitely large, it is convenient to exclude the part of the surface at infinity from the range of variation of the variable, proper account being taken of the exclusion. The reason is that the value of the integral for a path entirely at infinity (or for a point-path on Neumann's sphere) is not zero; $z=\infty$ is either a logarithmic or an algebraic infinity of the function. But, if the limit of zZ be zero for $z=\infty$, the exclusion of the part of the surface at infinity is unnecessary.

198. When, then, the region of variation of the variable is properly bounded, and the resolution of the surface into one that is simply connected has been made, each cross-cut or each portion of cross-cut, that is marked off either by the natural boundary or by termination in another cross-cut, determines a modulus of periodicity. The various moduli, for a given resolution, are therefore equal, in number, to the various portions of the cross-cuts. Again, a system of cross-cuts is susceptible of great variation, not merely as to the form of individual members of the system (which does not affect the value of the modulus), but in their relations to one another. The total number of cross-cuts, by which the surface can be resolved into one that is simply connected, is a constant for the surface and is independent of their configuration: but the number of distinct pieces, defined as above, is not independent of the configuration. Now each piece of cross-cut furnishes a modulus of periodicity; a question therefore arises as to the number of independent moduli of periodicity.

Let the connectivity of the surface be $N+1$, due regard being had to the exclusions, if any, of individual points in the surface: in order that account may be taken of infinite values of the variable, the surface will be assumed spherical. The number of cross-cuts necessary to resolve it into a surface that is simply connected is N; whatever be the number of portions of the cross-cuts, the number of these portions is not less than N.

When a cross-cut terminates in another, the modulus for the former and the moduli for the two portions of the latter are connected by a relation

$$\omega = \omega_1 \sim \omega_2,$$

so that the modulus for any portion can be expressed linearly in terms of the modulus for the earlier portion and of the modulus for the dividing cross-cut.

Similarly, when the directions of two cross-cuts intersect, the moduli of the four portions are connected by a relation

$$\omega_1 \sim \omega_1' = \omega_2 \sim \omega_2';$$

and by passing along one or other of the cross-cuts, some relation is obtainable between ω_1 and ω_1' or between ω_2 and ω_2', so that, again, the modulus of any portion can be expressed linearly in terms of the modulus for the earlier portion and of moduli independent of the intersection.

Hence it appears that a single constant must be associated with each cross-cut as an independent modular constant; and then the constants for the various portions can be linearly expressed in terms of these independent constants. *There are therefore N linearly independent moduli of periodicity:* but no system of moduli is unique, and any system can be modified partially or wholly, if any number of the moduli of the system be replaced by the same number of independent linear combinations of members of the system. These results are the analytical equivalent of geometrical results, which have already been proved, viz., that the number of independent simple irreducible circuits in a complete system is N, that no complete system of circuits is unique, and that the circuits can be replaced by independent combinations reconcileable with them.

199. If, then, the moduli of periodicity of a function U at the cross-cuts in a resolved surface be $I_1, I_2, \ldots, I_N$, all the values of the function at any point on the unresolved surface are included in the form

$$U + m_1 I_1 + m_2 I_2 + \ldots + m_N I_N,$$

where $m_1, m_2, \ldots, m_N$ are integers.

Some special examples, treated by the present method, will be useful in leading up to the consideration of integrals of the most general functions of position on a Riemann's surface.

Ex. 1. Consider the integral $\int \frac{dz}{z}$.

The subject of integration is uniform, so that the surface is one-sheeted. The origin is an accidental singularity and gives a logarithmic infinity for the integral; it is therefore excluded by a small circle round it. Moreover, the value of the integral round a circle of infinitely large radius is not zero: and therefore $z=\infty$ is excluded from the range of variation. The boundary of the single spherical sheet can be taken to be the point $z=\infty$; and the bounded sheet is of connectivity 2, owing to the small circle at the origin. The surface can be resolved into one that is simply connected by a single cross-cut drawn from the boundary at $z=\infty$ to the circumference of the small circle.

If a plane surface be used, this cross-cut is, in effect, a section (§ 103) of the plane made from the origin to the point $z=\infty$.

There is only one modulus of periodicity: its value is evidently $\int \frac{dz}{z}$, taken round the origin, that is, the modulus is $2\pi i$. Hence whenever the path of variation from a given point to a point z passes from A to B, the value of the integral increases by $2\pi i$; but if the path pass from B to A, the value of the integral decreases by $2\pi i$. Thus A is the negative edge, and B the positive edge of the cross-cut.

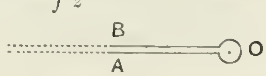

Fig. 69.

If, then, any one value of $\int_{z_0}^{z}\frac{dz}{z}$ be denoted by w, all values at the point in the unresolved surface are of the form $w+2m\pi i$, where m is an integer; when z is regarded as a function of w, it is a simply-periodic function, having $2\pi i$ for its period.

Ex. 2. Consider $\int\frac{dz}{z^2-a^2}$. The subject of integration is uniform, so that the surface consists of a single sheet. There are two infinities $\pm a$, each of the first order, because $(z\mp a)Z$ is finite at these two points: they must be excluded by small circles. The limit, when $z=\infty$, of $z/(z^2-a^2)$ is zero, so that the point $z=\infty$ does not need to be excluded. We can thus regard one of the small circles as the boundary of the surface, which is then doubly connected: a single cross-cut from the other circle to the boundary, that is, in effect, a cross-cut joining the two points a and $-a$, resolves the surface into one that is simply connected.

It is easy to see that the modulus of periodicity is $\frac{\pi i}{a}$: that A is the negative edge and B the positive edge of the cross-cut: and that, if w be a value of the integral in the unresolved surface at any point, all the values at that point are included in the form

$$w+n\frac{\pi i}{a},$$

where n is an integer.

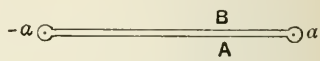

Fig. 70.

Ex. 3. Consider $\int(a^2-z^2)^{-\frac{1}{2}}dz$. The subject of integration is two-valued, so that the surface is two-sheeted. The branch-points are $\pm a$, and ∞ is not a branch-point, so that the single branch-line between the sheets may be taken as the straight line joining a and $-a$. The infinities are $\pm a$; but as $(z\mp a)(a^2-z^2)^{-\frac{1}{2}}$ vanishes at the points, they do not need to be excluded. As the limit of $z(a^2-z^2)^{-\frac{1}{2}}$, for $z=\infty$, is not zero, we exclude $z=\infty$ by small curves in each of the sheets.

Taking the surface in the spherical form, we assign as the boundary the small curve round the point $z=\infty$ in one of the sheets. The connectivity of the surface, through its dependence on branch-lines and branch-points, is unity: owing to the exclusion of the point $z=\infty$ by the small curve in the other sheet, the connectivity is increased by one unit: the surface is therefore doubly connected. A single cross-cut will resolve the surface into one that is simply connected: and this cross-cut must pass from the boundary at $z=\infty$ which is in one sheet to the excluded point $z=\infty$.

Since the (single) modulus of periodicity is the value of the integral along a circuit in the resolved surface from one edge of the cross-cut to the other, this circuit can be taken so that in the unresolved surface it includes the two branch-points; and then, by II. of § 195, the circuit can be deformed until it is practically a double straight line in the upper sheet on either side of the branch line, together with two small circles round a and $-a$ respectively. Let P be the origin, practically the middle point of these straight lines.

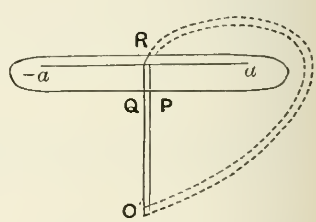

Fig. 71.

Consider the branch $(a^2-z^2)^{-\frac{1}{2}}$ belonging to the upper sheet. Its integral from P to a is

$$\int_0^a(a^2-z^2)^{-\frac{1}{2}}dz.$$

From a to $-a$ the branch is $-(a^2-z^2)^{-\frac{1}{2}}$; the point R is contiguous in the surface, not to P, but (as in § 189) to the point in the second sheet beneath P at which the branch is $-(a^2-z^2)^{-\frac{1}{2}}$, the other branch having been adopted for the upper sheet. Hence, from a to $-a$ by R, the integral is

$$\int_a^{-a} -(a^2-z^2) \quad dz.$$

From $-a$ to Q, the branch is $+(a^2-z^2)^{-\frac{1}{2}}$, the same branch as at P: hence from $-a$ to Q, the integral is

$$\int_{-a}^{0} (a^2-z^2)^{-\frac{1}{2}}\, dz.$$

The integral, along the small arcs round a and round a' respectively, vanishes for each. Hence the modulus of periodicity is

$$\int_0^a (a^2-z^2)^{-\frac{1}{2}} dz + \int_a^{-a} -(a^2-z^2)^{-\frac{1}{2}} dz + \int_{-a}^{0} (a^2-z^2)^{-\frac{1}{2}} dz,$$

that is, it is 2π.

This value can be obtained otherwise thus. The modulus is the same for all points on the cross-cut; hence its value, taken at O' where $z=\infty$, is

$$\int (a^2-z^2)^{-\frac{1}{2}} dz,$$

passing from one edge of the cross-cut at O' to the other, that is, round a curve in the plane everywhere at infinity. This gives

$$2\pi i \mathop{\text{Lt}}_{z=\infty} z(a^2-z^2)^{-\frac{1}{2}} = \frac{2\pi i}{i} = 2\pi,$$

the same value as before.

The latter curve round O', from edge to edge, can easily be deformed into the former curve round a and $-a$ from edge to edge of the cross-cut.

Again, let w_1 be a value of the integral for a point z_1 in one sheet and w_2 be a value for a point z_2 in the other sheet with the same algebraical value as z_1: take zero as the common lower limit of the integral, being the same zero for the two integrals. This zero may be taken in either sheet, let it be in that in which z_1 lies: and then

$$w_1 = \int_0^{z_1} (a^2-z^2)^{-\frac{1}{2}} dz.$$

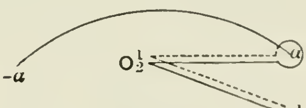

Fig. 72.

To pass from O to z_2 for w_2, any path can be justifiably deformed into the following: (i) a path round either branch-point, say a, so as to return to the point under O in the second sheet, say to O_2, (ii) any number m of irreducible circuits round a and $-a$, always returning to O_2 in the second sheet, (iii) a path from O_2 to z_2 lying exactly under the path from O to z_1 for w_1. The parts contributed by these paths respectively to the integral w_2 are seen to be

(i) a quantity $+\pi$, arising from $\int_0^a (a^2-z^2)^{-\frac{1}{2}} dz + \int_a^0 -(a^2-z^2)^{-\frac{1}{2}} dz$, for reasons similar to those above;

(ii) a quantity $m2\pi$, where m is an integer positive or negative;

(iii) a quantity $\int_{O_2}^{z_2} -(a^2-z^2)^{-\frac{1}{2}} dz$.

In the last quantity the minus sign is prefixed, because the subject of integration is everywhere in the second sheet. Now $z_2=z_1$, and therefore the quantity in (iii) is

$$-\int_0^{z_1} (a^2-z^2)^{-\frac{1}{2}} dz,$$

that is, it is $-w_1$; hence

$$w_2 = (2m+1)\pi - w_1.$$

If then we take $w = \int_0^z (a^2 - z^2)^{-\frac{1}{2}} dz$, the integral extending along some defined curve from an assigned origin, say along a straight line, the values of w belonging to the same algebraical value of z are $2n\pi + w$ or $(2m+1)\pi - w$; and the inversion of the functional relation gives

$$\begin{aligned}\phi(w) = z &= \phi\{(2n\pi + w) \\ &= \phi\{(2m+1)\pi - w\},\end{aligned}$$

where m and n are any integers.

Ex. 4. Consider $\int \frac{dz}{(z-c)(a^2-z^2)^{\frac{1}{2}}}$, assuming $|c| > |a|$. The surface is two-sheeted, with branch-points at $\pm a$ but not at ∞: hence the line joining a and $-a$ is the sole branch-line. The infinities of the subject of integration are a, $-a$, and c. Of these a and $-a$ need not be excluded, for the same reason that their exclusion was not required in the last example. But c must be excluded; and it must be excluded in both sheets, because $z = c$ makes the subject of integration infinite in both sheets. There are thus two points of accidental singularity of the subject of integration; in the vicinity of these points, the two branches of the subject of integration are

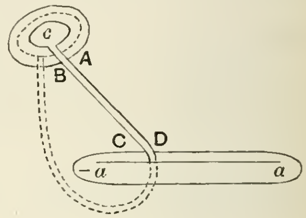

Fig. 73.

$$\frac{1}{z-c}(a^2-c^2)^{-\frac{1}{2}} + \ldots, \quad -\frac{1}{z-c}(a^2-c^2)^{-\frac{1}{2}} - \ldots,$$

the relation between the coefficients of $(z-c)^{-1}$ in them being a special case of a more general proposition (§ 210). And since $z/\{(z-c)(a^2-z^2)^{\frac{1}{2}}\}$ when $z = \infty$ is zero, ∞ does not need to be excluded.

The surface taken plane is doubly connected, as in the last example, one of the curves surrounding c, say that in the upper sheet, being taken as the boundary of the surface. A single cross-cut will suffice to make it simply connected: the direction of the cross-cut must pass from the c-curve in the lower sheet to the branch-line and thence to the boundary in the upper sheet.

There is only a single modulus of periodicity, being the constant for the single cross-cut. This modulus can be obtained by means of the curve AB in the first sheet; and, on contraction of the curve (by II, § 195) so as to be infinitesimally near c, it is easily seen to be $2\pi i(a^2-c^2)^{-\frac{1}{2}}$, or say $2\pi(c^2-a^2)^{-\frac{1}{2}}$. But the modulus can be obtained also by means of the curve CD; and when the curve is contracted, as in the previous example, so as practically to be a loop round a and a loop round $-a$, the value of the integral is

$$2\int_{-a}^{a} \frac{dz}{(z-c)(a^2-z^2)^{\frac{1}{2}}},$$

which is easily proved to be $2\pi(c^2-a^2)^{-\frac{1}{2}}$.

As in Ex. 4, a curve in the upper sheet which encloses the branch-points and the branch-lines can be deformed into the curve AB.

Ex. 5. Consider $w = \int (4z^3 - g_2 z - g_3)^{-\frac{1}{2}} dz = \int u dz.$

The subject of integration is two-valued, and therefore the Riemann's surface is two-sheeted. The branch-points are $z = \infty$, e_1, e_2, e_3 where e_1, e_2, e_3 are the roots of

$$4z^3 - g_2 z - g_3 = 0;$$

and no one of them needs to be excluded from the range of variation of the variable.

The connectivity of the surface is 3, so that two cross-cuts are necessary to resolve the surface into one that is simply connected. The configurations of the branch-lines and

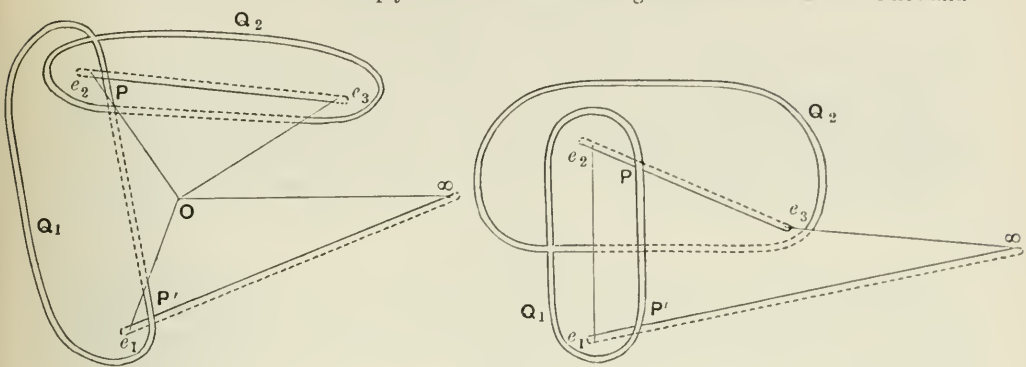

Fig. 74.

of the cross-cuts admit of some variety; two illustrations of branch-lines are given in Fig. 74, and a point on Q_1 in each diagram is taken as boundary.

The modulus for the cross-cut Q_1—say from the inside to the outside—can be obtained in two different ways. First, from P, a point on Q_1, draw a line to e_2 in the first sheet, then across the branch-line, then in the second sheet to e_3 and across the branch-line, then in the first sheet round e_3 and back to P: the circuit is represented by the double line between e_2 and e_3. The value of the integral is

$$\int_{e_2}^{e_3} udz + \int_{e_3}^{e_2} (-u)\,dz, \quad \text{that is,} \quad 2\int_{e_2}^{e_3} udz.$$

Again, it can be obtained by a line from P', another point on Q_1, to ∞, round the branch-point there and across the branch-line, then in the second sheet to e_1 and round e_1, then across the branch-line and back to P': the value of the integral is

$$E_1 = 2\int_{e_1}^{\infty} udz.$$

But the modulus is the same for P as for P': hence

$$E_1 = 2\int_{e_1}^{\infty} udz = 2\int_{e_2}^{e_3} udz.$$

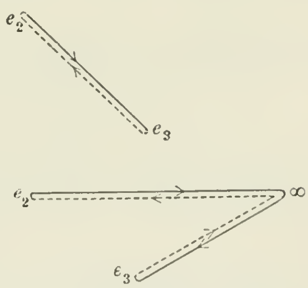

Fig. 75.

This relation can be expressed in a different form. The path from e_2 to e_3 can be stretched into another form towards $z=\infty$ in the first sheet, and similarly for the path in the second sheet, without affecting the value of the integral. Moreover as the integral is zero for $z=\infty$, we can, without affecting the value, add the small part necessary to complete the circuits from e_2 to ∞ and from e_3 to ∞. The directions of these circuits being given by the arrows, we have

$$2\int_{e_2}^{e_3} udz = 2\int_{e_2}^{\infty} udz + 2\int_{\infty}^{e_3} udz,$$

or, if

$$E_\lambda = 2\int_{e_\lambda}^{\infty} udz,$$

for $\lambda = 1, 2, 3$, we have*

$$E_1 = 2\int_{e_2}^{e_3} udz = E_2 - E_3,$$

say

$$E_2 = E_1 + E_3;$$

and E_1 is the modulus of periodicity for the cross-cut Q_1.

* See Ex. 6, § 104.

In the same way the modulus of periodicity for Q_2 is found to be

$$E_3 = 2\int_{e_3}^{\infty} udz \text{ and to be } 2\int_{e_2}^{e_1} udz,$$

the equivalence of which can be established as before.

Hence it appears that, if w be the value of the integral at any point in the surface, the general value is of the form $w + mE_1 + nE_3$, where m and n are integers. As the integral is zero at infinity (and for other reasons which have already appeared), it is convenient to take the fixed limit z_0 so as to define w by the relation

$$w = \int_z^{\infty} udz.$$

Now corresponding to a given algebraical value of z, there are two points in the surface and two values of w: it is important to know the relation to one another of these two values. Let z' denote the value in the lower sheet: then the path from z' to ∞ can be made up of

(i) a path from z' to ∞'; (ii) any number of irreducible circuits from ∞' to ∞'; and (iii) across the branch-line and round its extremity to ∞.

These parts respectively contribute to the integral

(i) a quantity $\int_{z'}^{\infty'} (-u)\, dz$, that is, $-\int_z^{\infty} udz$, or, $-w$; (ii) a quantity $mE_1 + nE_3$, where m and n are integers; (iii) a quantity zero, since the integral vanishes at infinity: so that $\quad w' = mE_1 + nE_3 - w.$

If now we regard z as a function of w, say $z = \wp(w)$, we have

$$\wp(w) = z = \wp(mE_1 + nE_3 + w), \quad \wp(w') = z'.$$

But $z' = z$ algebraically, so that we have

$$z = \wp(w) = \wp(mE_1 + nE_3 \pm w)$$

as the function expressing z in terms of w.

Similarly it can be proved that

$$\wp'(w) = \pm \wp'(mE_1 + nE_3 \pm w),$$

the upper and the lower signs being taken together. Now $\wp(w)$, by itself, determines a value of z, that is, it determines two points on the surface: and $\wp'(w)$ has different values for these two points. Hence *a point on the surface is uniquely determined by* $\wp(w)$ *and* $\wp'(w)$.

Ex. 6. Consider $w = \int_0^z \{(1-z^2)(1-k^2z^2)\}^{-\frac{1}{2}} dz = \int udz$. The subject of integration is two-valued, so that the surface is two-sheeted. The branch-points are ± 1, $\pm \frac{1}{k}$ but not ∞; no one of the branch-points need be excluded, nor need infinity.

The connectivity is 3, so that two cross-cuts will render the surface simply connected: let the branch-lines and the cross-cuts be taken as in the figure.

The details of the argument follow the same course as in the previous case.

The modulus of periodicity for Q_2 is $2\int_{-1}^{1} udz = 4\int_0^1 udz = 4K$, in the ordinary notation.

The modulus of periodicity for Q_1 is $2\int_1^{\frac{1}{k}} udz = 2iK'$, as before.

Hence, if w be a value of the integral for a point z in the first sheet, a more general value for that point is
$$w+m4K+n2iK'.$$

Let w' be a value of the integral for a point z' in the second sheet, where z' is algebraically equal to z—the point in the first sheet at which the value of the integral is w; then
$$w'=2K+m4K+n2iK'-w,$$
so that, if we invert the functional relation and take $z=\text{sn}\, w$, we have

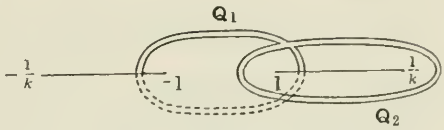

Fig. 76.

$$\begin{aligned}\text{sn}\, w = z &= \text{sn}\,(w+4mK+2niK')\\ &= \text{sn}\,\{(4m+2)\,K+2niK'-w\}.\end{aligned}$$

Ex. 7. Consider the integral $w=\int\frac{dz}{(z-c)\,u}$, where $u=\{(1-z^2)(1-k^2z^2)\}^{\frac{1}{2}}$.

As in the last case, the surface is two-sheeted: the branch-points are ± 1, $\pm\frac{1}{k}$ but no one of them need be excluded, nor need $z=\infty$. But the point $z=c$ must be excluded in both sheets; for expanding the subject of integration for points in the first sheet in the vicinity of $z=c$, we have
$$\frac{1}{z-c}\{(1-c^2)(1-k^2c^2)\}^{-\frac{1}{2}}+\ldots,$$
and for points in the second sheet in the vicinity of $z=c$, we have
$$-\frac{1}{z-c}\{(1-c^2)(1-k^2c^2)\}^{-\frac{1}{2}}-\ldots,$$
in each case giving rise to a logarithmic infinity for $z=c$.

We take the small curves excluding $z=c$ in both sheets as the boundaries of the surface. Then, by Ex. 4 § 178, (or because one of these curves may be regarded as a

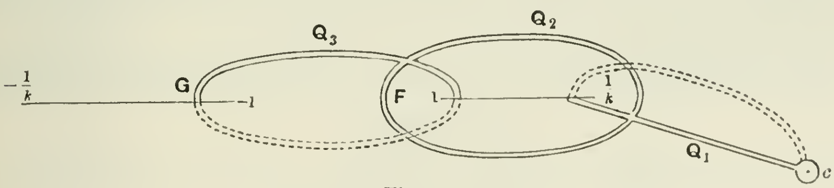

Fig. 77.

boundary of the surface in the last example, and the curve excluding the infinity in the other sheet is the equivalent of a loop-cut which (§ 161) increases the connectivity by unity), the connectivity is 4. The cross-cuts necessary to make the surface simply connected are three. They may be taken as in the figure; Q_1 is drawn from the boundary in one sheet to a branch-line and thence round $\frac{1}{k}$ to the boundary in the other sheet: Q_2 beginning and ending at a point in Q_1, and Q_3 beginning and ending at a point in Q_2.

The moduli of periodicity are:—

for Q_1, the quantity $(\Omega_1=)\,2\pi i\,\{(1-c^2)(1-k^2c^2)\}^{-\frac{1}{2}}$, obtained by taking a small curve round c in the upper sheet:

Q_2, the quantity $(\Omega_2=)\,2\int_{-1}^{1}\frac{dz}{(z-c)\,u}$, obtained by taking a circuit round 1 and $\frac{1}{k}$, passing from one edge of Q_2 to the other at F:

Q_3, the quantity $(\Omega_3=)\,2\int_{-1}^{-\frac{1}{k}}\frac{dz}{(z-c)\,u}$, obtained by taking a circuit round -1 and $-\frac{1}{k}$, passing from one edge of Q_3 to the other at G:

so that, if any value of the integral at a point be w, the general value at the point is

$$w+m_1\Omega_1+m_2\Omega_2+m_3\Omega_3,$$

where m_1, m_2, m_3 are integers.

Conversely, z is a triply-periodic function of w; but the function of w is not uniform (§ 108).

Ex. 8. As a last illustration for the present, consider

$$w=\int_0^z\left(\frac{1-k^2z^2}{1-z^2}\right)^{\frac{1}{2}}dz.$$

The surface is two-sheeted; its connectivity is 3, the branch-points being ± 1, $\pm\frac{1}{k}$ but not $z=\infty$. No one of the branch-points need be excluded, for the integral is finite round each of them. To consider the integral at infinity, we substitute $z=\frac{1}{z'}$, and then

$$\begin{aligned} w&=\int\left(\frac{k^2-z'^2}{1-z'^2}\right)^{\frac{1}{2}}\left(-\frac{dz'}{z'^2}\right)\\ &=-\int\frac{dz'}{z'^2}\left(k-\frac{k'^2}{2k}z'^2+\ldots\right)\\ &=\frac{k}{z'}+\frac{k'^2}{2k}z'+\ldots,\end{aligned}$$

giving for the function at infinity an accidental singularity of the first order in each sheet.

The point $z=\infty$ must therefore be excluded from each sheet: but the form of w, for infinitely large values of z, shews that the modulus for the cross-cut, which passes from one of the points (regarded as a boundary) to the other, is zero.

The figure in Ex. 6 can be used to determine the remaining moduli. The modulus for Q_2 is

$$\begin{aligned} &2\int_{-1}^{1}\left(\frac{1-k^2x^2}{1-x^2}\right)^{\frac{1}{2}}dx\\ &=4\int_0^1\frac{1-k^2x^2}{\{(1-x^2)(1-k^2x^2)\}^{\frac{1}{2}}}dx\\ &=4E,\end{aligned}$$

with the notation of Jacobian elliptic functions. The modulus for Q_1 is

$$\begin{aligned} &=2\int_1^{\frac{1}{k}}\left(\frac{1-k^2x^2}{1-x^2}\right)^{\frac{1}{2}}dx\\ &=2i\int_0^1\frac{k'^2y^2}{\{(1-y^2)(1-k'^2y^2)\}^{\frac{1}{2}}}dy,\end{aligned}$$

on transforming by the relation $k^2x^2+k'^2y^2=1$: the last expression can at once be changed into the form $2i\,(K'-E')$, with the same notation as before.

If then w be any value of the integral at a point on the surface, the general value there is

$$w+4mE+2ni\,(K'-E'),$$

where m and n are integers.

200. After these illustrations in connection with simple cases, we may proceed with the consideration of the integral of the most general function w' of position on a general Riemann surface, constructed in connection with the algebraical equation

$$f(w, z) = w^n + w^{n-1} g_1(z) + \ldots + w g_{n-1}(z) + g_n(z) = 0,$$

where the functions $g(z)$ are rational, integral and algebraical. Subsidiary explanations, which are merely generalised from those inserted in the preceding particular discussions, will now be taken for granted.

Taking w' in the form of § 193, we have

$$w' = \frac{1}{n} h_0(z) + \frac{h_1(z) w^{n-2} + \ldots + h_{n-1}(z)}{\dfrac{\partial f}{\partial w}} = \frac{1}{n} h_0(z) + \frac{U(w, z)}{\dfrac{\partial f}{\partial w}},$$

so that in taking the integral of w' we shall have a term $\frac{1}{n}\int h_0(z)\,dz$, where $h_0(z)$ is a rational algebraical function. This kind of integral has been discussed in Chapter II.; as it has no essential importance for the present investigation, it will be omitted, so that, without loss of generality merely for the present purpose*, we may assume $h_0(z)$ to vanish; and then *the numerator of w' is of degree not higher than $n-2$ in w.*

The value of z is insufficient to specify a point on the surface: the values of w and z must be given for this purpose, a requisite that was unnecessary in the preceding examples because the point z was spoken of as being in the upper or the lower of the two sheets of the various surfaces. Corresponding to a value a of z, there will be n points: they may be taken in the form (a_1, α_1), (a_2, α_2), ..., (a_n, α_n), where $a_1, \ldots, a_n$ are each algebraically equal to a, and $\alpha_1, \ldots, \alpha_n$ are the appropriately arranged roots of the equation

$$f(w, a) = 0.$$

The function w' to be integrated is of the form $\dfrac{U(w, z)}{\dfrac{\partial f}{\partial w}}$, where U is of degree $n-2$ in w, but though algebraical and rational it is not necessarily integral in z.

An ordinary point of w', which is neither an infinity nor a branch-point, is evidently an ordinary point of the integral.

The infinities of the subject of integration are of prime importance. They are:

(i) the infinities of the numerator,

(ii) the zeros of the denominator.

The former are constituted by (α), the poles of the coefficients of powers of w

* See § 207, where $h_0(z)$ is retained.

in $U(w, z)$, and (β), $z=\infty$: this value is included, because the only infinities of w, as determined by the fundamental equation, arise for infinite values of z, and infinite values of w and of z may make the numerator $U(w, z)$ infinite.

So far as concerns the infinities of w' which arise when $z=\infty$ (and therefore $w=\infty$), it is not proposed to investigate the general conditions that the integral should vanish there. The test is of course that the limit, for $z=\infty$, of $\dfrac{zU(w, z)}{\dfrac{\partial f}{\partial w}}$ should vanish for each of the n values of w.

But the establishment of the general conditions is hardly worth the labour involved; it can easily be made in special cases, and it will be rendered unnecessary for the general case by subsequent investigations.

201. The simplest of the instances, less special than the examples already discussed, are two.

The first, which is really that of most frequent occurrence and is of very great functional importance, is that in which $f(w, z)=0$ has the form

$$w^2 - S(z) = 0,$$

where $S(z)$ is of order $2m-1$ or $2m$ and all its roots are simple: then $\dfrac{\partial f}{\partial w} = 2w = 2\sqrt{S(z)}$. In order that the limit of $\dfrac{zU(w, z)}{\dfrac{\partial f}{\partial w}}$ may be zero when $z=\infty$, we see (bearing in mind that U, in the present case, is independent of w) that the excess of the degree of the numerator of U over its denominator may not be greater than $m-2$. In particular, if U be an integral function of z, a form of U which would leave $\int w'dz$ zero at $z=\infty$ is

$$U = c_0 z^{m-2} + c_1 z^{m-3} + \dots + c_{m-3} z + c_{m-2}.$$

As regards the other infinities of $U/\sqrt{S(z)}$, they are merely the roots of $S(z)=0$ or they are the branch-points, each of the first order, of the equation

$$w^2 - S(z) = 0.$$

By the results of § 101, the integral vanishes round each of these points; and each of the points is a branch-point of the integral function. The integral is finite everywhere on the surface: and *the total number of such integrals, essentially different from one another, is the number of arbitrary coefficients in U, that is, it is $m-1$, the same as the class of the Riemann's surface associated with the equation.*

202. The other important instance is that in which the fundamental equation is, so to speak, a generalised equation of a plane curve, so that $g_s(z)$ is an integral algebraical function of z of degree s: then it is easy to see that, at $z = \infty$, each branch $w \propto z$, so that $\frac{\partial f}{\partial w} \propto z^{n-1}$: hence $U(w, z)$ can vary only as z^{n-3}, in order that the condition may be satisfied. If then $U(w, z)$ be an integral function of z, it is evident that it can at most take a form which makes $U = 0$ the generalised equation of a curve of degree $n-3$; while, if it be $\frac{V(w, z)}{z-c}$, then $V(w, z)$, supposed integral in z, can at most take a form which makes $V = 0$ the generalised equation of a curve of degree $n-2$.

Other forms are easily obtainable for accidental singularities of coefficients of w in $U(w, z)$ that are of other orders.

As regards the other possible infinities of the integral, let c be an accidental singularity of a coefficient of some power of w in $U(w, z)$; it may be assumed not to be a zero of $\frac{\partial f}{\partial w}$. Denote the n points on the surface by (c_1, k_1), (c_2, k_2), ..., (c_n, k_n), where $c_1, c_2, \ldots, c_n$ are algebraically equal to c. In the vicinity of each of these points let w' be expanded: then, near (c_r, k_r) we have a set of terms of the type

$$\frac{A_{m,r}}{(z-c_r)^m} + \frac{A_{m-1,r}}{(z-c_r)^{m-1}} + \ldots + \frac{A_{2,r}}{(z-c_r)^2} + \frac{A_{1,r}}{z-c_r} + P(z-c_r),$$

where $P(z-c_r)$ is a converging series of positive integral powers of $z-c_r$. A corresponding expansion exists for every one of the n points.

The integral of w' will therefore have a logarithmic infinity at (c_r, k_r), unless $A_{1,r}$ is zero; and it will have an algebraical infinity, unless all the coefficients $A_{2,r}, \ldots\ldots, A_{m,r}$ are zero.

The simplest cases are

(i) that in which the integral has a logarithmic infinity but no algebraical infinity; and

(ii) that in which the integral has no logarithmic infinity.

For the former, w' is of the form $\dfrac{W(w, z)}{(z-c)\dfrac{\partial f}{\partial w}}$, and therefore in the vicinity of c_r we have

$$w' = \frac{A_{1,r}}{z-c_r} + P(z-c_r),$$

the value of $A_{1,r}$ being $\dfrac{W(k_r, c_r)}{\dfrac{\partial f}{\partial k_r}}$, and W is an integral function of k_r, of degree not higher than $n-2$. Hence

$$\sum_{r=1}^{n} A_{1,r} = \sum_{r=1}^{n} \frac{W(k_r, c_r)}{\dfrac{\partial f}{\partial k_r}}$$

$$= \sum_{r=1}^{n} \frac{W(k_r, c)}{\dfrac{\partial f}{\partial k_r}},$$

since c is the common algebraical value of the quantities $c_1, c_2, \ldots, c_n$. Now $k_1, k_2, \ldots, k_n$ are the roots of

$$f(w, c) = 0,$$

an equation of degree n, while W is of degree not higher than $n-2$; hence, by a known theorem*,

$$\sum_{r=1}^{n} \frac{W(k_r, c)}{\dfrac{\partial f}{\partial k_r}} = 0,$$

so that

$$\sum_{r=1}^{n} A_{1, r} = 0.$$

The validity of the result is not affected if some of the coefficients A vanish. But it is evident that a single coefficient A cannot be the only non-vanishing coefficient; and that, if all but two vanish, those two are equal and opposite.

This result applies to all those accidental singularities of coefficients of powers of w in the numerator of w' which, being of the first order, give rise solely to logarithmic infinities in the integral of w'. It is of great importance in regard to moduli of periodicity of the integral.

(ii) The other simple case is that in which each of the coefficients $A_{1, r}$ vanishes, so that the integral of w' has only an algebraical infinity at the point c_r, which is then an accidental singularity of order less by unity than its order for w'.

In particular, if in the vicinity of c_r, the form of w' be

$$\frac{A_{2, r}}{(z - c_r)^2} + P(z - c_r),$$

the integral has an accidental singularity of the first order.

It is easy to prove that

$$\sum_{r=1}^{n} A_{2, r} = 0,$$

so that a single coefficient A cannot be the only non-vanishing coefficient; but the result is of less importance than in the preceding case, for all the moduli of periodicity of the integral at the cross-cuts for these points vanish. And it must be remembered that in order to obtain the subject of integration in this form, some terms have been removed in § 200, the integral of which would give rise to infinities for either finite or infinite values of z.

It may happen that all the coefficients of powers of w in the numerator of w' are integral functions of z. Then $z = \infty$ is their only accidental singularity; this value has already been taken into account.

* Burnside and Panton, *Theory of Equations*, (3rd ed.), p. 319.

203. The remaining source of infinities of w', as giving rise to possible infinities of the integral, is constituted by the aggregate of the zeros of $\frac{\partial f}{\partial w} = 0$. Such points are the simultaneous roots of the equations

$$\frac{\partial f}{\partial w} = 0, \ f(w, z) = 0.$$

In addition to the assumption already made that $f = 0$ is the equation of a generalised curve of the nth order, we shall make the further assumptions that all the singular points on it are simple, that is, such that there are only two tangents at the point, either distinct or coincident, and that all the branch-points are simple.

The results of § 98 may now be used. The total number of the points given as simultaneous roots is $n(n-1)$: the form of the integral in the immediate vicinity of each of the points must be investigated.

Let (c, γ) be one of these points on the Riemann's surface, and let $(c + \zeta, \gamma + v)$ be any point in its immediate vicinity.

I. If $\frac{\partial f(w, z)}{\partial z}$ do not vanish at the point, then (c, γ) is a branch-point for the function w. We then have

$$f(w, z) = A'\zeta + B'v^2 + \text{quantities of higher dimensions},$$

for points in the vicinity of (c, γ), so that $v \propto \zeta^{\frac{1}{2}}$ when $|\zeta|$ is sufficiently small. Then

$$\begin{aligned}\frac{\partial f}{\partial w} &= 2B'v + \text{quantities of higher dimensions} \\ &\propto \zeta^{\frac{1}{2}},\end{aligned}$$

when $|\zeta|$ is sufficiently small. Hence, for such values, the subject of integration is a constant multiple of

$$\frac{U(\gamma, c) + \text{positive integral powers of } v \text{ and } \zeta}{\zeta^{\frac{1}{2}} + \text{powers of } \zeta \text{ with index} > \frac{1}{2}}$$

that is, of $\zeta^{-\frac{1}{2}}$, when $|\zeta|$ is sufficiently small. The integral is therefore a constant multiple of $\zeta^{\frac{1}{2}}$ when $|\zeta|$ is sufficiently small; and its value is therefore zero round the point, which is a branch-point for the function represented by the integral.

II. If $\frac{\partial f(w, z)}{\partial z}$ vanish at the point, we have (with the assumptions of § 98),

$$f(w, z) = A\zeta^2 + 2B\zeta v + Cv^2 + \text{terms of the third and higher degrees};$$

and there are two cases.

(i) If $B^2 \gtrless AC$, the point is not a branch-point, and we have

$$Cv + B\zeta = \zeta(B^2 - AC)^{\frac{1}{2}} + \text{integral powers } \zeta^2, \zeta^3, \ldots$$

as the relation between v and ζ deduced from $f=0$. Then

$$\frac{\partial f}{\partial w} = 2(B\zeta + Cv) + \text{terms of second and higher degrees}$$
$$= \lambda\zeta + \text{higher powers of } \zeta.$$

In the vicinity of (c, γ), the subject of integration is

$$\frac{U(\gamma, c) + Dv + E\zeta + \text{positive integral powers}}{\lambda\zeta + \text{higher powers of } \zeta}.$$

Hence when it is integrated, the first term is $\frac{U(\gamma, c)}{\lambda} \log \zeta$, and the remaining terms are positive integral powers of ζ: that is, such a point is a logarithmic infinity for the integral, unless $U(\gamma, c)$ vanish.

If, then, we seek integrals which have not the point for a logarithmic infinity and we begin with U as the most general function possible, we can prevent the point from being a logarithmic infinity by choosing among the arbitrary constants in U a relation such that

$$U(\gamma, c) = 0.$$

There are δ such points (§ 98); and therefore δ relations among the constants in the coefficients of U must be chosen, in order to prevent the integral

$$\int \frac{U(w, z)}{\frac{\partial f}{\partial w}} dz$$

from having a logarithmic infinity at these points, which are then ordinary points of the integral.

(ii) If $B^2 = AC$, the point is a branch-point; we have

$$B\zeta + Cv = \tfrac{1}{2}L\zeta^{\frac{3}{2}} + M\zeta^2 + N\zeta^{\frac{5}{2}} + \ldots$$

as the relation between ζ and v deduced from $f=0$. In that case,

$$\frac{\partial f}{\partial w} = 2(B\zeta + Cv) + \text{terms of the second and higher degrees}$$
$$= L\zeta^{\frac{3}{2}} + \text{powers of } \zeta \text{ having indices} > \tfrac{3}{2}.$$

In the vicinity of (c, γ), the subject of integration is

$$\frac{U(\gamma, c) + Dv + E\zeta + \text{higher powers}}{L\zeta^{\frac{3}{2}} + \text{higher powers of } \zeta}.$$

Hence when it is integrated, the first term is $-2\frac{U(\gamma, c)}{L}\zeta^{-\frac{1}{2}}$, and it can be proved that there is no logarithmic term; the point is an infinity for the integral, unless $U(\gamma, c)$ vanish.

If, however, among the arbitrary constants in U we choose a relation such that

$$U(\gamma, c) = 0,$$

then the numerator of the subject of integration

$$= D\upsilon + E\zeta + \text{higher positive powers}$$

$$= \lambda'\zeta + \mu'\zeta^{\frac{3}{2}} + \text{higher powers of } \zeta,$$

on substituting from the relation between υ and ζ derived from the fundamental equation. The subject of integration then is

$$\frac{\lambda'\zeta + \mu'\zeta^{\frac{3}{2}} + \dots}{L\zeta^{\frac{3}{2}} + M\zeta^2 + \dots},$$

that is,

$$\frac{\lambda' + \mu'\zeta^{\frac{1}{2}} + \dots}{L\zeta^{\frac{1}{2}} + M\zeta + \dots},$$

the integral of which is

$$2\frac{\lambda'}{L}\zeta^{\frac{1}{2}} + \text{positive powers.}$$

The integral therefore vanishes at the point: and the point is a branch-point for the integral. It therefore follows that we can prevent the point from being an infinity for the function by choosing among the arbitrary constants in U a relation such that

$$U(\gamma, c) = 0.$$

There are κ such points (§ 98): and therefore κ relations among the constants in the coefficients of U must be chosen in order to prevent the integral from becoming infinite at these points. Each of the points is a branch-point of the integral.

204. All the possible sources of infinite values of the subject of integration $w', = \dfrac{U(w, z)}{\dfrac{\partial f}{\partial w}}$, have now been considered. A summary of the preceding results leads to the following conclusions relative to $\int w' dz$:

(i) an ordinary point of w' is an ordinary point of the integral:

(ii) for infinite values of z, the integral vanishes if we assign proper limitations to the form of $U(w, z)$:

(iii) accidental singularities of the coefficients of powers of w in $U(w, z)$ are infinities, either algebraical or logarithmic or both algebraical and logarithmic, of the integral:

(iv) if the coefficients of powers of w in $U(w, z)$ have no accidental singularities except for $z = \infty$, then the integral is finite for infinite values of z (and of w) when $U(w, z)$ is the most general rational integral algebraical function of w and z of degree $n - 3$; but, if the coefficients of powers of w in $U(w, z)$ have an accidental singularity of order μ, then the integral will be finite for infinite values of z (and of w) when $U(w, z)$ is the most general rational integral algebraical function of w and z, the degree in w being not greater than $n-2$ and the dimensions in w and z combined being not greater than $n+\mu-3$:

(v) those points, at which $\partial f/\partial w$ vanishes and which are not branch-points of the function, can be made ordinary points of the integral, if we assign proper relations among the constants occurring in $U(w, z)$:

(vi) those points, at which $\partial f/\partial w$ vanishes and which are branch-points of the function, can, if necessary, be made to furnish zero values of the integral by assigning limitations to the form of $U(w, z)$; each such point is a branch-point of the integral in any case.

These conclusions enable us to select the simplest and most important classes of integrals of uniform functions of position on a Riemann's surface.

205. The first class consists of those integrals which do not acquire* an infinite value at any point; they are called integrals of the *first kind*†.

The integrals, considered in the preceding investigations, can give rise to integrals of the first kind, if the numerator $U(w, z)$ of the subject of integration satisfy various conditions. The function $U(w, z)$ must be an integral function of dimensions not higher than $n-3$ in w and z, in order that the integral may be finite for infinite values of z and for all finite values of z not specially connected with the equation $f(w, z)=0$; for certain points specially connected with the fundamental equation, being $\delta+\kappa$ in number, the value of $U(w, z)$ must vanish, so that there must be $\delta+\kappa$ relations among its coefficients. But when these conditions are satisfied, then the integral function is everywhere finite, it being remembered that certain limitations on the nature of $f(w, z)=0$ have been made.

Usually these conditions do not determine $U(w, z)$ uniquely save as to a constant factor; and therefore in the most general integral of the first kind a number of independent arbitrary constants will occur, left undetermined by the conditions to which U is subjected. Each of these constants multiplies an integral which, everywhere finite, is different from the other integrals so multiplied; and therefore the number of different integrals of the first kind is equal to the number of arbitrary independent constants, left undetermined in U. It is evident that any linear combination of these integrals, with

* They will be seen to be multiform functions even on the multiply connected Riemann's surface, and they do not therefore give rise to any violation of the theorem of § 40.

† The German title is *erster Gattung*; and similarly for the integrals of the second kind and the third kind.

constant coefficients, is also an integral of the first kind; and therefore a certain amount of modification of form among the integrals, after they have been obtained, is possible.

The number of these integrals, linearly independent of one another, is easily found. Because U is an integral algebraical function of w and z of dimensions $n-3$, it contains $\frac{1}{2}(n-1)(n-2)$ terms in its most general form; but its coefficients satisfy $\delta+\kappa$ relations, and these are all the relations that they need satisfy. Hence the number of undetermined and independent constants which it contains is

$$\tfrac{1}{2}(n-1)(n-2)-\delta-\kappa,$$

which, by § 182, is the class p of the Riemann's surface; and therefore, for the present case, *the number of integrals, which are finite everywhere on the surface and are linearly independent of one another, is equal to the class of the Riemann's surface.*

Moreover, the integral of the first kind has the same branch-points as the function w. Though the integral is finite everywhere on the surface, yet its derivative w' is not so: the infinities of w' are the branch-points.

The result has been obtained on the original suppositions of § 98, which were, that all the singular points of the generalised curve $f(w, z)=0$ are simple, that is, only two tangents (distinct or coincident) to the curve can be drawn at each such point, and that all the branch-points are simple. Other special cases could be similarly investigated. But it is superfluous to carry out the investigation for a series of cases, because the result just obtained, and the result of § 201, are merely particular instances of a general theorem which will be proved in Chapter XVIII., viz., that, *associated with a Riemann's surface of connectivity* $2p+1$, *there are* p *linearly independent integrals of the first kind which are finite everywhere on the surface.*

206. The functions, which thus arise out of the integral of an algebraical function and are finite everywhere, are not uniform functions of position on the unresolved surface. If the surface be resolved by $2p$ cross-cuts into one that is simply connected, then the function is finite, continuous and uniform everywhere in that resolved surface, which is limited by the cross-cuts as a single boundary. But at any point on a cross-cut, the integral, at the two points on opposite edges, has values that differ by any integral multiple of the modulus of the function for that cross-cut (and possibly also by integral multiples of the moduli of the function for the other cross-cuts).

Let the cross-cuts be taken as in § 181; and for an integral of the first kind, say W, let the moduli of periodicity for the cross-cuts be

$$\omega_1, \omega_2, \ldots, \omega_p, \text{ for } a_1, a_2, \ldots, a_p,$$

$$\text{and } \omega_{p+1}, \omega_{p+2}, \ldots, \omega_{2p}, \text{ for } b_1, b_2, \ldots, b_p,$$

respectively; the moduli for the portions of cross-cuts $c_2, c_3, \ldots, c_p$ have been proved to be zero.

Some of these moduli may vanish; but it will be proved later (§ 231) that all the moduli for the cross-cuts a, or all the moduli for the cross-cuts b, cannot vanish unless the integral is a mere constant. In the general case, with which we are concerned, we may assume that they do not vanish; and so it follows that, *if W be a value of an integral of the first kind at any point on the Riemann's surface, all its values at that point are of the form*

$$W + \sum_{r=1}^{2p} m_r \omega_r,$$

where the coefficients m are integers.

The foregoing functions, arising through integrals that are finite everywhere on the surface, will be found the most important from the point of view of Abelian transcendents: but other classes arise, having infinities on the surface, and it is important to indicate their general nature before passing to the proof of the Existence-Theorem.

207. First, consider an integral which has algebraical, but not logarithmic, infinities. Taking the subject of integration, as in the preceding case, to be the most general possible, so that arbitrary coefficients enter, we can, by assigning suitable relations among these coefficients, prevent any of the points, given as zeros of $\frac{\partial f}{\partial w} = 0$, from being infinities of the integral. It follows that then the only infinities of the integral will be the points that are accidental singularities of coefficients of powers of w in the numerator of the general expression for w'. These singularities must each be of the second order at least: and, in the expansion of w' in the vicinity of each of them, there must be no term of index -1, the index that leads, on integration, to a logarithm.

Such integrals are called integrals of the *second kind.*

The simplest integral of the second kind has an infinity for only a single point on the surface, and the infinity is of the first order only: the integral is then called an *elementary integral of the second kind.* After what has been proved in § 202 (ii), it is evident that an elementary integral of the second kind cannot occur in connection with the equation $f(w, z) = 0$, unless the term $h_0(z)$ of § 200 be retained in the expression for w'.

Ex. 1. Adopting the subject of integration obtained in § 200, we have

$$w' - \frac{1}{n} h_0(z) = \frac{U(w, z)}{\dfrac{\partial f}{\partial w}},$$

where U is of the character considered in the preceding sections, viz., it is of degree $n-2$ in w; various forms of w' lead to various forms of $h_0(z)$ and of $U(w, z)$.

If $\frac{1}{n}h_0(z) = \frac{-1}{(z-c)^2}$, and if c be not a singularity of the coefficient of any power of w in U, it is then evident that

$$\int w'dz = \frac{1}{z-c} + \int \frac{U(w, z)}{\frac{\partial f}{\partial w}} dz;$$

and the integral on the right-hand side can by choice among the constants be made an integral of the first kind. The integral is not, however, an elementary integral of the second kind, because $z=c$ is an infinity in each sheet.

Ex. 2. A special integral of the second kind occurs, when we take an accidental singularity, say $z=c$, of the coefficient of some power of w in $U(w, z)$ and we neglect $h_0(z)$; so that, in effect, the subject of integration w' is limited to the form

$$\frac{U(w, z)}{\frac{\partial f}{\partial w}},$$

U being of degree not higher than $n-2$ in w. To the value $z=c$, there correspond n points in the various sheets; if, in the immediate vicinity of any one of the points, w' be of the form

$$\frac{-A_r}{(z-c_r)^2} + P'(z-c_r),$$

in that vicinity the integral is of the form

$$\frac{A_r}{z-c_r} + P(z-c_r).$$

For such an integral the sum of the coefficients A_r is zero: the simplest case arises when all but two, say A_1 and A_2, of these vanish. The integral is then of the form

$$\frac{A}{z-c_1} + P_1(z-c_1)$$

in the vicinity of c_1, and of the form

$$\frac{-A}{z-c_2} + P_2(z-c_2)$$

in the vicinity of c_2. But the integral is not an elementary integral of the second kind.

208. To find the general value of an integral of the second kind, all the algebraically infinite points would be excluded from the Riemann's surface by small curves: and the surface would be resolved into one that is simply connected. The cross-cuts necessary for this purpose would consist of the set of $2p$ cross-cuts, necessary to resolve the surface as for an integral of the first kind, and of the k additional cross-cuts in relation with the curves excluding the algebraically infinite points.

Let the moduli for the former cross-cuts be

$$\epsilon_1, \epsilon_2, \ldots, \epsilon_p, \text{ for the cuts } a_1, a_2, \ldots, a_p,$$

$$\epsilon_{p+1}, \epsilon_{p+2}, \ldots, \epsilon_{2p} \text{ for the cuts } b_1, b_2, \ldots, b_p, \text{ respectively}:$$

the moduli for the cuts c are zero. It is evident from the form of the integral in the vicinity of any infinite point that, as the integral has only an algebraical infinity, the *modulus for each of the k cross-cuts, obtained by a curve from one edge to the other round the point, is zero.* Hence if one value of the integral of the second kind at a point on the surface be $E(z)$, all its values at that point are included in the form

$$E(z) + \sum_{r=1}^{2p} n_r \epsilon_r,$$

where $n_1, n_2, \ldots, n_{2p}$ are integers.

The importance of the elementary integral of the second kind, independently of its simplicity, is that *it is determined by its infinity, save as to an additive integral of the first kind.*

Let $E_1(z)$ and $E_2(z)$ be two elementary integrals of the second kind, having their single infinity common, and let a be the value of z at this point; then in its vicinity we have

$$E_1(z) = \frac{A_1}{z-a} + P_1(z-a), \qquad E_2(z) = \frac{A_2}{z-a} + P_2(z-a),$$

and therefore $A_1E_2(z) - A_2E_1(z)$ is finite at $z = a$. This new function is therefore finite over the whole Riemann's surface: hence it is an integral of the first kind, the moduli of periodicity of which depend upon those of $E_1(z)$ and $E_2(z)$.

Ex. It may similarly be proved that for the special case in Ex. 2, § 207, when the integral of the second kind has two simple infinities for the same algebraical value of z in different sheets, the integral is determinate save as to an additive integral of the first kind.

Let a_1 and a_2 be the two points for the algebraical value a of z; and let $F(z)$ and $G(z)$ be two integrals of the second kind above indicated having simple infinities at a_1 and a_2 and nowhere else.

Then in the vicinity of a_1 we have

$$F(z) = \frac{A}{z-a_1} + P_1(z-a_1), \quad G(z) = \frac{B}{z-a_1} + Q_1(z-a_1),$$

so that $BF(z) - AG(z)$ is finite in the vicinity of a_1.

Again, in the vicinity of a_2, we have, by § 202,

$$F(z) = \frac{-A}{z-a_2} + P_2(z-a_2), \quad G(z) = \frac{-B}{z-a_2} + Q_2(z-a_2),$$

so that $BF(z) - AG(z)$ is finite in the vicinity of a_2 also. Hence $BF(z) - AG(z)$ is finite over the whole surface, and it is therefore an integral of the first kind; which proves the statement.

It therefore appears that, if $F(z)$ be any such integral, every other integral of the same nature at those points is of the form $F(z) + W$, where W is an integral of the first kind. Now there are p linearly independent integrals of the first kind: it therefore follows that there are $p+1$ linearly independent integrals of the second kind, having simple infinities with equal and opposite residues at two points, (and at only two points), determined by one algebraical value of z.

From the property that an elementary integral of the second kind is determined by its infinity save as to an additive integral of the first kind, we infer that *there are $p+1$ linearly independent elementary integrals of the second kind with the same single infinity on the Riemann's surface.*

This result can be established in connection with $f(w, z)=0$ as follows. The subject of integration is

$$\frac{U(w, z)}{(z-a)^2 \dfrac{\partial f}{\partial w}},$$

where for simplicity it is assumed that a is neither a branch-point of the function nor a singular point of the curve $f(w, z)=0$, and in the present case U is of degree $n-1$ in w. To ensure that the integral vanishes for $z=\infty$, the dimensions of $U(w, z)$ may not be greater than $n-1$. Hence $U(w, z)$, in its most general form, is an integral, rational, algebraical function of w and z of degree $n-1$; the total number of terms is therefore $\frac{1}{2}n(n+1)$, which is also the total number of arbitrary constants.

In order that the integral may not be infinite at each of the $\delta+\kappa$ singularities of the curve $f(w, z)=0$, a relation $U(\gamma, c)=0$ must be satisfied at each of them; hence, on this score, there are $\delta+\kappa$ relations among the arbitrary constants.

Let the points on the surface given by the algebraical value a of z be (a_1, α_1), (a_2, α_2), ..., (a_n, α_n). The integral is to be infinite at only one of them; so that we must have

$$U(a_r, \alpha_r)=0,$$

for $r=2, 3, \ldots, n$; and $n-1$ is the greatest number of such points for which U can vanish, unless it vanish for all, and then there would be no algebraical infinity. Hence, on this score, there are $n-1$ relations among the arbitrary constants in U.

In the vicinity of $z=a$, $w=\alpha$, let

$$z=a+\zeta, \quad w=\alpha+v;$$

then we have

$$0=v\frac{\partial f}{\partial \alpha}+\zeta\frac{\partial f}{\partial a}+\ldots,$$

where $\dfrac{\partial f}{\partial \alpha}$ is the value of $\dfrac{\partial f}{\partial w}$ and $\dfrac{\partial f}{\partial a}$ that of $\dfrac{\partial f}{\partial z}$, for $z=a$ and $w=\alpha$. For sufficiently small values of $|v|$ and $|\zeta|$, we may take

$$0=v\frac{\partial f}{\partial \alpha}+\zeta\frac{\partial f}{\partial a}.$$

For such points we have

$$U(w, z)=U(\alpha, a)+v\frac{\partial U}{\partial \alpha}+\zeta\frac{\partial U}{\partial a}+\ldots$$

$$=U(\alpha, a)+\frac{\zeta}{\dfrac{\partial f}{\partial \alpha}}\frac{\partial(f, U)}{\partial(\alpha, a)}+\ldots,$$

and

$$\frac{\partial f}{\partial w}=\frac{\partial f}{\partial \alpha}+\frac{\zeta}{\dfrac{\partial f}{\partial \alpha}}\frac{\partial\left(f, \dfrac{\partial f}{\partial \alpha}\right)}{\partial(\alpha, a)}+\ldots.$$

Then unless

$$\frac{1}{U(\alpha, a)}\frac{\partial(f, U)}{\partial(\alpha, a)}=\frac{1}{\dfrac{\partial f}{\partial \alpha}}\frac{\partial\left(f, \dfrac{\partial f}{\partial \alpha}\right)}{\partial(\alpha, a)}$$

for (a_1, α_1), and

$$\frac{\partial(f, U)}{\partial(\alpha, a)}=0$$

for (a_2, a_2), (a_3, a_3), ..., (a_n, a_n), there will be terms in $\frac{1}{\zeta}$ in the expansion of the subject of integration in the vicinity of the respective points, and consequently there will be logarithmic infinities in the integral. Such infinities are to be excluded; and therefore their coefficients, being the residues, must vanish, so that, on this score, there appear to be n relations among the arbitrary constants in U. But, as in § 210, the sum of the residues for any point is zero: and therefore, when $n-1$ of them vanish, the remaining residue also vanishes. Hence, from this cause, there are only $n-1$ relations among the arbitrary constants in U.

The tale of independent arbitrary constants in $U(w, z)$, remaining after all the conditions are satisfied, is

$$\tfrac{1}{2}n(n+1)-(\delta+\kappa)-(n-1)-(n-1)$$
$$=p+1;$$

as each constant determines an integral, the inference is that there are $p+1$ linearly independent elementary integrals of the second kind with a common infinity.

209. Next, consider integrals which have logarithmic infinities, independently of or as well as algebraical infinities. They are called integrals of the *third kind.* As in the case of integrals of the first kind and the second kind, we take the subject of integration to be as general as possible so that it contains arbitrary coefficients; and we assign suitable relations among the coefficients to prevent any of the points, given as zeros of $\partial f/\partial w$, from becoming infinities of the integral. It follows that the only infinities of the integral are accidental singularities of coefficients of powers of w in the numerator of the general expression for w'; and that, when w' is expanded for points in the immediate vicinity of such an expression, the term with index -1 must occur.

To find the general value of an integral of the third kind, we should first exclude from the Riemann's surface all the infinite points, say

$$l_1, l_2, \ldots, l_\mu,$$

by small curves; the surface would then be resolved into one that is simply connected. The cross-cuts necessary for this purpose would consist of the set of $2p$ cross-cuts, necessary to resolve the surface for an integral of the first kind, and of the additional cross-cuts, μ in number and drawn from the boundary (taken at some ordinary point of the integral) to the small curves that surround the infinities of the function.

The moduli for the former set may be denoted by

$$\varpi_1, \varpi_2, \ldots, \varpi_p \text{ for the cuts } a_1, a_2, \ldots, a_p,$$

$$\text{and } \varpi_{p+1}, \varpi_{p+2}, \ldots, \varpi_{2p} \text{ for the cuts } b_1, b_2, \ldots, b_p \text{ respectively;}$$

they are zero for the cuts c. Taking the integral from one edge to the other of any one of the remaining cross-cuts $l_1, l_2, \ldots, l_q$, (where l_q is the cross-cut drawn from the curve surrounding l_q to the boundary), its value is given by the value of the integral round the small curve and therefore it is $2\pi i \lambda_q$, where the expansion of the subject of integration in the immediate vicinity of $z = l_q$ is

$$\ldots\ldots + \frac{A_2}{(z-l_q)^2} + \frac{\lambda_q}{z-l_q} + P(z-l_q).$$

Then, if Π be any value of the integral of the third kind at a point on the unresolved Riemann's surface, all its values at the point are included in the form

$$\Pi + \sum_{r=1}^{2p} m_r \varpi_r + 2\pi i \sum_{q=1}^{\mu} n_q \lambda_q,$$

where the coefficients $m_1, \ldots, m_{2p}, n_1, \ldots, n_\mu$ are integers.

210. It can be proved that *the quantities λ_q are subject to the relation*

$$\lambda_1 + \lambda_2 + \ldots + \lambda_\mu = 0.$$

Let the surface be resolved by the complete system of $2p + \mu$ cross-cuts: the resolved surface is simply connected and has only a single boundary. The subject of integration, w', is uniform and continuous over this resolved surface: it has no infinities in the surface, for its infinities have been excluded; hence

$$\int w' dz = 0,$$

when the integral is taken round the complete boundary of the resolved surface.

This boundary consists of the double edges of the cross-cuts a, b, c, L, and the small curves round the μ points l; the two edges of the same cross-cut being described in opposite directions in every instance.

Since the integral is zero and the function is finite everywhere along the boundary, the parts contributed by the portions of the boundary may be considered separately.

First, for any cross-cut, say a_q: let O be the point where it is crossed by b_q, and let the positive direction of description of the whole boundary be indicated by the arrows (fig. 81, p. 438). Then, for the portion $Ca\ldots E$, the part of the integral is $\int_C^E w' dz$, or, if $Ca\ldots E$ be the negative edge (as in § 196), the part of the integral may be denoted by

$$\int_C^E w' dz.$$

The part of the integral for the portion $F\ldots aD$, being the positive edge of the cross-cut, is $\int_F^D w' dz$, which may be denoted by $-\int_D^F w' dz$. The course and the range for the latter part are the same as those for the former, and w' is the same on the two edges of the cross-cut; hence the sum of the two is

$$= \int_C^E (w' - w')\, dz,$$

which evidently vanishes*. Hence the part contributed to $\int w'dz$ by the two edges of the cross-cut a_q is zero.

Similarly for each of the other cross-cuts a, and for each of the cross-cuts b, c, L.

The part contributed to the integral taken along the small curve enclosing l_q is $2\pi i \lambda_q$, for $q = 1, 2, \ldots, \mu$: hence the sum of the parts contributed to the integral by all these small curves is

$$2\pi i \sum_{q=1}^{\mu} \lambda_q.$$

All the other parts vanish, and the integral itself vanishes; hence

$$2\pi i \sum_{q=1}^{\mu} \lambda_q = 0,$$

establishing the result enunciated.

Corollary. *An integral of the third kind, that is, having logarithmic infinities on a Riemann's surface, must have at least two logarithmic infinities.*

If it had only one logarithmic infinity, the result just proved would require that λ_1 should vanish, and the infinity would then be purely algebraical.

211. The simplest instance is that in which there are only two logarithmic infinities; their constants are connected by the equation

$$\lambda_1 + \lambda_2 = 0.$$

If, in addition, the infinities be purely logarithmic, so that there are no algebraically infinite terms in the expansion of the integral in the vicinity of either of the points, the integral is then called an *elementary integral of the third kind*. If two points C_1 and C_2 on the surface be the two infinities, and if they be denoted by assigning the values c_1 and c_2 to z; and if $\lambda_1 = 1 = -\lambda_2$ (as may be assumed, for the assumption only implies division of the integral by a constant factor), the expansion of the subject of integration for points in the vicinity of c_1 is

$$\frac{1}{z - c_1} + P_1(z - c_1),$$

* It vanishes from two independent causes, first through the factor $w' - w'$, and secondly because $z_E = z_C$, the breadth of any cross-cut being infinitesimal.

The same result holds for each of the cross-cuts a and b.

For each of the cross-cuts c and L, the sum of the parts contributed by opposite edges vanishes only on account of the factor $w' - w'$; in these cases the variable z is not the same for the upper and lower limit of the integral.

and for points in the vicinity of c_2 the expansion is

$$\frac{-1}{z-c_2}+P_2(z-c_2).$$

Such an integral may be denoted by Π_{12}: its modulus, consequent on the logarithmic infinity, is $2\pi i$.

Ex. 1. Prove that, if Π_{12}, Π_{23}, Π_{31} be three elementary integrals of the third kind having c_1, c_2; c_2, c_3; c_3, c_1 for their respective pairs of points of logarithmic discontinuity, then $\Pi_{12}+\Pi_{23}+\Pi_{31}$ is either an integral of the first kind or a constant.

Clebsch and Gordan pass from this result to a limit in which the points c_1 and c_2 coincide and obtain an expression for an elementary integral of the second kind in the form of the derivative of Π_{13} with regard to c_1. Klein, following Riemann, passes from an elementary integral of the second kind to an elementary integral of the third kind by integrating the former with regard to its parametric point*.

Ex. 2. Reverting again to the integrals connected with the algebraical equation $f(w, z)=0$, when it can be interpreted as the equation of a generalised curve, an integral of the third kind arises when the subject of integration is

$$w'=\frac{V(w, z)}{(z-c)\dfrac{\partial f}{\partial w}},$$

where $V(w, z)$ is of degree $n-2$ in w. If $V(w, z)$ be of degree in z not higher than $n-2$, the integral of w' is not infinite for infinite values of z; so that $V(w, z)$ is a general integral algebraical function of w of degree $n-2$.

Corresponding to the algebraical value c of z, there are n points on the surface, say (c_1, k_1), (c_2, k_2), ..., (c_n, k_n); and the expansion of w' in the vicinity of (c_r, k_r) is

$$\frac{V(k_r, c_r)}{\dfrac{\partial f}{\partial k_r}}\frac{1}{z-c_r}+\dots$$

the coefficients of the infinite terms being subject to the relation

$$\sum_{r=1}^{n}\frac{V(k_r, c_r)}{\dfrac{\partial f}{\partial k_r}}=0,$$

because $V(w, z)$ is only of degree $n-2$ in w. The integral of w' will have a logarithmic infinity at each point, unless the corresponding coefficient vanish.

Not more than $n-2$ of these coefficients can be made to vanish, unless they all vanish; and then the integral has no logarithmic infinity. Let $n-2$ relations, say

$$V(k_r, c_r)=0$$

for $r=2, 3, \dots, n$, be chosen; and let the $\delta+\kappa$ relations be satisfied which secure that the integral is finite at the singularities of the curve $f(w, z)=0$. Then the integral is an elementary integral of the third kind, having (c_1, k_1) and (c_2, k_2) for its points of logarithmic discontinuity.

Ex. 3. Prove that there are $p+1$ linearly independent elementary integrals of the third kind, having the same logarithmic infinities on the surface.

* Clebsch und Gordan, (l.c., p. 361, note), pp. 28—33; Klein-Fricke, *Vorlesungen über die Theorie der elliptischen Modulfunctionen*, t. i, pp. 518—522; Riemann, p. 100.

Ex. 4. Shew that, in connection with the fundamental equation

$$w^3+z^3=1$$

any integral of the first kind is a constant multiple of

$$\int\frac{dz}{w^2};$$

that an integral of the second kind, of the class considered in Ex. 2, § 207, is given by

$$\int\frac{1-w}{z^2w^2}\,dz;$$

and that an elementary integral of the third kind is given by

$$\int\frac{1-w}{zw^2}\,dz.$$

Ex. 5. An elementary (Jacobian) elliptic integral of the third kind occurs in Ex. 7, p. 385; and a (Jacobian) elliptic integral of the second kind occurs in Ex. 8, p. 386.

Shew that an elementary (elliptic) integral of the second kind, associated with the equation

$$w^2=4z^3-g_2z-g_3,$$

and having its infinity at (c_1, γ_1), is

$$\int\frac{\gamma_1(w+\gamma_1)+(6c_1^2-\tfrac{1}{2}g_2)(z-c_1)}{(z-c_1)^2w}\,dz;$$

and that an elementary (elliptic) integral of the third kind, associated with the same equation and having its two infinities at (c_1, γ_1), (c_2, γ_2), is

$$\tfrac{1}{2}\int\left(\frac{w+\gamma_1}{z-c_1}-\frac{w+\gamma_2}{z-c_2}\right)\frac{dz}{w}.$$

A sufficient number of particular examples, and also of examples with a limited generality, have been adduced to indicate some of the properties of functions arising, in the first instance, as integrals of multiform functions of a variable z (or as integrals of uniform functions of position on a Riemann's surface). The succeeding investigation establishes, from the most general point of view, the existence of such functions on a Riemann's surface: they will no longer be regarded as defined by integrals of multiform functions.

CHAPTER XVII.

Schwarz's proof of the Existence-Theorem.

212. The investigations in the preceding chapter were based on the supposition that a fundamental equation was given, the appropriate Riemann's surface being associated with it. The general expression of uniform functions of position on the surface was constructed, and the integrals of such functions were considered. These integrals in general were multiform on the surface, the deviation from uniformity consisting in the property that the difference between any two of the infinite number of values could be expressed as a linear combination of integral multiples of certain constants associated with the function. Infinities of the functions defined by the integrals, and the classification of the functions according to their infinities, were also considered.

But all these investigations were made either in connection with very particular forms of the fundamental equation, or with a form of not unlimited generality: and, for the latter case, assumptions were made, justified by the analysis so far as it was carried, but not established generally.

In order to render the consideration of the propositions complete, it must be made without any limitations upon the general form of fundamental equation.

Moreover, the second question of § 192, viz., the existence of functions (both uniform and multiform) of position on a surface given independently of any algebraical equation, is as yet unconsidered.

The two questions, in their generality, can be treated together. In the former case, with the fundamental equation there is associated a Riemann's surface, the branching of which is determined by that fundamental equation; in the latter case, the Riemann's surface with assigned branching is supposed given*. We shall take the surface as having one boundary and being otherwise closed; the connectivity is therefore an uneven integer, and it will be denoted by $2p+1$.

213. The problem can be limited initially, so as to prevent unnecessary complications. All the functions to be discussed, whether they be algebraical functions or integrals of algebraical functions, can be expressed in the form $u+iv$, where u and v are two real functions of two independent real variables x and y. It has already (§ 10) been proved that both u and v satisfy the equation

$$\nabla^2\theta = \frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} = 0,$$

and that, if either u or v be known, the other can be derived by a quadrature at most, and is determinate save as to an additive arbitrary constant. Since therefore w is determined by u, save as to an additive constant, we shall, in the first place, consider the properties of the real function u only.

The result is valid so long as v can be determined, that is, so long as the function u has differential coefficients. It will appear, in the course of the present chapter, that no conditions are attached to the derivatives of u along the boundary of an area, so that the determination of v along such a boundary seems open to question.

It has been (§ 36) proved, in a theorem due to Schwarz, that, if w a function of z be defined for a half-plane and if it have real finite continuous values along any portion of the axis of x, it can be symmetrically continued across that portion of the axis. The continuation is therefore possible for the real part u of the function w; and the values of u are the real finite continuous values of w along that portion of the axis.

It will be seen, in Chapters XIX., XX. that, by changing the independent variables, the axis of x can be changed into a circle or other analytical line (§ 221); so that a function u, defined for an interior and having real finite continuous values along any portion of the boundary, can be continued across that portion of the boundary, which is therefore not the limit of existence† of u.

* The surface is supposed given; we are not concerned with the quite distinct question as to how far a Riemann's surface is determinate by the assignment of its number of sheets, its branch-points (and consequently of its connectivity), and of its branch-lines. This question is discussed by Hurwitz, *Math. Ann.*, t. xxxix, (1891), pp. 1—61. He shews that, if Ω denote the ramification (§ 179) of the surface which, necessarily an even integer, is defined as the sum of the orders of its branch-points, a two-sheeted surface is made uniquely determinate by assigned branch-points; the number of essentially distinct three-sheeted surfaces, made determinate by assigned branch-points, is $\frac{1}{2}(3^{\Omega-2}-1)$; and so on. It is easy to verify that the number of distinct three-sheeted surfaces, with 4 assigned points as simple branch-points, is 4: an example suggested to me by Mr Burnside.

† The continuation indicated will be carried out for the present case by means of the combination of areas (§ 222), and without further reference to the transformation indicated or to Schwarz's theorem on symmetrical continuation.

The derivatives of u can then be obtained in the extended space and so v can be determined for the boundary*.

And, what is more important, it will be found that, under conditions to be assigned, the number of functions u that are determined is double the number of functions w that are determined; the complete set of functions u lead to all the parts u and v of the functions w (§ 234, note).

214. The infinities of u at any point are given by the real parts of the terms which indicate the infinities of w. Conversely, when the infinities of u are assigned in functional form, those of w can be deduced, the form of the associated infinities of v first being constructed by quadratures.

The periods of w, being the moduli at the cross-cuts, lead to real constants as differences of u at opposite edges of cross-cuts, or, if we choose, as constant differences of values of u at points on definite curves, conveniently taken for reference as lines of possible cross-cuts. Conversely, a real constant modulus for u is the real part† of the corresponding modulus of w.

Hence a function, w, of position on a Riemann's surface is, except as to an additive constant, determined by a real function u of x and y (where $x + iy$ is the independent variable for the surface), if u be subject to the conditions:—

(i) it satisfies the equation $\nabla^2 u = 0$ at all points on the surface where its derivatives are not infinite:

(ii) if it be multiform, its values at any point on the surface differ by linear combinations of integral multiples of real constants: otherwise, it is uniform:

(iii) it may have specified infinities, of given form in the vicinity of assigned points on the surface.

In addition to these general conditions imposed upon the function u, it is convenient to admit as a further possible condition, for portions of the surface, that the function u shall assume, along a closed curve, values which are always finite. But it must be understood that this condition is used only for subsidiary purposes: it will be seen that it causes no limitation on the final result, all that is essential in its limitations being merged in the three dominant conditions.

The questions for discussion are therefore (i), the existence of functions‡ satisfying the above conditions in connection with a given Riemann's

* See Phragmén, *Acta Math.*, t. xiv, (1890), pp. 225—227, for some remarks upon this question.

† The imaginary parts of the moduli of w are determinate with the imaginary part of w: see remark at end of § 213, and the further reference there given.

‡ The functions u (and also v) are of great importance in mathematical physics for two-dimensional phenomena in branches such as gravitational attraction, electricity, hydrodynamics and heat. In all of them, the function represents a potential; and, consequently, in the general theory of functions, it is often called a *potential function*.

surface, the connectivity of which is $2p+1$ as dependent upon its branching and the number of its sheets; and (ii), assuming that the functions exist, their determination by the assigned conditions.

215. There are many methods for the discussion of these questions. The potential function, both for two and for three dimensions in space, first arose in investigations connected with mathematical physics: and, so far as concerns such subjects, its theory was developed by Poisson, Green, Gauss, Stokes, Thomson, Maxwell and others. Their investigations have reference to applications to mathematical physics, and they do not tend towards the solution of the questions just propounded in relation to the general theory of functions.

Klein uses considerations drawn from mathematical and experimental physics to establish the existence of potential functions under the assigned conditions. The proof that will be adopted brings the stages of the investigation into closer relations with the preceding and the succeeding parts of the subject than is possible if Klein's method be followed*.

To establish the existence of the functions under the assigned conditions, Riemann† uses the so-called Dirichlet's Principle‡; but as Riemann's proof of the principle is inadequate, his proof of the existence-theorem cannot be considered complete.

There are two other principal, and independent, methods of importance, each of which effectively establishes the existence of the functions, due to Neumann and to Schwarz respectively; each of them avowedly dispenses§ with the use of Dirichlet's Principle.

The courses of the methods have considerable similarity. Both begin with the construction of the function for a circular area. Neumann uses what is commonly called the method of the arithmetic mean, for gradual approximation to the value of the potential function for a region bounded by a convex curve: Schwarz uses the method of conformal representation, to deduce from results previously obtained, the potential function for regions bounded by analytical curves; and both authors use certain methods for combination of areas, for each of which the potential function has been constructed‖.

* Klein's proof occurs in his tract, already quoted, *Ueber Riemann's Theorie der algebraischen Functionen und ihrer Integrale*, (Leipzig, Teubner, 1882), and it is modified in his memoir "Neue Beiträge zur Riemann'schen Functionentheorie," *Math. Ann.*, t. xxi, (1883), pp. 141—218, particularly pp. 160—162.

† *Ges. Werke*, pp. 35—39, pp. 96—98.

‡ Riemann enunciates it, (l.c.), pp. 34, 92.

§ Neumann, *Vorlesungen über Riemann's Theorie der Abel'schen Integrale*, (2nd ed., 1884), p. 238; Schwarz, *Ges. Werke*, ii, p. 171.

‖ Neumann's investigations are contained in various memoirs, *Math. Ann.*, t. iii, (1871), pp. 325—349; ib., t. xi, (1877), pp. 558—566; ib., t. xiii, (1878), pp. 255—300; ib., t. xvi, (1880), pp. 409—431; and the methods are developed in detail and amplified in his treatise

What follows in the present chapter is based upon Schwarz's investigations: the next chapter is based upon the investigations of both Schwarz and Neumann, and, of course, upon Riemann's memoirs.

The following summary of the general argument will serve to indicate the main line of the proof of the establishment of potential functions satisfying assigned conditions.

I. A potential function u is uniquely determined by the conditions: that it, as well as its derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y^2}$ (which satisfy the equation $\nabla^2 u = 0$), shall be uniform, finite and continuous, for all points within the area of a circle; and that, along the circumference of the circle, the function shall assume assigned values that are always finite, uniform and, except at a limited number of isolated points where there is a sudden (finite) change of value, continuous. (§§ 216—220.)

II. By using the principle of conformal representation, areas bounded by curves other than circles—say by analytical curves—are obtained, over which the potential function is uniquely determined by general conditions within the area and assigned values along its boundary. (§ 221.)

III. The method of combination of areas, dependent upon an alternating process, leads to the result that a function exists for a given region, satisfying the general conditions in that region and acquiring assigned finite values along the boundary, when the region can be obtained by combinations of areas that can be conformally represented upon the area of a circle. (§ 222.)

IV. The theorem is still valid when the region (supposed simply connected) contains a branch-point; the winding-surface is transformed by a relation

$$z - c = RZ^m$$

into a single-sheeted surface, for which the theorem has already been established.

When the surface is multiply connected, we resolve it by cross-cuts into one that is simply connected, before discussing the function. (§ 223.)

V. Real functions exist on a Riemann's surface, which are everywhere finite and

Ueber das logarithmische und Newton'sche Potential (Leipzig, Teubner, 1877) and in his treatise quoted in the preceding note. In this connection, as well as in relation to Schwarz's investigations, and also in view of some independence of treatment, Harnack's treatise, *Die Grundlagen der Theorie des logarithmischen Potentiales und der eindeutigen Potentialfunction in der Ebene* (Leipzig, Teubner, 1887), and a memoir by Harnack, *Math. Ann.*, t. xxxv, (1890), pp. 19—40, may be consulted.

A modification of Neumann's proof, due to Klein, is given in the first volume (pp. 508—522) of the treatise cited on p. 403, note.

Schwarz's investigations are contained in various memoirs occurring in the second volume of his *Gesammelte Werke*, pp. 108—132, 133—143, 144—171, 175—210, 303—306: their various dates and places of publication are there stated. A simple and interesting general statement of the gist of his results will be found in a critical notice of the two volumes of his collected works, written by Henrici in *Nature* (Feb. 5, 12, 1891, pp. 321—323, 349—352). There is a comprehensive memoir by Ascoli, based upon Schwarz's method, "Integration der Differentialgleichung $\nabla^2 u = 0$ in einer beliebigen Riemann'schen Fläche," (*Bih. t. kongl. Svenska Vet. Akad. Handl.*, bd. xiii, 1887, Afd. 1, n. 2; 83 pp.); a thesis by Jules Riemann, *Sur le problème de Dirichlet*, (Thèse, Gauthier-Villars, Paris, 1888), discusses a number of Schwarz's theorems (see, however, Schwarz, *Ges. Werke*, t. ii, pp. 356—358); and an independent memoir by Prym, *Crelle*, t. lxxiii, (1871), pp. 340—364, may be consulted.

The literature of this part of the subject is very wide in extent: many other references are given by the authors already quoted.

uniquely determinate by arbitrarily assigned real moduli of periodicity at the cross-cuts. (§§ 224—227.)

VI. Functions exist, satisfying the conditions in (V) except that they may have at isolated points on the surface, infinities of an assigned form. (§ 229.)

216. We shall, in the first place, treat of potential functions that have no infinities, either algebraical or logarithmic, over some continuous area on the surface limited by a simple closed boundary, or by a number of non-intersecting simple closed curves constituting the boundary; for the present, the area thus enclosed will be supposed to lie in one and the same sheet, so that we may regard the area as lying in a simple plane.

At all points within the area and on its boundary, the function u is finite and will be supposed uniform and continuous; for all points within the area (but not necessarily for points on the boundary), the derivatives

$$\frac{\partial u}{\partial x},\quad \frac{\partial u}{\partial y},\quad \frac{\partial^2 u}{\partial x^2},\quad \frac{\partial^2 u}{\partial y^2}$$

are uniform, finite and continuous and they satisfy the equation $\nabla^2 u = 0$. These may be called the *general* conditions.

Two cases occur according as the character of the derivatives at points in the area is or is not assigned for points on the boundary; if the character be assigned, there will then be what may be called *boundary* conditions. The two cases therefore are:

(A) When a function u is required to satisfy the general conditions, and its derivatives are required to satisfy the boundary conditions:

(B) When the only requirement is that the function shall satisfy the general conditions.

Before proceeding to the establishment of what is the fundamental proposition in Schwarz's method, it is convenient to prove three lemmas and to deduce some inferences that will be useful.

LEMMA I. *If two functions u_1 and u_2 satisfy the general conditions for two regions T_1 and T_2 respectively, which have a common portion T that is more than a point or a line, and if u_1 and u_2 be the same for the common portion T, then they define a single function for the whole region composed of T_1 and T_2.*

This proposition can be made to depend upon the continuation of analytical functions*, whether in a plane (§ 34) or, in view of a subsequent transformation (§ 223), on a Riemann's surface.

The real function u_1 defines a function w_1 of the complex variable z, for any point in the region T_1; and for points within this region, the function w_1 is uniquely determined by means of its own value and the values of its derivatives at any point within T_1, obtained, if necessary, by a succession of elements

* For other proofs, see Schwarz, ii, pp. 201, 202 and references there given.

in continuation. Hence the value of w_1 and its derivatives at any point within T defines a function existing over the whole of T_1.

Similarly the real function u_2 defines a function w_2 within T_2, and this function is uniquely determined over the whole of T_2 by its value and the value of its derivatives at any point within T.

Now the values of u_1 and u_2 are the same at all points in T, and therefore the values of w_1 and w_2 are the same at all points in T, except possibly for an additive (imaginary) constant, say $i\alpha$, so that

$$w_1 = w_2 + i\alpha.$$

Hence for all points in T, (supposed not to be a point, so that we may have derivatives in every direction (§ 8): and not to be a line, so that we may have derivatives in all directions from a point on the line), the derivatives of w_1 agree with those of w_2; and therefore the quantities necessary to define the continuation of w_1 from T over T_1 agree with the quantities necessary to define the continuation of w_2 from T over T_2, except only that w_1 and w_2 differ by an imaginary constant. Hence, having regard to the form of the elements, w_1 and w_2 can be continued over the region composed of T_1 and T_2, and their values differ (possibly) by the imaginary constant. When we take the real parts of the functions, we have u_1 and u_2 defining a single function existing over the whole region occupied by the combination of T_1 and T_2.

The other two lemmas relate to integrals connected with potential functions.

Lemma II. *Let u be a function required to satisfy the general conditions, and let its derivatives be required to satisfy the boundary conditions, for an area S bounded by simple non-intersecting curves: then*

$$\int \frac{\partial u}{\partial n} ds = 0:$$

where the integral is extended round the whole boundary in the direction that is positive with regard to the bounded area S; and dn is an element of the normal to a boundary-line drawn towards the interior of the space enclosed by that boundary-line regarded merely as a simple closed curve.*

Let P and Q be any two functions, which, as well as their first and second derivatives with regard to x and to y, are uniform finite and continuous for all points within S and on its boundary. Then, proceeding as in § 16 and taking account of the conditions to which P and Q are subject, we have

$$\iint P\nabla^2 Q dx dy = \int P \left(\frac{\partial Q}{\partial x} dy - \frac{\partial Q}{\partial y} dx\right) - \iint \left(\frac{\partial P}{\partial x}\frac{\partial Q}{\partial x} + \frac{\partial P}{\partial y}\frac{\partial Q}{\partial y}\right) dx dy;$$

* The element dn of the normal is, by this definition, measured inwards to, or outwards from, the area S according as the particular boundary-line is described in the positive, or in the negative, trigonometrical sense. Thus, if S be the space between two concentric circles, the element dn at each circumference is drawn *towards* its centre; the directions of integration are as in § 2.

where ∇^2 denotes $\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$, the double integrals extend over the area of S, and the single integral is taken round the whole boundary of S in the direction that is positive for the bounded area S.

Let ds be an element PT of arc of the boundary at a point (x, y), and dn be an element TQ of the normal at T drawn to the interior of the space included by the boundary-line regarded as a simple closed curve; and let ψ be the inclination of the tangent at T. Then in (i), as TQ is drawn to the interior of the area included by the curve, the direction of integration being indicated by the arrow (so that S lies within the curve), we have

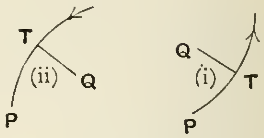

Fig. 78.

$$dx = ds\cos\psi - dn\sin\psi,\quad dy = ds\sin\psi + dn\cos\psi;$$

and therefore it follows that, for any function R,

$$\frac{\partial R}{\partial n} = -\frac{\partial R}{\partial x}\sin\psi + \frac{\partial R}{\partial y}\cos\psi.$$

Now for variations along the boundary we have $dn = 0$, so that

$$-\frac{\partial R}{\partial n}ds = \frac{\partial R}{\partial x}dy - \frac{\partial R}{\partial y}dx.$$

And in (ii), as TQ is drawn to the interior of the area included by the curve, the direction of integration being indicated by the arrow (so that S lies without the curve), we have

$$dx = (-ds)\cos\psi + dn\sin\psi,\quad dy = (-ds)\sin\psi - dn\cos\psi,$$

and therefore $$\frac{\partial R}{\partial n} = \frac{\partial R}{\partial x}\sin\psi - \frac{\partial R}{\partial y}\cos\psi,$$

so that, as before, for variations along the boundary,

$$-\frac{\partial R}{\partial n}ds = \frac{\partial R}{\partial x}dy - \frac{\partial R}{\partial y}dx.$$

Hence, with the conventions as to the measurement of dn and ds, we have

$$\int P\left(\frac{\partial Q}{\partial x}dy - \frac{\partial Q}{\partial y}dx\right) = -\int P\frac{\partial Q}{\partial n}ds,$$

both integrals being taken round the whole boundary of S in a direction that is positive as regards S. Therefore

$$\iint P\nabla^2 Q\,dxdy = -\int P\frac{\partial Q}{\partial n}ds - \iint\left(\frac{\partial P}{\partial x}\frac{\partial Q}{\partial x} + \frac{\partial P}{\partial y}\frac{\partial Q}{\partial y}\right)dxdy.$$

In the same way, we obtain the equation

$$\iint Q\nabla^2 P\,dxdy = -\int Q\frac{\partial P}{\partial n}ds - \iint\left(\frac{\partial P}{\partial x}\frac{\partial Q}{\partial x} + \frac{\partial P}{\partial y}\frac{\partial Q}{\partial y}\right)dxdy;$$

and therefore $$\iint(P\nabla^2 Q - Q\nabla^2 P)\,dxdy = \int\left(Q\frac{\partial P}{\partial n} - P\frac{\partial Q}{\partial n}\right)ds,$$

where the double integral extends over the whole of S, and the single integral is taken round the whole boundary of S in the direction that is positive for the bounded area S.

Now let u be a potential function defined as in the lemma; then u satisfies all the conditions imposed on P, as well as the condition $\nabla^2 u = 0$ throughout the area and on the boundary. Let $Q = 1$; so that $\nabla^2 Q = 0$, $\frac{\partial Q}{\partial n} = 0$. Each element of the left-hand side is zero, and there is no discontinuity in the values of P and Q; the double integral therefore vanishes, and we have

$$\int \frac{\partial u}{\partial n} ds = 0,$$

the result which was to be proved.

But if the derivatives of u are not required to satisfy the boundary conditions, the foregoing equation may not be inferred; we then have the following proposition.

Lemma III. *Let u be a function, which is only required to satisfy the general conditions for an area S; and let u' be any other function, which is required to satisfy the general conditions for that area and may or may not be required to satisfy the boundary conditions. Let A be an area entirely enclosed in S and such that no point of its whole boundary lies on any part of the whole boundary of S; then*

$$\int \left(u \frac{\partial u'}{\partial n} - u' \frac{\partial u}{\partial n} \right) ds = 0,$$

where the integral is taken round the whole boundary of A in a direction which is positive with regard to the bounded area A, and the element dn of the normal to a boundary-line is drawn towards the interior of the space enclosed by that boundary-line, regarded merely as a simple closed curve.

The area A is one over which the functions u and u' satisfy the general conditions. The derivatives of these functions satisfy the boundary-conditions for A, because they are uniform, finite and continuous for all points inside S, and the boundary of A is limited to lie entirely within S. Hence

$$\iint (u \nabla^2 u' - u' \nabla^2 u)\, dx dy = - \int \left(u \frac{\partial u'}{\partial n} - u' \frac{\partial u}{\partial n} \right) ds,$$

the integrals respectively referring to the area of A and its boundary in a direction positive as regards A. But, for every point of the area, $\nabla^2 u = 0$, $\nabla^2 u' = 0$; and u and u' are finite. Hence the double integral vanishes, and therefore

$$\int \left(u \frac{\partial u'}{\partial n} - u' \frac{\partial u}{\partial n} \right) ds = 0,$$

taken round the whole boundary of A in the positive direction.

One of the most effective modes of choosing a region A of the above character is as follows. Let a simple curve C_1 be drawn lying entirely within the area S, so that it does not meet the boundary of S; and let another simple curve C_2 be drawn lying entirely within C_1, so that it does not meet C_1 and that the space between C_1 and C_2 lies in S. This space is an area of the character of A, and it is such that for all internal points, as well as for all points on the whole of its boundary (which is constituted by C_1 and C_2), the conditions of the preceding lemma apply. The curve C_2 in the above integration is described positively relative to the area which it includes: the curve C_1 is described, as in § 2, negatively relative to the area which it includes. Hence, for such a space, the above equation is

$$\int\left(u\frac{\partial u'}{\partial n} - u'\frac{\partial u}{\partial n}\right) ds_1 - \int\left(u\frac{\partial u'}{\partial n} - u'\frac{\partial u}{\partial n}\right) ds_2 = 0,$$

if the integrals be now extended round the two curves in a direction that is positive relative to the area enclosed by each, and if in each case the normal element dn be drawn from the curve towards the interior.

217. We now proceed to prove that *a function u, required to satisfy the general conditions for an area included within a circle, is uniquely determined by the series of values assigned to u along the circumference of the circle.*

Let the circle S be of radius R and centre the origin. Take an internal point $z_0 = re^{\phi i}$, and its inverse $z_0' = r'e^{\phi i}$ (such that $rr' = R^2$): so that z_0' is external to the circle. Then the curves determined by

$$\left|\frac{z - z_0}{z - z_0'}\right| = \frac{r}{R}\lambda,$$

for real values of λ, are circles which do not meet one another. The boundary of S is determined by $\lambda = 1$, and $\lambda = 0$ gives the point z_0 as a limiting circle: and the whole area of S is obtained by making the real parameter λ change continuously from 0 to 1.

Lemma III. may be applied. We choose, as the ring-space, the area included between the two circles determined by λ_1 and λ_2, where

$$1 > \lambda_1 > \lambda_2 > 0;$$

and then we have

$$\int\left(u\frac{\partial u'}{\partial n} - u'\frac{\partial u}{\partial n}\right) ds_1 = \int\left(u\frac{\partial u'}{\partial n} - u'\frac{\partial u}{\partial n}\right) ds_2,$$

where the integrals are taken round the two circumferences in the trigonometrically positive direction (dn being in each case a normal element drawn towards the centre of its own circle), and the function u' satisfies the general and the boundary conditions for the ring-area considered. Moreover, the area between the circles, determined by λ_1 and λ_2, is one for which u satisfies the general conditions, and its derivatives certainly satisfy the boundary conditions: hence

$$\int \frac{\partial u}{\partial n} ds_1 = 0, \qquad \int \frac{\partial u}{\partial n} ds_2 = 0.$$

Now the function u' is at our disposal, subject to the general conditions for the area between the two λ-circles and the boundary conditions for each of those circles. All these conditions are satisfied by taking u' as the real part of $\log\left(\frac{z - z_0}{z - z_0'}\right)$, that is, in the present case,

$$u' = \log \left| \frac{z - z_0}{z - z_0'} \right|.$$

For all points on the outer circle u' is equal to the constant $\log\left(\frac{r}{R}\lambda_1\right)$, so that

$$\int u' \frac{\partial u}{\partial n} ds_1 = 0:$$

and similarly for all points on the inner circle u' is equal to the constant $\log\left(\frac{r}{R}\lambda_2\right)$, so that

$$\int u' \frac{\partial u}{\partial n} ds_2 = 0.$$

Again, for a point z on the outer circle, whose angular coordinate is ψ, the value of $\frac{\partial u'}{\partial n}$ for an inward drawn normal is (§ 11)

$$-\frac{(R^2 - r^2\lambda_1^2)^2}{\lambda_1 R (R^2 - r^2)\{R^2 - 2Rr\lambda_1 \cos(\psi - \phi) + r^2\lambda_1^2\}};$$

and because the radius of that outer circle is $\lambda_1 R (R^2 - r^2)/(R^2 - r^2\lambda_1^2)$, we have

$$ds_1 = \frac{\lambda_1 R (R^2 - r^2)}{R^2 - r^2\lambda_1^2} d\psi.$$

Denoting by $f(\lambda_1, \psi)$ the value of u at this point ψ on the circle determined by λ_1, we have

$$\int u \frac{\partial u'}{\partial n} ds_1 = -\int_0^{2\pi} f(\lambda_1, \psi) \frac{R^2 - r^2\lambda_1^2}{R^2 - 2Rr\lambda_1 \cos(\psi - \phi) + r^2\lambda_1^2} R d\psi.$$

Similarly for the inner circle, the normal element again being drawn towards its centre, we have

$$\int u \frac{\partial u'}{\partial n} ds_2 = -\int_0^{2\pi} f(\lambda_2, \psi) \frac{R^2 - r^2\lambda_2^2}{R^2 - 2Rr\lambda_2 \cos(\psi - \phi) + r^2\lambda_2^2} R d\psi.$$

Combining these results, we have

$$\int_0^{2\pi} f(\lambda_1, \psi) \frac{R^2 - r^2\lambda_1^2}{R^2 - 2Rr\lambda_1 \cos(\psi - \phi) + r^2\lambda_1^2} d\psi$$

$$= \int_0^{2\pi} f(\lambda_2, \psi) \frac{R^2 - r^2\lambda_2^2}{R^2 - 2Rr\lambda_2 \cos(\psi - \phi) + r^2\lambda_2^2} d\psi.$$

In the analysis which has established this equation, λ_1 and λ_2 can have all values between 1 and 0: the limiting value 0 is excluded because then u' is not finite, and the limiting value 1 is excluded because no supposition has been made as to the character of the derivatives of u at the circumference of S.

The equation which has been obtained involves only the values of u but not the values of its derivatives. Since the values of u are finite both for $\lambda = 0$ and $\lambda = 1$, and the integrals are finite, the exclusion of the limiting values of λ need not be applied to the equation, although the exclusion was necessary during the proof, owing to the presence of quantities that have since disappeared. Hence the equation is valid when we take $\lambda_1 = 1$, $\lambda_2 = 0$.

When $\lambda_2 = 0$, the corresponding circle collapses to the point z_0: the value of $f(\lambda_2, \psi)$ is then the value of u at z_0 say $u(r, \phi)$; and the integral connected with the second circle is $2\pi u(r, \phi)$.

When $\lambda_1 = 1$, the corresponding circle is the circle of radius R; the value of $f(\lambda_1, \psi)$ is then the assigned value of u at the point ψ on the circumference, say the function $f(\psi)$. Substituting these values, we have

$$u(r, \phi) = \frac{1}{2\pi}\int_0^{2\pi} f(\psi)\frac{R^2 - r^2}{R^2 - 2Rr\cos(\psi - \phi) + r^2}\,d\psi,$$

the integral being taken positively round the circumference of the circle S.

It therefore appears that the function u, subjected to the general conditions for the area of the circle, is uniquely determined by the values assigned to it along the circumference of the circle.

The general conditions for u imply certain restrictions on the boundary values. These values must be finite, continuous and uniform: and therefore $f(\psi)$, as a function of ψ, must be finite, continuous, uniform and periodic in ψ of period 2π.

218. It is easy to verify that, when the boundary values $f(\psi)$ are not otherwise restricted, all the conditions attaching to u are satisfied by the function which the integral represents.

Since the real part of $(Re^{\psi i} + z)/(Re^{\psi i} - z)$ is the fraction

$$(R^2 - r^2)/\{R^2 - 2Rr\cos(\psi - \phi) + r^2\},$$

it follows that u is the real part of the function $F(z)$, defined by the equation

$$F(z) = \frac{1}{2\pi}\int \frac{Re^{\psi i} + z}{Re^{\psi i} - z} f(\psi)\,d\psi.$$

For all values of z such that $|z| < R$, the fraction can be expanded in a series of positive integral powers of z, which converges unconditionally and uniformly; and therefore $F(z)$ is a uniform, continuous, analytical function, everywhere finite for such values of z. Hence all its derivatives are uniform, continuous, analytical functions, finite for those values of z; and these properties are possessed by the real and the imaginary parts of such derivatives. Now $\dfrac{\partial^{m+n} u}{\partial x^m \partial y^n}$ is the real part of $i^n \dfrac{d^{m+n} F(z)}{dz^{m+n}}$; and therefore, for all integers m and n positive or zero, it is a uniform, finite and continuous function for points such that $|z| < R$, that is, for points within the circle. Moreover, since u is the real part of a function of z, and has its differential coefficients uniform, finite and continuous, it satisfies the differential equation $\nabla^2 u = 0$.

To infer the continuity of approach of $u(r, \phi)$ to $f(\phi)$ as r is made equal to R, we change the integral expression for $u(r, \phi)$ into

$$\frac{1}{2\pi}\int_{-\phi}^{2\pi-\phi} \frac{R^2 - r^2}{R^2 - 2Rr\cos\theta + r^2} f(\theta+\phi)\, d\theta.$$

Moreover for all values of $r < R$ (but not for $r = R$), we have

$$\frac{1}{2\pi}\int_{-\phi}^{2\pi-\phi} \frac{R^2 - r^2}{R^2 - 2Rr\cos\theta + r^2}\, d\theta = \frac{1}{\pi}\left[\tan^{-1}\left\{\frac{R+r}{R-r}\tan\tfrac{1}{2}\theta\right\}\right]_{-\phi}^{2\pi-\phi} = 1;$$

and therefore

$$\begin{aligned} I &= u(r, \phi) - f(\phi) \\ &= \frac{1}{2\pi}\int_{-\phi}^{2\pi-\phi} \{f(\theta+\phi) - f(\phi)\}\, \frac{R^2 - r^2}{R^2 - 2Rr\cos\theta + r^2}\, d\theta. \end{aligned}$$

Let Θ denote the subject of integration in the last integral. Then, as r is made to approach indefinitely near to R in value, Θ becomes infinitesimal for all values of θ except those which are extremely small, say for values of θ between $-\delta$ and $+\delta$. Dividing the integral into the corresponding parts, we have

$$I = \frac{1}{2\pi}\int_{-\phi}^{-\delta} \Theta d\theta + \frac{1}{2\pi}\int_{\delta}^{2\pi-\phi} \Theta d\theta + \frac{1}{2\pi}\int_{-\delta}^{\delta} \Theta d\theta.$$

Let M be the greatest value of $f(\psi)$ for points along the circle. Then the first integral and the second integral are less than

$$\frac{\phi - \delta}{2\pi} 2M \frac{R^2 - r^2}{(R-r)^2 + 2Rr(1-\cos\delta)} \quad\text{and}\quad \frac{2\pi - \delta - \phi}{2\pi} 2M \frac{R^2 - r^2}{(R-r)^2 + 2Rr(1-\cos\delta)}$$

respectively; by taking r indefinitely near to R in value, these quantities can be made as small as we please. For the third integral, let k be the greatest value of $f(\phi+\theta) - f(\phi)$ for values of θ between δ and $-\delta$: then the third integral is less than

$$\frac{k}{2\pi}\int_{-\delta}^{\delta} \frac{R^2 - r^2}{R^2 - 2Rr\cos\theta + r^2}\, d\theta,$$

that is, it is less than $\dfrac{2k}{\pi}\tan^{-1}\left(\tfrac{1}{2}\dfrac{R+r}{R-r}\delta\right)$; so that, when r is made nearly equal to R, the third integral is less than k.

If then k be infinitesimal, as is the case when $f(\phi)$ is everywhere finite and continuous, the quantity I can be diminished indefinitely; hence $u(r, \phi)$ continuously changes into the function $f(\phi)$ as r is made equal to R. The verification that the function, defined by the integral, does satisfy the general conditions for the area of the circle and assumes the assigned values along the circumference is thus complete.

Ex. Shew that, if M denote the maximum value (supposed positive) of $f(\psi)$ for points along the circumference of the circle and if $u(0)$ denote the value of the function at the centre, then

$$|u(r, \phi) - u(0)| < \frac{4}{\pi} M \sin^{-1} \frac{r}{R};$$

also that, if $u(0)$ vanish, then

$$u(r, \phi) < \frac{4}{\pi} M \tan^{-1} \frac{r}{R}. \qquad \text{(Schwarz.)}$$

219. But in view of subsequent investigations, it is important to consider the function represented by the integral when the periodic function $f(\phi)$ which occurs therein is not continuous, though still finite, for all points on the circumference. The contemplated modification in the continuity is that which is caused by a sudden change in value of $f(\phi)$ as ϕ passes through a value α: we shall have

$$f(\alpha + \epsilon) - f(\alpha - \epsilon) = A,$$

when ϵ is ultimately zero. Then the following proposition holds:

Let a function $f(\phi)$ be periodic in 2π, finite everywhere along the circle, and continuous save at an assigned point α where it undergoes a sudden increase in value: a function u can be obtained, which satisfies the general conditions for the circle except at such a point of discontinuity in the value of $f(\phi)$, and acquires the values of $f(\phi)$ along the circumference.

Let ρ be a quantity $< R$: then along the circumference of a circle of radius ρ, the general conditions are everywhere satisfied for the function u, so that, if $u(\rho, \psi)$ be the value at any point of its circumference, the value of u at any internal point is given by

$$u(r, \phi) = \frac{1}{2\pi} \int_0^{2\pi} u(\rho, \psi) \frac{\rho^2 - r^2}{\rho^2 - 2\rho r \cos(\psi - \phi) + r^2} d\psi.$$

Now ρ can be gradually increased towards R, because the general conditions are satisfied; but, when ρ is actually equal to R, the continuity of $u(\rho, \psi)$ is affected at the point α. We therefore divide the integral into three parts, viz., 0 to $\alpha - \epsilon$, $\alpha - \epsilon$ to $\alpha + \epsilon$, and $\alpha + \epsilon$ to 2π, when ρ is very nearly equal to R. For the first and the third of these parts, ρ can, as in the preceding investigation, be changed continuously into R without affecting the value of the integral. If we denote by p the integral

$$\frac{1}{2\pi} \int_0^{2\pi} u'(R, \psi) \frac{R^2 - r^2}{R^2 - 2Rr \cos(\psi - \phi) + r^2} d\psi,$$

where the range of integration does not include the part from $\alpha - \epsilon$ to $\alpha + \epsilon$, and where the values $f(\alpha-\epsilon)$, $f(\alpha+\epsilon)$ are assigned to $u'(R, \alpha-\epsilon)$, $u'(R, \alpha+\epsilon)$, respectively; the sum of the integrals for the first and the third intervals is $p+\Delta$, where Δ is a quantity that vanishes with $R-\rho$, because the subject of integration is everywhere finite. For the second interval, the integral is equal to $q+\Delta'$, where

$$q=\frac{1}{2\pi}\int_{\alpha-\epsilon}^{\alpha+\epsilon} f(\psi)\frac{R^2-r^2}{R^2-2Rr\cos(\psi-\phi)+r^2}d\psi,$$

and Δ' is a quantity vanishing with $R-\rho$ because the subject of integration is everywhere finite. So far as concerns q, let M be the greatest value of $|f(\psi)|$: then

$$|q|<\frac{M}{2\pi}\frac{R+r}{R-r}2\epsilon,$$

a quantity which, because M is finite (but only if M be finite), can be made infinitesimal with ϵ, provided r is never actually equal to R. If then, an infinitesimal arc from $\alpha-\epsilon$ to $\alpha+\epsilon$ be drawn so as, except at its assigned extremities, to lie within the area of the circle, the last proviso is satisfied: and the effect is practically to exclude the point α from the region of variation of u as a point for which the function is not precisely defined. With this convention, we therefore have

$$u(r, \phi)-\frac{1}{2\pi}\int_0^{2\pi} u'(R, \psi)\frac{R^2-r^2}{R^2-2Rr\cos(\psi-\phi)+r^2}d\psi=\Delta+\Delta'+q,$$

so that, by making ρ ultimately equal to R and ϵ as small as we please, the difference between $u(r, \phi)$ and the integral defined as above can be made zero. Hence *the integral is, as before, equal to the function* $u(r, \phi)$, provided that the point α be excluded from the range of integration, the value $f(\alpha-\epsilon)$ just before $\psi=\alpha$ and the value $f(\alpha+\epsilon)$ just after $\psi=\alpha$ being assigned to $u'(R, \psi)$.

It therefore appears that discontinuities may occur in the boundary values when the change is a finite change at a point, provided that all the values assigned to the boundary function be finite.

COROLLARY. *The boundary value may have any limited number of points of discontinuity, provided that no value of the function be infinite and that at all points other than those of discontinuity the periodic function be uniform, finite and continuous: and the integral will then represent a potential function satisfying the general conditions.*

The above analysis indicates why discontinuities, in the form of infinite values at the boundary, must be excluded: for, in the vicinity of such a point, the quantity M can have an infinite value and the corresponding integral does not then necessarily vanish. Hence, for example, the real part of

$$\frac{1}{e^{Re^{\psi i}-Re^{\alpha i}}}$$

is not a function that, under the assigned conditions, can be made a boundary value for the function u.

It is easy to construct a function with permissible discontinuities. We know (§ 3) that the argument of a point experiences a sudden change by π when the path of the point passes through the origin. Let a point P on a circle be considered relative to A: the inclination of AP to the normal, drawn inwards at A, is $\frac{\pi}{2}-\frac{1}{2}(a-\phi)$, and of AQ to the same line is $-\left[\frac{\pi}{2}-\frac{1}{2}(a-\phi')\right]$, so that there is a sudden change by π in that inclination. Now, taking a function

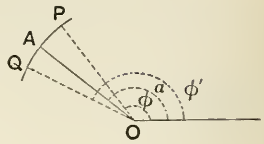

Fig. 79.

$$g(\phi)=-\frac{A}{\pi}\tan^{-1}\left[\tan\left\{\frac{\pi}{2}-\frac{1}{2}(a-\phi)\right\}\right],$$

and limiting the angle, defined by the inverse function, so that it lies between $-\frac{1}{2}\pi$ and $+\frac{1}{2}\pi$, as may be done in the above case and as is justifiable with an argument determined inversely by its tangent, the function $g(\phi)$ undergoes a sudden change A as ϕ increases through the value a. Moreover, all the values of $g(\phi)$ are finite: hence $g(\phi)$ is a function which can be made a boundary value for the function u. Let the function thence determined be denoted by u_a.

By means of the functions u_a, we can express the value of a function u whose boundary value $f(\phi)$ has a limited number of permissible discontinuities. Let the increases in value be $A_1, \ldots, A_m$ at the points $a_1, a_2, \ldots, a_m$ respectively: then, if $g_n(\phi)$ denote

$$-\frac{A_n}{\pi}\tan^{-1}\left[\tan\left\{\frac{\pi}{2}-\frac{1}{2}(a_n-\phi)\right\}\right],$$

we have $g_n(a_n+\epsilon)-g_n(a_n-\epsilon)=A_n$, when ϵ is infinitesimal. Hence

$$f(a_n+\epsilon)-f(a_n-\epsilon)-\{g_n(a_n+\epsilon)-g_n(a_n-\epsilon)\}$$

has no discontinuity at a_n, that is, $f(\phi)-g_n(\phi)$ has no discontinuity at a_n.

Hence also $f(\phi)-\sum_{n=1}^{m} g_n(\phi)$ has no discontinuity at $a_1, \ldots, a_m$, and therefore it is uniform, finite, and continuous everywhere along the circle; and it is periodic in 2π. By § 218, it determines a function U which satisfies the general conditions.

Each of the functions $g_n(\phi)$ determines a function u_n satisfying the general conditions: hence, as u is determined by $f(\phi)$, we have

$$u-\sum_{n=1}^{m} u_n=U,$$

which gives an expression for u in terms of the simpler functions u_n and of a function U determined by simpler conditions as in § 218.

Ex. Shew that, if $f(\psi)=1$ from $-\frac{1}{2}\pi$ to $+\frac{1}{2}\pi$ and $=0$ from $+\frac{1}{2}\pi$ to $\frac{3}{2}\pi$, then u is the real part of the function

$$\frac{1}{i\pi}\log\frac{1+iz}{i+z}.$$

The general inference from the investigation therefore is, that a function of two real variables x and y is uniquely determined for all points within a circle by the following conditions:

(i) at all points within the circle, the function u and its derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y^2}$ must be uniform, finite and continuous, and must satisfy the equation $\nabla^2 u=0$:

(ii) if $f(\phi)$ denote a function, which is periodic in ϕ of period 2π, is finite everywhere as the point ϕ moves along the circumference, is continuous and uniform at all except a limited number of isolated points on the circle, and at those excepted points undergoes a sudden prescribed (finite) change of value, then to u is assigned the value $f(\phi)$ at all points on the circumference except at the limited number of points of discontinuity of that boundary function.

And an analytical expression has been obtained, the function represented by which has been verified to satisfy the above conditions.

220. We now proceed to obtain some important results relating to a function u, defined by the preceding conditions.

I. *The value of u at the centre of the circle is the arithmetic mean of its values along the circumference.*

For, by taking $r = 0$, we have

$$u(0) = \frac{1}{2\pi}\int_0^{2\pi} f(\psi)\,d\psi,$$

the right-hand side being the arithmetic mean along the circumference.

II. *If the function be a uniform constant along the circumference, it is equal to that constant everywhere in the interior.*

For, let C denote the uniform constant; then

$$\begin{aligned} u(r, \phi) &= \frac{C}{2\pi}\int_0^{2\pi} \frac{R^2 - r^2}{R^2 - 2Rr\cos(\psi - \phi) + r^2}\,d\psi \\ &= C \end{aligned}$$

for all values of r less than R, that is, everywhere in the interior.

But if the function, though not varying continuously along the circumference, should have different constant values in different finite parts, as, for instance, in the example in § 219, then the inference can no longer be drawn.

III. *If the function be uniform, finite and continuous everywhere in the plane, it is a constant.*

Since the function is everywhere uniform, finite and continuous, the radius R of the circle of definition can be made infinitely large: then, as the limit of the fraction $(R^2 - r^2)/\{R^2 - 2Rr\cos(\psi - \phi) + r^2\}$ is unity, we have

$$u(r, \phi) = \frac{1}{2\pi}\int_0^{2\pi} u(\infty, \psi)\,d\psi,$$

the integral being taken round a circle of infinite radius whose centre is the origin. But, by (I) above, the right-hand integral is $u(0)$, the value at the centre of the circle; so that

$$u(r, \phi) = u(0),$$

and therefore u has the same value everywhere.

This is practically a verification of the proposition in § 40, that a uniform, finite and continuous function w, which has no infinity anywhere, is a constant.

IV. *A uniform, finite and continuous function u cannot have a maximum value or a minimum value at any point in the interior of a region over which, subject to the general conditions as to the differential coefficients, it satisfies the differential equation* $\nabla^2 u = 0$.

If there be any such point not on the boundary, it can be surrounded by an infinitesimal circle for the interior of which, as well as for the circumference of which, u satisfies both the general and the boundary conditions; hence

$$\int \frac{\partial u}{\partial n} ds = 0,$$

the integral being taken round the circumference. But in the immediate vicinity of such a point, $\frac{\partial u}{\partial n}$ has everywhere the same sign, so that the integral cannot vanish: hence there is no such point in the interior.

In the same way, it may be proved that there cannot be a line of maximum value or a line of minimum value within the surface: and that there cannot be an area of maximum value or an area of minimum value within the surface.

V. It therefore follows that *the maximum values for any region are to be found on its boundary: and so also are the minimum values.*

If M be the maximum value, and if m be the minimum value of the function for points along the boundary, then the value of the function for an interior point is $< M$ and is $> m$ and can therefore be represented in the form $Mp + m(1 - p)$, where p is a real positive proper fraction, varying from point to point.

In particular, let a function have the value zero for a part of the boundary and have the value unity for the rest: the value that it has for points along a line in the interior is always positive and has an upper limit q, a proper fraction. But q will vary from one line to another. If the region be a circle and q be the proper fraction for a line in the circle, then the value along that line of a function u, which is still zero over the former part of the boundary but has a varying positive value $\leqslant \mu$ along the remainder, is evidently $\leqslant q\mu$. This fraction q may be called the *fractional factor* for the line in the supposed distribution of boundary values.

VI. It may be noted that the second of these propositions can now be deduced for any simply connected surface. For when a function is constant along the boundary, its maximum value and its minimum value are the same, say λ: then its value at any point in the interior is $\lambda p + \lambda(1 - p)$, that is, λ, the same as at the boundary. Consequently if two functions u_1 and u_2 satisfy the general conditions over any region, and if they have the same value at all points along the boundary, then they are the same for all points of the region. For their difference satisfies the general conditions: it is zero everywhere along the boundary: hence it is zero over the whole of the bounded region.

If, then, a function u satisfy the general conditions for any region, it is unique for assigned boundary values that are everywhere finite, uniform, and continuous except at isolated points.

221. The explicit expression of u with boundary values, that are arbitrary within the assigned limits, has been determined for the area enclosed by a circle: the determination being partially dependent upon the form assumed in § 217 for the subsidiary function u'. The assumption of other forms for u', leading to other curves dependent upon a parametric constant, would lead by a similar process to the determination of u for the area limited by such families of curves.

But without entering into the details of such alternative forms for u', we can determine the value of u, under corresponding conditions, for curves derivable from the circle by the principle of conformal representation*. Suppose that, by means of a relation

$$z = \Phi(\zeta) = \Phi(\xi + i\eta),$$

or, say

$$x + iy = p(\xi, \eta) + iq(\xi, \eta),$$

where p and q are real functions of ξ and η, the area contained within the circle is transformed, point by point, into the area contained within another curve which is the transformation of the circle: then the function $u(x, y)$ becomes, after substitution for x and y in terms of ξ and η, a function, say U, of ξ and η.

Owing to the character of the geometrical transformation, p and q (and their derivatives with regard to ξ and η) are uniform, finite and continuous within corresponding areas. Hence

$$U(\xi, \eta) = u(x, y);$$

$$\frac{\partial U}{\partial \xi} = \frac{\partial u}{\partial x}\frac{\partial p}{\partial \xi} + \frac{\partial u}{\partial y}\frac{\partial q}{\partial \xi}, \qquad \frac{\partial U}{\partial \eta} = \frac{\partial u}{\partial x}\frac{\partial p}{\partial \eta} + \frac{\partial u}{\partial y}\frac{\partial q}{\partial \eta};$$

and

$$\frac{\partial^2 U}{\partial \xi^2} + \frac{\partial^2 U}{\partial \eta^2} = \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)\left\{\left(\frac{\partial p}{\partial \xi}\right)^2 + \left(\frac{\partial p}{\partial \eta}\right)^2\right\},$$

so that the function U satisfies the general conditions for the new area bounded by the new curve.

Moreover, u has assigned values along the circular boundary which is transformed, point by point, into the new boundary; hence U has those assigned values at the corresponding points along the new boundary. Thus the function U is uniquely determined for the new area by conditions which are exactly similar to those that determine u for a circle: and therefore *the*

* The general idea of the principle, and some illustrations of it, as expounded in Chapters XIX and XX, will be assumed known in the argument which follows: see especially §§ 265, 266.

potential function is uniquely determined for any area, which can be conformally represented on the area of a circle, by the general conditions of § 216 and the assignment of values that are finite and, except at a limited number of isolated points where they may suffer sudden (finite) changes of value, uniform and continuous at all points along the boundary of the area.

One or two examples of very special cases are given, merely by way of illustration. The general theory of the transformation of a circle or an infinite straight line into an analytical curve will be considered in Chapter XX. But, meanwhile, it is sufficient to indicate that, by the principle of conformal representation, we can pass from the circle to more general curves as the boundary of an area within which the potential function is defined by conditions similar to those for a circle: in particular that, by assuming the result of §§ 265, 266, we can pass from the circle to an analytical curve as the boundary of such an area.

Ex. 1. A function u satisfying the general conditions for a circle of radius unity and centre the origin, and having assigned values $f(\psi)$ along the circumference, is determined at any internal point by the equation

$$u(r, \phi) = \frac{1}{2\pi}\int_0^{2\pi} f(\psi) \frac{1-r^2}{1-2r\cos(\psi-\phi)+r^2}\, d\psi.$$

Now the circle and its interior are transformed by the equation

$$z+1=\frac{2}{\zeta^{\frac{1}{2}}}$$

into a parabola and the excluded area (Ex. 7, § 257): so that, if R, θ be polar coordinates of any point in that excluded area, we have

$$r\cos\phi = 2R^{-\frac{1}{2}}\cos\tfrac{1}{2}\theta - 1, \qquad r\sin\phi = -2R^{-\frac{1}{2}}\sin\tfrac{1}{2}\theta.$$

Corresponding to the circle $r=1$, we have the parabola

$$R\cos^2\tfrac{1}{2}\theta = 1;$$

if Θ determine the point on the parabola, which corresponds to ψ on the circle, we have

$$\cos\psi = 2\cos^2\tfrac{1}{2}\Theta - 1,$$

or $\psi = \Theta$.

Hence the function $U(R, \Theta)$ assumes the values $f(\Theta)$ along the boundary of the parabola.

Also
$$1-r^2 = \frac{4}{R}(R^{\frac{1}{2}}\cos\tfrac{1}{2}\theta - 1),$$

$$1-2r\cos(\psi-\phi)+r^2 = \frac{4}{R}\left[R\cos^2\tfrac{1}{2}\Theta - 2R^{\frac{1}{2}}\cos\tfrac{1}{2}\Theta\cos\tfrac{1}{2}(\Theta+\theta)+1\right];$$

and therefore we have the following result:

A function which satisfies the general conditions for the area bounded by and lying on the convex side of the parabola $R\cos^2\frac{1}{2}\Theta=1$ *and is required to assume the value* $f(\Theta)$ *at points along the parabola, is defined uniquely for a point* (r, θ) *external to the parabola by the integral*

$$\frac{1}{2\pi}\int_0^{2\pi} f(\Theta)\, \frac{r^{\frac{1}{2}}\cos\frac{1}{2}\theta - 1}{1-2r^{\frac{1}{2}}\cos\frac{1}{2}\Theta\cos\frac{1}{2}(\Theta+\theta)+r\cos^2\frac{1}{2}\Theta}\, d\Theta.$$

The function $f(\Theta)$ may suffer finite discontinuities in value at isolated points: elsewhere it must be finite, continuous and uniform.

Ex. 2. Obtain an expression for u at points within the area of the same parabola, by using

$$z=\tan^2\left(\tfrac{1}{4}\pi\zeta^{\frac{1}{2}}\right)$$

as the equation of transformation of areas (§ 257).

Ex. 3. When the equation

$$z=\frac{i-\zeta}{i+\zeta}$$

is used, then, if $z=x+iy$ and $\zeta=X+iY$, we have

$$x+iy=\frac{1-X^2-Y^2+i2X}{X^2+(1+Y)^2}.$$

If the point ζ describe the whole length of the axis of X from $-\infty$ to $+\infty$, so that we may take $\zeta=X=\tan\phi$ with ϕ increasing from $-\frac{1}{2}\pi$ to $+\frac{1}{2}\pi$, we have $x=\cos 2\phi$, $y=\sin 2\phi$; and z describes the whole circumference of a circle, centre the origin and radius unity, in a trigonometrically positive direction beginning at the point $(-1, 0)$. We easily find

$$\frac{r\cos\theta}{1-R^2}=\frac{r\sin\theta}{2R\cos\Theta}=\frac{r^2}{1-2R\sin\Theta+R^2}=\frac{1}{1+2R\sin\Theta+R^2},$$

where $\xi=R\cos\Theta$, $\eta=R\sin\Theta$. Moreover, for variations along the circumference, we have $\psi=2\phi$; whence, substituting and denoting by $F(x)$, $=f(2\tan^{-1}x)$, the value of the potential at a point on the axis of real quantities whose abscissa is x, we ultimately find

$$u(R, \Theta)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{R\sin\Theta}{R^2-2xR\cos\Theta+x^2}F(x)\,dx,$$

as the value of the potential-function u at a point (R, Θ) in the upper half of the plane, when it has assigned values $F(x)$ at points along the axis of real variables.

222. The function u has now been determined, by means of the general conditions within an area and the assigned boundary values, for each space obtained by the method indicated in § 221. But the determination is unique and distinct for each space thus derived; and, if two such spaces have a common part, there are distinct functions u. We now proceed to shew that when two spaces, for each of which alone a function u can be determined, have a common part which is not merely a point or a line, then *the function u is uniquely determined for the combined area by the assignment of finite, uniform and continuous values* (*or partially discontinuous values, as in* § 219) *along the boundary of the combined area.*

Let the spaces be T_1 and T_2 having a common part T, so that the whole space can be taken in the form T_1+T_2-T. Let the part of the boundary of T_1 without T_2 be L_0, and the part within T_2 be L_2: and similarly, for the boundary of T_2, let L_1 denote the part within T_1 and L_3 the part without it. Then the boundary of

$$T_1+T_2-T$$

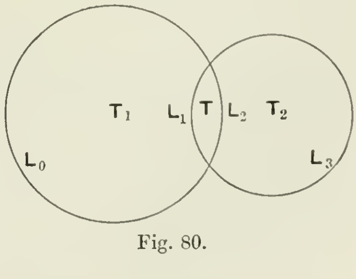

Fig. 80.

is made up of L_0 and L_3: the boundary of T is made up of L_1 and L_2.

With an assignment of zero value along L_0 and unit value along L_2, let the fractional factor (§ 220, V), for the line L_1 in the region T_1 be q_1; and with an assignment of zero value along L_3 and unit value along L_1, let the fractional factor along the line L_2 in the region T_2 be q_2. Then q_1 and q_2 are positive proper fractions.

Let any series of values be assigned along L_0 and L_3 subject to the conditions of being uniform, finite everywhere, and discontinuous, if at all, only at a limited number of isolated points; these values are the boundary values of the function u to be determined for the whole area, and will be called the *assigned values*. Let the maximum of the values be M and the minimum be m; and denote $M - m$ by μ, so that μ is positive.

Assume, for a boundary value along L_2, the minimum m of the assigned values for the function along L_0 and L_3. Let the function, which is uniquely determined for the region T_1 by the general conditions for the area and by values along the boundary, constituted by the assigned values along L_0 and the assumed value m along L_2, be denoted by u_1. The values assumed by u_1 along the line L_1 in this region are uniform, finite and continuous; and they may be denoted by $m + p\mu$, where p is a positive proper fraction varying from point to point along the line.

Let the function, which is uniquely determined for the region T_2 by the general conditions for the area and by values along the boundary, constituted by the assigned values along L_3 and by the values of u_1 along L_1, be denoted by u_2. Then the uniform, finite, continuous values which it assumes along L_2 are of the form $m + q\mu$, where q is a positive proper fraction varying from point to point along the line; let the greatest of these values be $m + Q\mu$, where Q lies between 0 and 1.

For the region T_1 determine a function* u_3 by means of boundary values, consisting of the assigned values along L_0 and the values of u_2, viz., $m + Q\mu$, along L_2. Then the function $u_3 - u_1$ satisfies the general conditions; its value along the part L_0 of the boundary is zero, and its value along the other part L_2 of the boundary is $\leqslant Q\mu$ and is greater than zero. Hence $u_3 - u_1$ is always positive within T_1, and along L_1 we have $u_3 - u_1 \leqslant q_1 Q\mu$.

For the region T_2 determine a function u_4 by means of boundary values, consisting of the assigned values along L_3 and the values of u_3 along L_1. Then the function $u_4 - u_2$ satisfies the general conditions; its value is zero along L_3; and its value along L_1 is that of $u_3 - u_1$, that is, a positive quantity which is not greater than $q_1 Q\mu$. Hence $u_4 - u_2$ is always positive within T_2, and along L_2 we have $u_4 - u_2 \leqslant q_2 q_1 Q\mu$.

* All the succeeding functions will be determined subject to the general conditions for the respective areas; the specific mention of the general conditions will be omitted.

For the region T_1 determine a function u_5 by means of boundary values, consisting of the assigned values along L_0 and the values of u_4 along L_2. Then the function $u_5 - u_3$ satisfies the general conditions; its value is zero along L_0; and its value along L_2 is that of $u_4 - u_2$, that is, a positive quantity which is not greater than $q_2 q_1 Q\mu$. Hence $u_5 - u_3$ is always positive within T_1, and along L_1 we have $u_5 - u_3 \leqslant q_2 q_1^2 Q\mu$.

Proceeding in this manner for the regions alternately, we obtain functions u_{2n+1} for the region T_1, such that u_{2n+1} has the assigned values along L_0 and the values of u_{2n} along L_2; and functions u_{2n} for the region T_2, such that u_{2n} has the assigned values along L_3 and the values of u_{2n-1} along L_1. And the functions are such that

$$u_{2n+1} - u_{2n-1} > 0 \text{ in } T_1 \text{ and } \leqslant q_1^n q_2^{n-1} Q\mu \text{ along } L_1; \text{ and}$$

$$u_{2n+2} - u_{2n} > 0 \text{ in } T_2 \text{ and } \leqslant q_1^n q_2^n Q\mu \text{ along } L_2.$$

Hence, both for functions with an uneven suffix and for functions with an even suffix, there are limits to which the functions approach along L_1 and L_2 respectively; let these limits be u' and u''.

Both of these limits are finite; for along L_1, we have

$$\begin{aligned} u' &= u_1 + (u_3 - u_1) + (u_5 - u_3) + \ldots \text{ ad inf.} \\ &\leqslant m + q_1 Q\mu + q_1^2 q_2 Q\mu + q_1^3 q_2^2 Q\mu + \ldots \\ &\leqslant m + \frac{q_1 Q\mu}{1 - q_1 q_2}, \end{aligned}$$

so that this expression, which is finite, is an upper limit and m is a lower limit for u'. And, along L_2, we have

$$\begin{aligned} u'' &= u_2 + (u_4 - u_2) + (u_6 - u_4) + \ldots \text{ ad inf.} \\ &\leqslant m + Q\mu + q_1 q_2 Q\mu + q_1^2 q_2^2 Q\mu + \ldots \\ &\leqslant m + \frac{Q\mu}{1 - q_1 q_2}, \end{aligned}$$

so that this expression, which is finite, is an upper limit and m is a lower limit for u''. Hence both u' and u'' are finite.

Now in determining u' for T_1 and regarding it as the limit of u_{2n+1}, we have its values along L_2 as the values of u_{2n}, that is, of u'' in the limit; and in determining u'' for T_2 and regarding it as the limit of u_{2n+2}, we have its values along L_1 as the values of u_{2n+1}, that is, of u' in the limit. Hence over the whole boundary of T, the region common to T_1 and T_2, we have $u' = u''$; and therefore (by § 220, VI) we have $u' = u''$ over the whole area of the common region T.

Lastly, let a function u be determined for the region T_1, having the assigned values along L_0 and the values of u' along L_2. Then the function $u - u'$ satisfies the general conditions; it has zero values round the whole boundary of T_1, and therefore (by § 220, VI) it is zero over the whole region T_1. Hence u' is the function for T_1.

Similarly, determining a function u for T_2, having the assigned values along L_3 and the values of u'' along L_1, we have $u = u''$ everywhere in T_2, so that u'' is the function for T_1.

The functions u' and u'' satisfy the general conditions for T_1 and T_2 respectively; and these two regions have a common portion T over which u' and u'' have been proved to be the same. Hence, by Lemma I. of § 216, they determine one and the same function for the whole region combined of T_1 and T_2; this function u satisfies the general conditions and, along the boundary of the whole region, assumes values that are assigned arbitrarily subject only to the general limitations of being everywhere finite and, except for finite discontinuities at isolated points, uniform and continuous. The proposition is therefore established.

This method of combination, dependent upon the alternating process whereby a function determined separately for two given regions having a common part is determined for the combination of the regions, is capable of repeated application. Hence it follows that *a function exists, subject to the general conditions within a given region and acquiring assigned finite values along the boundary of the region,* when the region can be obtained by combinations of areas that can be conformally represented upon the area of a circle.

Note. Let A, B, C be three non-intersecting simple closed curves, such that C lies within B and B within A. The area bounded by the curves A and C can, by a similar method, be combined with the whole area enclosed by B; and we can make the same inference as above, as to the existence of a function u for the whole area enclosed by A, when it exists for the areas that are combined.

223. At the beginning of the discussion it was assumed that the areas, in which the existence of the function is to be proved, lie in a single sheet (§ 216) or, in other words, that no branch-point occurs within the area.

It is now necessary to take the alternative possibility into consideration: a simple example will shew that the theorem just proved is valid for an area containing a branch-point except in one unessential particular.

Let the area be a winding surface consisting of m sheets: the region in each sheet will be taken circular in form, and the centre c of the circles will be the winding-point, of order $m-1$. Such a surface is simply connected (§ 178); and its boundary consists of the m successive circumferences which, owing to the connection, form a single simple closed curve. Using the substitution

$$z - c = RZ^m,$$

we have a new Z-surface which consists of a circle, centre the Z-origin and radius unity: it lies in one sheet in the Z-region and has no branch-points; its circumference is described once for a single description of the complete boundary of the winding-surface. The correspondence between the two regions is point-to-point: and therefore the assigned values along the boundary of the winding-surface lead to assigned values along the Z-circumference. Any function w of z changes into a function W of Z: hence u changes into a real function U satisfying the general conditions in the Z-region; and conversely.

But a function U, satisfying the general conditions over the area of a plane circle and acquiring assigned finite values along the circumference, is uniquely determinate; and hence the function u is uniquely determined on the circular winding-surface by satisfying the general conditions over the area and by assuming assigned values along its boundary.

It is thus obvious that the multiplicity of sheets, connected through branch-lines terminated at branch-points and (where necessary) on the single boundary of the surface consisting of the sheets, does not affect the validity of the result obtained earlier for the simpler one-sheeted area; and therefore *the function u, acquiring assigned values along the boundary of the simply connected surface and satisfying the general conditions throughout the area of the surface which may consist of more than a single sheet is uniquely determinate.*

There is, as already remarked, one unessential particular in which deviation from the theorem occurs when the region contains a branch-point. At a branch-point a function may be finite*, but all its derivatives are not necessarily finite; and therefore at such a point a possible exception to the general conditions arises as to the finiteness of value of the derivatives and the consequent satisfying of the equation $\nabla^2 u = 0$: no exception, of course, arises as regards the uniformity of the derivatives on the Riemann's surface. The exception does not necessarily occur; but, when it does occur, it is only at isolated points, and its nature does not interfere with the validity of the proposition. We shall therefore assume that, in speaking of the general conditions through the area, the exception (if necessary) from the general conditions, of finiteness of value of the derivatives at a branch-point, is tacitly implied.

Hence we infer, by taking combinations of circles in a manner somewhat similar to the process adopted for successive circles of convergence in the continuation of a function in § 34, that *a function u exists, subject to the general conditions within any simply connected surface and acquiring assigned finite values along the boundary of the surface.*

* Infinities of the function itself at a branch-point will fall under the general head of infinities of the function, discussed afterwards (in § 229).

224. The functions which have been discussed so far in the present connection are functions which have no infinities and, except possibly at points on the boundaries of the regions considered, no discontinuities: they are uniform functions. And the regions have, hitherto, been supposed simply connected parts of a Riemann's surface, or simply connected surfaces. When the surface is multiply connected, we resolve it by a canonical system (§ 181) of cross-cuts and proceed as follows.

We now proceed to introduce the cross-cut constants, and so to consider the existence of functions which have the multiform character of the integrals of uniform functions of position on the Riemann's surface. The functions will still be considered to be uniform, finite and continuous except at the cross-cuts: their derivatives will be supposed uniform, finite, and continuous everywhere in the region, and subject to the equation $\nabla^2 u = 0$: and boundary values will be assigned of the same character as in the previous cases. As moduli of periodicity are to be introduced, the unresolved surface is no longer one of simple connection: we shall begin with a doubly connected surface.

Let such a surface T be resolved, in two different ways, into a simply connected surface: say into T_1 by a cross-cut Q_1, and into T_2 by a cross-cut Q_2. Mark on T_1 and on T_2 the directions of Q_2 and of Q_1 respectively: the

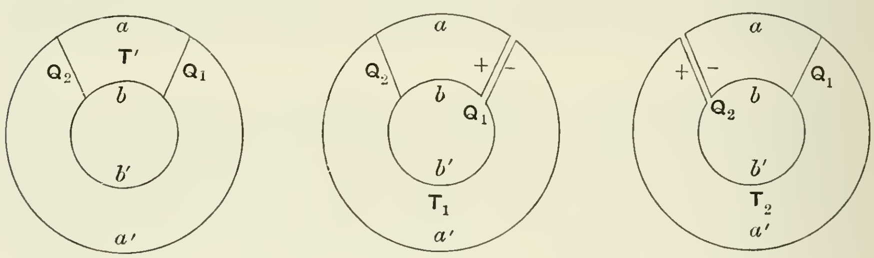

Fig. 81.

notations of the boundaries are indicated in the figures, and T' is the region between the lines of Q_1 and Q_2.

It will be shewn that a function u exists, determined uniquely by the following conditions:

(i) The first and the second derivatives are throughout T to be uniform, finite and continuous, and to satisfy $\nabla^2 u = 0$: but no conditions for them are assigned at points on the boundary:

(ii) The (single) modulus of periodicity is to be K, which will be taken as an arbitrary, real, positive constant: the value of any branch of u at a point on the positive edge is therefore to be greater by K than its value at the opposite point on the negative edge:

(iii) Some selected branch of u is to assume assigned values along

a' and b', typically represented by H, and assigned values along a and b, typically represented by G. These boundary values are to be finite everywhere, though they may be discontinuous at a finite number of isolated points on the boundary; such discontinuity will arise through the modulus.

In T_1, for zero values along a, b, a', b' and for unit values along Q_1^- and Q_1^+, let the fractional factor for the line Q_2 be q_1: and similarly in T_2, for zero values along a, b, a', b' and for unit values along Q_2^- and Q_2^+, let the fractional factor for the line Q_1 be q_2, where q_1 and q_2 are positive proper fractions.

For the simply connected region* T_1 determine a function u_1, satisfying the general conditions and having as its boundary values, H along a' and b', G along a and b, arbitrarily assumed values represented by θ (the maximum value being M_1 and the minimum value being m_1) along Q_1^- and values $\theta+K$ along Q_1^+: the function so obtained is unique. Let the values along the line Q_2 in T_1 be denoted by u_1'.

For the region T_2 determine a function u_2, satisfying the general conditions and having as its boundary values, H along a' and b', $G-K$ along a and b, $u_1'-K$ along Q_2^- and u_1' along Q_2^+: the function so obtained is unique. Let its values along the line Q_1 in T_2 be denoted by u_2', the maximum value being M_2 and the minimum value being m_2.

For the region T_1 determine a function u_3, satisfying the general conditions and having as its boundary values, H along a' and b', G along a and b, u_2' along Q_1^- and $u_2'+K$ along Q_1^+: the function so obtained is unique. Let its values along the line Q_2 in T_1 be denoted by u_3'. Then the function u_3-u_1 satisfies the general conditions in T_1; it is zero along a' and b', a and b: it is $u_2'-\theta$ along Q_1^- and also along Q_1^+, and $u_2'-\theta \leqslant M_2-m_1$ and $\geqslant m_2-M_1$.

225. A difference of limits for $u_3'-u_1'$ arises according to the relative values of M_2 and m_1, of m_2 and M_1; evidently $M_2-m_1 > m_2-M_1$.

(i) If m_2-M_1 be positive, then M_2-m_1 is positive and equal, say, to λ; the boundary values for u_3-u_1 may range from 0 to λ and we have $u_3'-u_1' > 0 < q_1\lambda$ along Q_2.

(ii) If m_2-M_1 be negative and equal to $-\epsilon$, then M_2-m_1 is either positive or negative.

(*a*) If M_2-m_1 be negative, then the boundary values for u_3-u_1 may range from 0 to $-\epsilon$, that is, boundary values for u_1-u_3 may range from 0 to ϵ and we have $u_1'-u_3' > 0 < q_1\epsilon$ along Q_2, which may be expressed in the form

$$|u_3'-u_1'| < q_1\epsilon,$$

where ϵ is the greatest modulus of values along the boundary.

* In the special case, when T_1 is bounded by concentric circles and the cross-cut is made along a diameter, the region can be represented conformally on the area of a circle: see a paper by the author, *Quart. Journ. Math.*, Vol. xxvi, (1892), pp. 145—148.

(*b*) If $M_2 - m_1$ be positive, let its value be denoted by η: then the boundary values for $u_3 - u_1$ may range from η to $-\epsilon$. The boundary values for $u_3 - u_1 + \epsilon$ may range from 0 to $\eta + \epsilon$, and it is a function satisfying all the internal conditions: hence $u_3 - u_1 + \epsilon \leqslant q_1(\eta + \epsilon)$, and therefore

$$u_3 - u_1 \leqslant q_1\eta - (1 - q_1)\epsilon \leqslant q_1\eta.$$

Again, the boundary values of $u_1 - u_3 + \eta$ may range from $\eta + \epsilon$ to 0, and it is a function satisfying all the internal conditions: hence $u_1 - u_3 + \eta \leqslant q_1(\eta + \epsilon)$, and therefore

$$u_1 - u_3 \leqslant q_1\epsilon - (1 - q_1)\eta \leqslant q_1\epsilon.$$

Hence at points where $u_3 > u_1$, so that $u_3 - u_1$ is positive, we have $u_3 - u_1 \leqslant q_1\eta$; and at points where $u_3 < u_1$, so that $u_1 - u_3$ is positive, we have $u_1 - u_3 \leqslant q_1\epsilon$.

Every case can be included in the following result*: If μ be the greatest modulus of the values of $u_2' - \theta$ along the two edges of Q_1 in T_1, then

$$|u_3' - u_1'| \leqslant q_1\mu,$$

along Q_2, so that $q_1\mu$ is certainly the greatest modulus of $u_3' - u_1'$ along Q_2.

226. For the region T_2 determine a function u_4, satisfying the general conditions and having as its boundary values, H along a' and b', $G - K$ along a and b, $u_3' - K$ along Q_2^- and u_3' along Q_2^+: the function so obtained is unique. Let its values along the line Q_1 be denoted by u_4'. Then the function $u_4 - u_2$ satisfies the general conditions in T_2: it is zero along a' and b', a and b: it is $u_3' - u_1'$ along Q_2^- and also along Q_2^+, and along Q_2 we have

$$|u_3' - u_1'| \leqslant q_1\mu.$$

Hence, after the preceding explanations, we have along Q_1 in T_2

$$|u_4' - u_2'| \leqslant q_2 q_1\mu.$$

Proceeding in this way for the regions alternately, we have for T_1 a function u_{2n+1}, the boundary values of which are, H, along a' and b', G along a and b, u_{2n}' along Q_1^- and $u_{2n}' + K$ along Q_1^+: and along Q_2

$$|u_{2n+1}' - u_{2n-1}'| \leqslant q_1^n q_2^{n-1}\mu;$$

and for T_2, a function u_{2n+2}, the boundary values of which are, H along a' and b', $G - K$ along a and b, $u_{2n+1}' - K$ along Q_2^- and u_{2n+1}' along Q_2^+: and along Q_1

$$|u_{2n+2}' - u_{2n}'| \leqslant q_1^n q_2^n\mu.$$

Thus both the function u_{2n+1} along Q_2 and the function u_{2n} along Q_1 approach limiting values; let them be u' and u'' respectively.

These limiting values are finite. For

$$u_{2n+1} = u_1 + (u_3 - u_4) + (u_5 - u_3) + \dots + (u_{2n+1} - u_{2n-1});$$

* Another method of proceeding, different from the method in the text, depends upon the introduction of another fractional factor for Q_2, having the same relation to minimum values as q_1 to maximum values; but it is more cumbersome, as it requires the continuous consideration of the separate cases indicated.

in the limit, when n is infinitely large, the sum of the moduli of the terms of the series at points along Q_2

$$< (M_1 + K) + q_1\mu + q_1^2 q_2 \mu + q_1^3 q_2^2 \mu + \dots$$

$$< M_1 + K + \frac{q_1\mu}{1 - q_1 q_2};$$

so that the series converges and the limit of u_{2n+1}, viz. u', is finite. Similarly for u''.

Now consider the functions in the portions $T - T'$ and T' of the region T.

For $T - T'$ we have u_{2n}, (that is, u'' in the limit), with values H along a' and b', u' along Q_2^+: and also u_{2n+1}, (that is, u' in the limit), with values H along a' and b' and u'' along Q_1^-: thus u' and u'' have the same values over the whole boundary of $T - T'$ and, therefore, throughout that portion we have $u' = u''$.

For T' we have u_{2n}, (that is, u'' in the limit), with values $G - K$ along a and b and $u' - K$ along Q_2^-: and also u_{2n+1}, (that is, u' in the limit), with values G along a and b and $u'' + K$ along Q_1^+. Thus over the whole boundary of T' we have $u' - u'' = K$: and therefore within the portion T' we have

$$u' = u'' + K.$$

Lastly, for the whole region T we take $u = u'$. In the portion $T - T'$ we have $u = u' = u''$, and in the portion T' we have $u = u' = u'' + K$; that is, the *function is such that in the region T_1 the value changes from u'' at Q_1^- to $u'' + K$ at Q_1^+*, or the modulus of periodicity is K.

Hence the function is uniquely determined for a doubly connected surface by the general conditions, by the assigned boundary values and by the arbitrarily assumed real modulus of periodicity.

227. We now consider the determination of the function, when the surface S is triply connected and has a single boundary.

Let S be resolved, in two different ways, into a doubly connected surface. Let Q_1 be a cross-cut, which changes the surface into one of double connectivity and gives two pieces of boundary: and let Q_2 be another cross-cut, not meeting the direction of Q_1 anywhere but continuously deformable into Q_1, so that it also changes the surface into one of double connectivity with two pieces of boundary. Then, in each of these doubly connected surfaces, any number of functions can be uniquely determined which satisfy the general conditions, each of which assumes assigned boundary values, that is, along the boundary of S and the new boundary, and possesses an arbitrarily assigned modulus of periodicity.

The combination of these functions, by an alternate process similar to that for the preceding case, leads to a unique function which has an assigned modulus of periodicity for the cross-cut Q_1. The conditions which determine it are: (i), the general conditions: (ii), the values along the boundary of the given surface, (iii) the value of the modulus of periodicity for the cross-cut, which resolves the surface into one of double connectivity, and the modulus of periodicity for the cross-cut, which resolves the latter into a simply connected surface, that is, by assigned moduli of periodicity for the two cross-cuts necessary to resolve the original surface S into one that is simply connected.

Proceeding in this synthetic fashion, we ultimately obtain the result that a real function u exists for a surface of connectivity $2p+1$ with a single boundary, uniquely determined by the following conditions:—

(i) its derivatives within the surface are everywhere uniform, finite and continuous, and they satisfy the equation $\nabla^2 u = 0$;

(ii) it assumes, along the boundary of the surface, assigned values which are always finite but may be discontinuous at a limited number of isolated points on the boundary;

(iii) the function within the surface is everywhere finite and, except at the positions of cross-cuts, is everywhere uniform and continuous: the discontinuities in value in passing from one edge to another of the cross-cuts are arbitrarily assigned real quantities.

Now the surfaces under consideration are of odd connectivity: the function thus determinate is everywhere finite, so that no points need to be excluded from the range of variation of the independent variable; the single boundary of the closed surface can be made a point. The boundary value is then a value assigned to the function at this point*; it may be dependent upon a value assigned to w at some point, in order to obtain the arbitrary additive imaginary constant in w subject to which it is dependent upon u. Hence we infer that *real functions exist on a Riemann's surface, finite everywhere on the surface and uniquely determined by their moduli of periodicity at the cross-cuts, which moduli are arbitrarily assigned real quantities.* It will be proved that the moduli cannot all be zero (§ 231).

228. The following important proposition may now be deduced:—

Of the real functions, which satisfy the general conditions and are finite everywhere on the Riemann's surface, and are determined by arbitrarily assigned moduli of periodicity, there are 2p and no more that are linearly independent of one another; and every other such function can be expressed, except as to an additive constant, as a linear combination of multiples of these functions with constant coefficients.

Taking into account only real functions, which satisfy the general conditions and are everywhere finite, we can obtain an infinite number of functions by assigning arbitrary moduli of periodicity.

* Or, if we please, the constant value along the circumference of a small circle round the point; in the absence of the conditions of uniformity and continuity, the proposition VI. of § 220 does not apply to this case.

When one function u_1 has been obtained, with $\omega_{1,1}, \omega_{1,2}, \dots, \omega_{1,2p}$ as its arbitrarily assigned moduli, another function u_2 can be obtained with

$$\omega_{2,1}, \omega_{2,2}, \dots, \omega_{2,2p}$$

as its arbitrarily assigned moduli of periodicity, which are not the moduli of $k_1 u_1$, where k_1 is a constant. A third function u_3 can then be obtained, with $\omega_{3,1}, \omega_{3,2}, \dots, \omega_{3,2p}$ as its arbitrarily assigned moduli of periodicity, which are not the moduli of $k_1 u_1 + k_2 u_2$, where k_1 and k_2 are constants; and so on, provided that the number of functions obtained, say q, is less than $2p$. When $q < 2p$, another function can be obtained whose moduli of periodicity are different from those of $\sum_{r=1}^{q} k_r u_r$. But when $q = 2p$, so that $2p$ definite functions, linearly independent of one another, have been obtained, it is possible to determine constants $k_1, k_2, \dots, k_{2p}$, so that

$$\sum_{r=1}^{2p} k_r \omega_{r,m} = \Omega_m$$

(for $m = 1, 2, \dots, 2p$), where $\Omega_1, \Omega_2, \dots, \Omega_{2p}$ are arbitrary constants.

Let U be the potential function, which satisfies the general conditions and is finite everywhere on the surface and is determined by the arbitrarily assigned constants $\Omega_1, \Omega_2, \dots, \Omega_{2p}$; then the function

$$U - \sum_{r=1}^{2p} k_r u_r$$

has all its moduli of periodicity zero, it is everywhere finite and, because its moduli are zero, it is uniform and continuous everywhere on the surface. It is therefore, by § 220, a constant; and therefore

$$U = \sum_{r=1}^{2p} k_r u_r + A,$$

proving the proposition.

229. The only remaining condition of § 214 to be considered is the possible possession, by the function u, of infinities of assigned forms, at assigned positions on the surface.

Let the infinity at a point on the surface, where z is equal to c_r, be represented by the real part of $\phi(z, c_r)$, where

$$\phi(z, c_r) = \frac{A_{r,m}}{(z-c_r)^m} + \frac{A_{r,m-1}}{(z-c_r)^{m-1}} + \dots\dots + \frac{A_{r,1}}{z-c_r} + B_r \log(z-c_r),$$

and let this real part be denoted* by $\Re\phi(z, c_r)$; then $u - \Re\phi(z, c_r)$ has no infinity at $z = c_r$. Proceeding in the same manner with the other assigned infinities at all the assigned points, we have a function

$$U = u - \sum_{r=1} \Re\phi(z, c_r),$$

* The form of $\phi(z, c_r)$ implies that the series giving the infinite terms has negative integral exponents; the case, in which the exponents are proper fractions so that the point is a branch-point, is covered by the transformation of § 223 when the modified form of ϕ explicitly satisfies the tacit implication as to form.

which has no infinities on the surface. Its derivatives everywhere (save at branch-points) are finite, uniform and continuous and satisfy the equation $\nabla^2 u = 0$. If Υ be a typical representation of the assigned boundary values of u and Φ be the corresponding typical representation of the assigned boundary values of $\sum_{r=1} \Re\phi(z, c_r)$, then $\Upsilon - \Phi$ is a typical representation of the boundary values of U.

The moduli of periodicity of U may arise through two sources: (i) arbitrarily assigned real moduli of periodicity at the $2p$ cross-cuts of the canonical system (§ 181), that are necessary to resolve the original surface into one that is simply connected: (ii) the various moduli $\Re(2\pi i B_r)$, arising from the infinities c_r in the surface, the occurrence of which infinities renders these additional moduli necessary for the various additional cross-cuts that must be made before the surface can be resolved. Then U has all these moduli as its moduli of periodicity: it is finite everywhere on the surface and, except for its moduli of periodicity, it is uniform and continuous on the surface; hence it is a function uniquely determinate, which is a constant if all the moduli be zero.

It therefore follows that the determination of u is unique, that is, that *a real function u on the Riemann's surface is determined by the general conditions at all points on the surface except infinities, by the assignment of specified forms of infinities at isolated points, and by the possession of arbitrarily assigned moduli of periodicity at the cross-cuts which must be made to resolve the surface into one that is simply connected.* And, when all the moduli are zero, the real function u is uniform.

Now $w, = u + iv$, is determined by u save as to an arbitrary additive constant. Hence, summarising the preceding results, we infer the existence of the following classes of functions on the surface:—

(*A*) Functions which are finite everywhere on the surface and, except at the lines of the cross-cuts which suffice to resolve the surface into one that is simply connected, uniform and continuous; and which have, at these cross-cuts, moduli of periodicity, the real parts of which are arbitrarily assigned constants:—

(*B*) Functions which have a limited number of assigned singularities (either algebraical, or logarithmic, or both) at assigned isolated points, and which otherwise have the characteristics of the functions defined in (*A*).

The existence of the various kinds of functions, considered in the preceding chapter in connection with a special form of Riemann's surface, will now be established for any given surface.

CHAPTER XVIII.

Applications of the Existence-Theorem.

230. We proceed to make some applications of the existence-theorem as established in the preceding chapter in connection with any Riemann's surface, that is supposed given geometrically in an arbitrary way; and we shall first consider it in relation with the functions usually known as Abelian transcendents.

The existence of various classes of functions of position has been established. Let functions which, satisfying the general conditions, are finite everywhere on the Riemann's surface and have assigned moduli of periodicity at the $2p$ cross-cuts, be called *functions of the first kind*, in analogy with the nomenclature of §§ 205—211; let functions which, satisfying the general conditions, have assigned algebraical infinities on the Riemann's surface and have assigned moduli of periodicity at the $2p$ cross-cuts, be called *functions of the second kind;* and let functions which, satisfying the general conditions, have assigned logarithmic and algebraical infinities* and have assigned moduli of periodicity at the $2p$ cross-cuts as well as the proper moduli in connection with the logarithmic infinities, be called *functions of the third kind.* These classes of functions evidently contain the integrals of the respective kinds which arise through algebraical functions.

First, let P and Q be two functions of x and y, the derivatives of which are finite, uniform and continuous at all points (except possibly branch-points) on the given Riemann's surface and satisfy the equation $\nabla^2 u = 0$. Let the functions themselves be finite and, except at cross-cuts, uniform and continuous on the surface: and let their moduli of periodicity be $A_1, \ldots, A_p, B_1, \ldots, B_p$; $A_1', \ldots, A_p', B_1', \ldots, B_p'$, for the cross-cuts $a_1, \ldots, a_p, b_1, \ldots, b_p$ respectively, the moduli for the cross-cuts c being zero. (If P and Q should have infinities on the surface, as will be the case in later applications, so that in their vicinity portions of the surface are excluded, thereby requiring other cross-cuts for the resolution of the surface into one that is simply connected, other moduli will be required; but, in the first instance, P and Q have merely the $2p$ assigned moduli.)

When the surface is resolved by the $2p$ cross-cuts into one that is simply

* The logarithmic infinities must be at least two in number, by § 210.

connected, the functions P and Q are uniform, finite and continuous over that resolved surface. Proceeding as in § 16 and § 216, we have

$$\iint\left(\frac{\partial P}{\partial x}\frac{\partial Q}{\partial y}-\frac{\partial Q}{\partial x}\frac{\partial P}{\partial y}\right)dxdy=\int P\frac{\partial Q}{\partial y}dy-\iint P\frac{\partial^2 Q}{\partial x\partial y}dxdy$$
$$-\left\{-\int P\frac{\partial Q}{\partial x}dx-\iint P\frac{\partial^2 Q}{\partial x\partial y}dxdy\right\}$$
$$=\int P\frac{\partial Q}{\partial s}ds=\int PdQ,$$

where the double integrals extend over the whole area of the resolved surface, and the single integrals extend positively round the whole boundary. This boundary is composed of a single curve, composed of both edges of each of the cross-cuts; and the positive directions of the description are indicated in the figure, at a point of intersection of two cross-cuts.

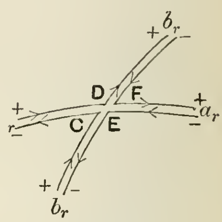

Fig. 82.

As explained in § 196, the negative edge of the cross-cut a_r is CE and the positive edge is DF; the negative edge of the cross-cut b_r is EF and the positive edge is CD. Then we have

$$P_D-P_F=P_C-P_E=B_r,\quad P_F-P_E=P_D-P_C=A_r;$$

and similarly for the function Q.

Consider the integral $\int PdQ$, taken along the two edges of the cross-cut a_r: let P_- and P_+ denote the functions along the negative and the positive edges respectively, so that $P_+-P_-=A_r$. The value of the integral for the two edges is

$$\int_F^D P_+dQ, \text{ taken in the direction } F\ldots D$$
$$+\int_C^E P_-dQ, \text{ taken in the direction } C\ldots E$$
$$=\int_F^D (P_+-P_-)\,dQ, \text{ taken in the direction } F\ldots D$$
$$=A_r\int_F^D dQ=A_r(Q_D-Q_F)=A_rB_r'.$$

Similarly, when the value of the integral for the two edges of the cross-cut b_r is taken, we have

$$\int_D^C P_+dQ, \text{ taken in the direction } D\ldots C$$
$$+\int_E^F P_-dQ, \text{ taken in the direction } E\ldots F$$
$$=\int_D^C (P_+-P_-)\,dQ, \text{ taken in the direction } D\ldots C$$
$$=B_r\int_D^C dQ=B_r(Q_C-Q_D)=-B_rA_r'.$$

And the value of the integral for the combination of the two edges of any cross-cut c is zero.

Hence summing for the whole boundary of the resolved surface, we have

$$\int P dQ = \sum_{r=1}^{p} (A_r B_r' - B_r A_r'),$$

and therefore

$$\iint \left(\frac{\partial P}{\partial x} \frac{\partial Q}{\partial y} - \frac{\partial Q}{\partial x} \frac{\partial P}{\partial y} \right) dxdy = \sum_{r=1}^{p} (A_r B_r' - B_r A_r'),$$

subject to the assigned conditions.

This theorem is of considerable importance: and the conditions, subject to which it is valid, permit P and Q (or either of them) to be real or complex potential functions of x and y or to be a function of z.

231. As a first application, let P and Q be real potential functions such that $P + iQ$ is a function of z, say w, evidently a function of the first kind. Let its moduli for the cross-cuts be

$$\omega_s + iv_s \text{ at } a_s, \text{ for } s = 1, 2, \ldots, p;$$

and

$$\omega_s' + iv_s' \text{ at } b_s, \text{ for } s = 1, 2, \ldots, p.$$

Since $P + iQ$ is a function of $x + iy$, we have, by §§ 7, 8,

$$\frac{\partial P}{\partial x} = \frac{\partial Q}{\partial y}, \quad -\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

The double integral then becomes

$$\iint \left\{ \left(\frac{\partial P}{\partial x} \right)^2 + \left(\frac{\partial P}{\partial y} \right)^2 \right\} dxdy,$$

which cannot be negative, because P is real; it is a quantity that is positive except only when P (and therefore w) is a constant everywhere. In the present case

$$A_r = \omega_r, \ B_s = \omega_s'; \quad A_r' = v_r, \ B_s' = v_s',$$

so that $\sum_{r=1}^{p} (\omega_r v_r' - \omega_r' v_r)$ is always positive. Hence:

If a function w, everywhere finite on a Riemann's surface, have $\omega_s + iv_s$ at a_s (for $s = 1, 2, \ldots, p$) and $\omega_s' + iv_s'$ at b_s (for $s = 1, 2, \ldots, p$) as its moduli, the cross-cuts a and b being the $2p$ cross-cuts necessary to resolve the surface into one that is simply connected, then

$$\sum_{r=1}^{p} (\omega_r v_r' - \omega_r' v_r)$$

is always positive, unless w is a constant: and then it is zero.

This proposition has the following corollaries.

COROLLARY I. *A function of z of the first kind cannot have its moduli of periodicity for $a_1, \ldots, a_p$ all zero.*

For if all these moduli were to vanish, then each of the quantities ω_r and each of the quantities v_r would be zero: the sum $\sum_{r=1}^{p} (\omega_r v_r' - \omega_r' v_r)$ would then vanish, which cannot occur unless w be a constant.

COROLLARY II. *A function of z of the first kind cannot have its moduli of periodicity for $b_1, \ldots, b_p$ all zero; it cannot have its moduli of periodicity all purely real, or all purely imaginary, or some zero and all the rest either purely real or purely imaginary.*

The different cases can be proved as in the preceding Corollary.

Note. One important inference can at once be derived, relative to functions of the first kind that have only two moduli of periodicity, Ω_1 and Ω_2.

Neither of the moduli may vanish; for if one, say Ω_1, were to vanish, then w/Ω_2 would be a function having one modulus zero and the other unity.

The ratio of the moduli may not be real. If it were real, then w/Ω_1 would be a function having one modulus unity and the other real. Both of these inferences are contrary to Corollary II.; and therefore the ratio of the two moduli is a complex constant, the real part of which may vanish but not the imaginary part.

The association of this result with the doubly-periodic functions is immediate.

Ex. Shew that, if two functions of the first kind have the same moduli of periodicity, their difference is a constant: and that, if W be a value, at any point of the surface, of a function of the first kind with moduli $\omega_1, \omega_2, \ldots, \omega_{2p}$, all the functions of the first kind, which have those moduli, are included in the form

$$W + \sum_{r=1}^{2p} m_r \omega_r + A,$$

where the coefficients m are integers and A is a constant.

232. As a second application, let P be a function of z and Q also a function of z; evidently, with the restriction of the proposition, P and Q must be functions of the first kind, when no part of the surface is excluded from the range of variation of z. Then

$$i\frac{\partial P}{\partial x} = \frac{\partial P}{\partial y}, \quad i\frac{\partial Q}{\partial x} = \frac{\partial Q}{\partial y},$$

so that at every point on the surface we have

$$\frac{\partial P}{\partial x}\frac{\partial Q}{\partial y} - \frac{\partial Q}{\partial x}\frac{\partial P}{\partial y} = 0.$$

Consequently the double integral

$$\iint \left(\frac{\partial P}{\partial x}\frac{\partial Q}{\partial y} - \frac{\partial Q}{\partial x}\frac{\partial P}{\partial y} \right) dxdy = 0:$$

and therefore, *if a function of the first kind have moduli* $A_1, \ldots, A_p, B_1, \ldots, B_p$, *and if any other function of the first kind have moduli* $A_1', \ldots, A_p', B_1', \ldots, B_p'$ *at the cross-cuts a and b respectively, then*

$$\sum_{r=1}^{p} (A_r B_r' - B_r A_r') = 0.$$

233. Next, let Q be a function of z of the first kind, as in the preceding case; but now let P be a function of z of the second kind, so that all its infinities are algebraical. The points where the function is infinite must be excluded from the surface: a corresponding number of cross-cuts will be necessary for the resolution of the surface into one that is simply connected. The modulus of periodicity of P for each of these cross-cuts is zero, (as in Ex. 8 of § 199, which is an instance of a function of this kind), no additional modulus being necessary with an algebraical infinity.

Then over the resolved surface, thus modified, the functions $P(z)$ and $Q(z)$ are everywhere uniform, finite and continuous: and therefore, as before

$$\iint \left(\frac{\partial P}{\partial x} \frac{\partial Q}{\partial y} - \frac{\partial Q}{\partial x} \frac{\partial P}{\partial y} \right) dxdy = \int PdQ,$$

the double integral extending over the whole of the resolved surface and the single integral extending round its whole boundary. But, at all points in the resolved surface, we have

$$\frac{\partial P}{\partial x} \frac{\partial Q}{\partial y} - \frac{\partial Q}{\partial x} \frac{\partial P}{\partial y} = 0,$$

and therefore, as before, the double integral vanishes. Hence $\int PdQ$, taken round the whole boundary, vanishes.

The boundary is made up of the double edges of all the cross-cuts a, b, and those, say l, which are introduced through the infinities, and of the small curves round the infinities.

As in § 230, the value of the integral for the two edges of a_r is $A_r B_r'$; and its value for the two edges of b_r is $-B_r A_r'$. The value of the integral for the two edges of any cross-cut l is zero, because the subject of integration is the same along the edges which are described in opposite directions.

To find the value round one of the small curves, say that which encloses an infinity represented analytically by a value c_s of z, we take, in the immediate vicinity of c_s,

$$P(z) = \frac{H_s}{z - c_s} + p(z - c_s),$$

where $p(z - c_s)$ is a converging series of positive integral powers of $z - c_s$. In that vicinity, let

$$Q = Q_s + (z - c_s) Q_s' + \text{higher powers of } z - c_s,$$

so that Q_s' is dQ/dz for $z = c_s$; thus

$$dQ = (Q_s' + \text{positive powers of } z - c_s)\, dz.$$

Hence along the small curve

$$PdQ = H_s Q_s' \frac{dz}{z - c_s} + q(z - c_s)\, dz,$$

where $q(z - c_s)$ is a converging series of positive integral powers of $z - c_s$. The value of the integral round the curve is $2\pi i H_s Q_s'$.

Summing these various parts of the integral and remembering that the whole integral is zero, we have

$$\sum_{r=1}^{p} (A_r B_r' - B_r A_r') + 2\pi i \Sigma H_s Q_s' = 0,$$

there being as many terms in the last summation as there are simple infinities of P.

The equation

$$\sum_{r=1}^{p} (A_r B_r' - B_r A_r') + 2\pi i \sum_s H_s \left(\frac{dQ}{dz}\right)_{z=c_s} = 0$$

is the relation which subsists between the moduli A', B' of a function $Q(z)$ of the first kind and the moduli A, B of a function $P(z)$ of the second kind, all the infinities of which are simple.

The simplest illustration is furnished by the integrals that were considered in Ex. 6 and Ex. 8 of § 199.

Let P be the function of Ex. 8, usually denoted by $E(z)$, being the elliptic integral of the second kind; it is infinite for $z = \infty$ in each sheet. In the upper sheet we have, for large values of $|z|$,

$$P = E(z) = kz\left(1 + \text{positive integral powers of } \frac{1}{z}\right);$$

and for the same in the lower, we have

$$P = E(z) = -kz\left(1 + \text{positive integral powers of } \frac{1}{z}\right).$$

Let Q be the function of Ex. 6, usually denoted by $F(z)$, being the elliptic integral of the first kind, finite everywhere. We easily find, for large values of $|z|$ in the upper sheet, that

$$dQ = dF(z) = \frac{1}{kz^2}\left(1 + \text{positive integral powers of } \frac{1}{z}\right) dz,$$

and, for large values of $|z|$ in the lower, that

$$dQ = dF(z) = -\frac{1}{kz^2}\left(1 + \text{positive integral powers of } \frac{1}{z}\right) dz.$$

Then for large values of $|z|$ in the upper sheet, we have

$$PdQ = \frac{dz}{z}\left(1 + \text{positive integral powers of } \frac{1}{z}\right)$$

$$= -\frac{dz'}{z'}(1 + \text{positive integral powers of } z'),$$

where $zz'=1$; and we may consider the Riemann's surface spherical. Hence the value round the excluding curve in the upper sheet is $-2\pi i$.

Similarly the value round the excluding curve in the lower sheet is $-2\pi i$.

Now A_1 and B_1, the moduli of P, are $4E$ and $2i(K'-E')$ respectively; A_1' and B_1', the moduli of Q, are $4K$ and $2iK'$ respectively. Hence

$$4E \, . \, 2iK' - 4K \, . \, 2i(K'-E') - 4\pi i = 0,$$

leading to the Legendrian equation

$$EK' + E'K - KK' = \tfrac{1}{2}\pi.$$

234. Before proceeding to the relations affecting the moduli of periodicity of functions of the third kind, we shall make some inferences from the preceding propositions.

It has been proved that functions of the first kind, special examples of which arose as integrals of algebraic functions, exist on a Riemann's surface. They are everywhere finite and, except for additive multiples of the moduli, they are uniform and continuous; and when, in addition to these properties, the real parts of their moduli of periodicity are arbitrarily assigned, the functions are uniquely determinate. Hence the number of such functions is unlimited: they are, however, subject to the following proposition:—

The number of linearly independent functions of the first kind, that exist on a given Riemann's surface, is equal to p; where $2p+1$ is the connectivity of the surface. And every function of the first kind on that surface is of the form $C + \sum_{q=1}^{p} c_q w_q$, where C is a constant, the coefficients $c_1, \ldots, c_p$ are constants, and $w_1, \ldots, w_p$ are p linearly independent functions.

Let q series of linearly independent real quantities, each series containing $2p$ non-vanishing constants, be arbitrarily assigned as the real parts of the moduli of periodicity of functions of the first kind, which are thence uniquely determined. Let the functions be $w_1, w_2, \ldots, w_q$; and let the real parts of their moduli be $(\omega_{1,1}, \omega_{1,2}, \ldots, \omega_{1,2p}), (\omega_{2,1}, \omega_{2,2}, \ldots, \omega_{2,2p}), \ldots, (\omega_{q,1}, \omega_{q,2}, \ldots, \omega_{q,2p})$. The modulus of w_r at the cross-cut C_m has its real part denoted by $\omega_{r,m}$: when the modulus is divided into real and imaginary parts, let it be $\omega_{r,m} + i\omega'_{r,m}$.

If any set of q arbitrary complex constants be denoted by $c_1, \ldots, c_q$, where c_s is of the form $\alpha_s + i\beta_s$, then, at the cross-cut C_m, the real part of the modulus of $\sum_{r=1}^{q} c_r w_r$ is the real part of $\sum_{r=1}^{q} c_r (\omega_{r,m} + i\omega'_{r,m})$, that is, it is equal to

$$\alpha_1\omega_{1,m} + \ldots + \alpha_q\omega_{q,m} - \beta_1\omega'_{1,m} - \ldots - \beta_q\omega'_{q,m},$$

holding for $m = 1, 2, \ldots, 2p$ and therefore giving $2p$ expressions in all.

Now let a series of real arbitrary quantities $A_1, A_2, \ldots, A_{2p}$ be assigned as the real parts of the moduli of periodicity of a function of the first kind, which is uniquely determined by them; and consider the equations

$$\left.\begin{array}{l} A_1 = \alpha_1\omega_{1,1} + \alpha_2\omega_{2,1} + \dots + \alpha_q\omega_{q,1} - \beta_1\omega'_{1,1} - \beta_2\omega'_{2,1} - \dots - \beta_q\omega'_{q,1} \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ A_{2p} = \alpha_1\omega_{1,2p} + \alpha_2\omega_{2,2p} + \dots + \alpha_q\omega_{q,2p} - \beta_1\omega'_{1,2p} - \beta_2\omega'_{2,2p} - \dots - \beta_q\omega'_{q,2p} \end{array}\right\}.$$

First, let $q < p$: the $2q$ constants α and β can be determined so as to make the right-hand sides respectively equal to $2q$ arbitrarily assigned constants A. The right-hand sides of the remaining equations are then determinate constants; and therefore the remaining equations will not be satisfied when the remaining constants A are arbitrarily assigned.

The function determined by the moduli A has some of its moduli different from those of the function Σcw, when $q < p$; hence, when q functions $w_1, \dots$ $\dots, w_q$, where $q < p$, have been obtained, we can obtain another function, and so on; until $q = p$.

But, when $q = p$, then the foregoing $2p$ equations determine the quantities α and β, whatever be the quantities A. Let W be the function of the first kind, determined by the constants A as the arbitrarily assigned real parts of its moduli of periodicity: then

$$W - \sum_{s=1}^{p} c_s w_s,$$

where the coefficients c are constants, has the real parts of all its moduli of periodicity zero: it is therefore, by Cor. II. § 231, a constant, so that

$$W = c_1 w_1 + \dots + c_p w_p + C,$$

where C is a constant. Therefore the number of linearly independent functions of the first kind is p; and every function of the first kind is of the form

$$C + \sum_{s=1}^{p} c_s w_s.$$

It has been assumed in what precedes that the determinant of the quantities ω and ω' does not vanish. This possibility is not excluded merely by the arbitrary choice of the quantities ω; because the quantities ω' are determined for w, and w is dependent on v. If, however, the determinant should vanish, then, by taking the quantities α and β proportional to the minors of ω and ω' respectively in the determinant, all the quantities

$$\sum_{s=1}^{p} (\alpha_s\omega_{sm} - \beta_s\omega'_{sm})$$

can be made to vanish. These quantities are the real parts of the moduli of periodicity of $\sum_{s=1}^{p} c_s w_s$ which, because they all vanish, is a constant, that is, the quantities w_s are not linearly independent of one another—an inference contrary to their construction. Hence *the determinant of the quantities ω does not vanish.*

Note. It may be remarked, in passing, that each function w, being of the first kind, gives rise to two real potential functions, which are everywhere finite and have moduli of periodicity at the cross-cuts: one of the functions being the real part of w, the other arising from its imaginary part. Hence from the p linearly independent functions of the first kind, there are altogether $2p$ linearly independent real potential functions. This number is the same as the total number of real potential functions considered in § 228: hence each of them can be expressed as a linear function of the members of that former system, save possibly as to an additive constant. Conversely, it follows that linear combinations of the members of that former system can be taken in pairs, so as to furnish p (and not more than p) linearly independent functions of z of the first kind.

235. The functions so far obtained are very general: it is convenient to have a set of functions of the first kind in normal forms. The foregoing analysis indicates that linear combinations of constant multiples of the functions, being themselves functions of the first kind, are conveniently considered from the point of view of their moduli of periodicity: and the simpler the aggregate of these moduli is, the simpler will be the functions determined by them. Some conditions have been shewn (§ 231) to attach to the aggregate of the moduli for any one function of the first kind, and a condition (§ 232) for the moduli of different functions; these are the conditions that limit the choice of linear combinations.

Let $c_1w_1 + \ldots + c_pw_p$ be a linear combination of the functions $w_1, \ldots, w_p$ which have $\omega_{r1}, \ldots, \omega_{rp}$ $(r = 1, \ldots, p)$ as the moduli of periodicity for the cross-cuts $a_1, \ldots, a_p$. Then Δ, where Δ is the determinant

$$\Delta = \begin{vmatrix} \omega_{11}, & \omega_{12}, \ldots\ldots, & \omega_{1p} \\ \omega_{21}, & \omega_{22}, \ldots\ldots, & \omega_{2p} \\ \ldots & \ldots & \ldots \\ \omega_{p1}, & \omega_{p2}, \ldots\ldots, & \omega_{pp} \end{vmatrix},$$

cannot vanish: for otherwise by taking constants $c_1, \ldots, c_p$ proportional to the first minors, we should obtain a function $\sum_{s=1}^{p} c_s w_s$, having all its moduli for the cross-cuts $a_1, \ldots, a_p$ zero and therefore, by § 231, merely a constant, so that $w_1, \ldots, w_p$ would not be linearly independent. Hence Δ *does not vanish.*

Next, we can choose constants c so that the moduli of periodicity vanish for the function $\sum_{s=1}^{p} c_s w_s$ at all the cross-cuts a, except at one, say a_r, and that there it has any assigned value, say πi. For, solving the equations,

$$0 = c_1\omega_{s,1} + c_2\omega_{s,2} + \ldots + c_p\omega_{s,p}, \quad (\text{for } s \gtrless r = 1, 2, \ldots, p);$$

$$\pi i = c_1\omega_{r,1} + c_2\omega_{r,2} + \ldots + c_p\omega_{r,p},$$

the determinant of the right-hand side does not vanish, and the constants c, say $c_{r,1}, c_{r,2}, \dots, c_{r,p}$, are determinate. The function $c_{r,1}w_1 + c_{r,2}w_2 + \dots + c_{r,p}w_p$, say W_r, then has its moduli zero for $a_1, \dots, a_{r-1}, a_{r+1}, \dots, a_p$: it has the modulus πi for a_r; it has moduli, say $B_{r,1}, \dots, B_{r,p}$ at $b_1, \dots, b_p$ respectively. And the function is determinate save as to an additive constant.

This combination can be effected for each of the values $1, \dots, p$ of r: and thus p new functions will be obtained. These *p functions are linearly independent*: for, if there were a relation of the form

$$C_1 W_1 + C_2 W_2 + \dots\dots + C_p W_p = \text{constant},$$

the modulus of the function $\sum_{r=1}^{p} C_r W_r$ at the cross-cut a_s should be zero because the function is a constant; and it is $C_s \pi i$, so that all the coefficients C would be zero.

The functions W, thus obtained, have the moduli:—

	a_1	a_2				a_p	b_1	b_2				b_p
W_1	πi	0				0	$B_{1,1}$	$B_{1,2}$				$B_{1,p}$
W_2	0	πi				0	$B_{2,1}$	$B_{2,2}$				$B_{2,p}$
......	......	......	......	......	......	......	......	......	......	......	......	......
......	......	......	......	......	......	......	......	......	......	......	......	......
W_p	0	0				πi	$B_{p,1}$	$B_{p,2}$				$B_{p,p}$

These functions are called *normal functions of the first kind*: they are a complete system linearly independent of one another, and are such that every function of the first kind is, except as to an additive constant, a linear combination of constant multiples of them.

The quantities B are not completely independent of one another. Since W_j, $W_{j'}$ are functions of the first kind we have, by § 232,

$$\sum_{r=1}^{p} (A_{r,j}B_{r,j'} - B_{r,j}A_{r,j'}) = 0,$$

which, for the normal functions, takes the form

$$\pi i B_{jj'} - \pi i B_{j'j} = 0,$$

that is, $B_{jj'} = B_{j'j}$. Hence *the moduli B with the same integers for suffix are equal to one another.*

This is a first relation among the moduli. Another is given by the following theorem :—

Let $B_{m,n}=\rho_{m,n}+i\sigma_{m,n}$, *(so that* $\rho_{m,n}=\rho_{n,m}$, *and* $\sigma_{m,n}=\sigma_{n,m}$*): then, if* $c_1, \ldots, c_p$ *be any real quantities, the expression*

$$\rho_{11}c_1^2+2\rho_{12}c_1c_2+\rho_{22}c_2^2+\ldots+\rho_{pp}c_p^2,$$

is negative, unless the quantities c *vanish together.*

The function $c_1W_1+c_2W_2+\ldots+c_pW_p+C$ is a function of the first kind with moduli (say) ω_r+iv_r at a_r, where $r=1, \ldots, p$, and moduli $\omega_s'+iv_s'$ at b_s, where $s=1, \ldots, p$. Then, by § 231, the sum $\sum_{r=1}^{p}(\omega_r v_r'-\omega_r' v_r)$ is positive, except when the function is a constant, that is, except when $c_1, \ldots, c_p$ all vanish. But

$$\omega_r+iv_r=c_r\pi i,$$

so that $\omega_r=0$, $v_r=\pi c_r$; and

$$\omega_s'+iv_s'=c_1B_{1,s}+c_2B_{2,s}+\ldots+c_pB_{p,s},$$

so that

$$\omega_s'=c_1\rho_{1,s}+c_2\rho_{2,s}+\ldots+c_p\rho_{p,s}.$$

Hence the sum

$$\sum_{r=1}^{p}-c_r\pi(c_1\rho_{1,r}+c_2\rho_{2,r}+\ldots+c_p\rho_{p,r})$$

is positive and therefore the sum $\sum_{r=1}^{p}\sum_{s=1}^{p}\rho_{rs}c_rc_s$ is negative. This (with the property $\rho_{mn}=\rho_{nm}$) is the required result.

These properties of the periods, all due to Riemann, are useful in the construction of the Theta-Functions.

For the ordinary Jacobian elliptic functions in which $p=1$, there is only one integral which is everywhere finite: its periods are $4K$, $2iK'$. To express it in the normal form, we take $cF(z)$, choosing c so that the period at a_1 is purely imaginary and $=\pi i$; hence $c=\frac{\pi i}{4K}$, and the normal integral is

$$\frac{\pi i F(z)}{4K}.$$

The other period of this function is $-\frac{\pi K'}{2K}$, which, when k is real and less than unity, is a negative quantity; it is the value of ρ_{11} and satisfies the condition that $\rho_{11}c_1^2$ is negative for all real quantities c.

236. It has been proved that functions exist on a Riemann's surface, having assigned algebraical infinities and assigned real parts of its moduli of periodicity, but otherwise uniform, finite and continuous. The simplest instance of these functions of the second kind occurs when the infinity is an accidental singularity of the first order.

Let the single infinity on the surface be represented by $z=c$: let $E_c(z)$ be the function having $z=c$ as its algebraical infinity, and having the real parts of its moduli of periodicity assigned. If $E_c'(z)$ be any other function with that single infinity and the real parts of its moduli the same, then

$E_c(z) - E_c'(z)$ is a function all the real parts of whose moduli are zero; it does not have c for an infinity and therefore it is everywhere finite: by § 231, it is a constant. Hence *an elementary function of the second kind is determined, save as to an additive constant, by its infinity and the real parts of its moduli.*

Again, it can be proved, as for the special case in § 208, that an elementary function of the second kind is determined, save as to an additive function of the first kind, by its infinity alone: hence, if $E(z)$ be any elementary function, having its infinity represented by $z = c$, we have

$$E(z) = E_c(z) + \lambda_1 W_1 + \ldots + \lambda_p W_p + A,$$

where $\lambda_1, \ldots, \lambda_p$, A are constants, the values of which depend on the special function chosen. Let $E_c(z)$ have $\pi i C_1, \ldots, \pi i C_p$ for its moduli at the cross-cuts $a_1, \ldots, a_p$ respectively: and let the function $E(z)$ be chosen so as to have all its moduli at $a_1, \ldots, a_p$ equal to zero: then $\lambda_r = -C_r$ and $E(z)$ is given by

$$E_c(z) - C_1 W_1 - \ldots - C_p W_p + A.$$

The special function of the second kind, which has all its moduli at the cross-cuts $a_1, \ldots, a_p$ equal to zero, is called *the normal function of the second kind.* It is customary to take unity as the coefficient of the infinite term, that is, the residue of the normal function.

This normal function is determined, save as to an additive constant, by its infinity alone. For if $E(z)$ and $E'(z)$ be two such normal functions, the function

$$E(z) - E'(z)$$

is finite everywhere; its moduli are zero at $a_1, \ldots, a_p$; hence (§ 231) it is a constant.

Normal functions of the second kind will be used later (§ 241) in the construction of functions with any number of simple infinities on the surface.

Let the moduli of this normal function $E(z)$ of the second kind be $B_1, \ldots, B_p$ for the cross-cuts $b_1, \ldots, b_p$. Then applying the proposition of § 233 and considering the integral $\int E dW_r$, we have $A_1 = \ldots = A_p = 0$; also

$$A_1' = \ldots = A'_{r-1} = A'_{r+1} = \ldots = A_p' = 0,$$

and $A_r' = \pi i$. The relation therefore is

$$-B_r \pi i + 2\pi i \left(\frac{dW_r}{dz}\right)_{z=c} = 0,$$

where, in the immediate vicinity of $z = c$,

$$E(z) = \frac{1}{z-c} + p(z-c),$$

p being a converging series of positive powers. Thus

$$B_r = 2\left(\frac{dW_r}{dz}\right)_{z=c},$$

or, as $\frac{dW_r}{dz}$ is an algebraical function (§ 241) on the surface, *the periods of a* *normal function of the second kind at the cross-cuts b are algebraical functions of its single infinity.*

In the case of the Jacobian elliptic integrals, the integral of the second kind has at $z=\infty$ an infinity of the first order in each sheet (Ex. 8, § 199). The moduli of this integral, denoted by $E(z)$, are $4E$ and $2i(K'-E')$ for a_1 and b_1 respectively; hence the normal integral of the second kind is

$$E(z)-\frac{E}{K}F(z),$$

$F(z)$ being the (one) integral of the first kind. This is the function $Z(z)$; its modulus is zero for a_1, and for b_1 it is

$$2i(K'-E')-\frac{E}{K}2iK',$$

which is $\frac{2i}{K}(KK'-E'K-EK')$, that is, it is $-\frac{i\pi}{K}$.

237. The other simple class of function which exists on a Riemann's surface with assigned infinities and assigned real parts of its moduli is that which is represented by the elementary integral of the third kind. It has two points of logarithmic infinity on the surface*, say P_1 and P_2; let these be represented by the values c_1 and c_2 of z. On division by a proper constant, the function, which may be denoted by Π_{12}, takes the forms

$$-\log(z-c_1)+p_1(z-c_1), \quad +\log(z-c_2)+p_2(z-c_2),$$

in the immediate vicinities of P_1 and of P_2 respectively, where p_1 and p_2 are converging series of positive integral powers.

The points P_1 and P_2 can be taken as boundaries of the surface, as in Ex. 7 in § 199. A cross-cut from P_2 to P_1 is then necessary for the resolution of the surface: and the period for the cross-cut is $2\pi i$, being the increase of the function in passing from the negative to the positive edge of the cross-cut.

Then with this assignment of infinities and with the real parts of the moduli at the cross-cuts $a_1, \ldots, a_p, b_1, \ldots, b_p$ arbitrarily assigned, functions Π_{12} exist on the Riemann's surface.

As in the case of the function of the second kind, it is easy to prove that a function Π_{12} of the third kind is determined, save as to an additive constant, by its two infinities and the assignment of its moduli: and that it is determined, save as to an additive function of the first kind, by its infinities alone.

Among the infinitude of elementary functions of the third kind, having the same logarithmic infinities, a normal form can be chosen in the same manner as for the functions of the second kind. Let Π_{12} be an elementary function of the third kind, having P_1 and P_2 for its logarithmic infinities: let its moduli of periodicity be $2\pi i$ for the cross-cut P_1P_2; $\pi iC_1, \ldots, \pi iC_p$ for $a_1, \ldots, a_p$ respectively; and other quantities for $b_1, \ldots, b_p$ respectively. Then

$$\varpi_{12}=\Pi_{12}-C_1W_1-\ldots-C_pW_p$$

* The representation of a single point on the Riemann's surface by means solely of the value of z at the point will henceforward be adopted, without further explanation, in instances when it cannot give rise to ambiguity. Otherwise, the representation in full detail of statement will be adopted.

is an elementary function of the third kind, having zero as its modulus of periodicity at each of the cross-cuts $a_1, \ldots, a_p$. This function is the *normal form of the elementary function of the third kind.*

If ϖ_{12}' and ϖ_{12} be two normal elementary functions of the third kind with the same logarithmic infinities and the same period $2\pi i$ at the cross-cut P_1P_2, then

$$\varpi_{12}' - \varpi_{12}$$

is a function without infinities on the surface; its modulus for P_1P_2 is zero, and its modulus for each of the cross-cuts $a_1, \ldots, a_p$ is zero; and therefore it is a constant. Hence *a normal elementary function of the third kind is, save as to an additive constant, determined by its infinities alone.*

Ex. The sum of three normal elementary functions of the third kind, having as their logarithmic infinities the respective pairs that can be selected from three points, is a constant.

238. A relation among the moduli of an elementary function of the third kind can be constructed in the same way as, in § 233, for the function of the second kind.

Let the surface be resolved by the $2p$ cross-cuts $a_1, \ldots, a_p, b_1, \ldots, b_p$ and by the cross-cut P_1P_2, joining the excluded infinities of an elementary function Π_{12} of the third kind. Let w be any function of the first kind; then over the resolved surface, we have

$$\frac{\partial \Pi_{12}}{\partial x}\frac{\partial w}{\partial y} - \frac{\partial \Pi_{12}}{\partial y}\frac{\partial w}{\partial x}$$

everywhere zero; and therefore $\int \Pi_{12}\, dw$ round the whole boundary of the resolved surface is zero, as in § 233.

Let the moduli of Π_{12} be $A_1, \ldots, A_p, B_1, \ldots, B_p$, and those of w be $A_1', \ldots, A_p', B_1', \ldots, B_p'$ for the $2p$ cross-cuts a and b respectively.

The whole boundary is made up of the two edges of the cross-cuts a, the two edges of the cross-cuts b, the two edges of the cross-cut P_1P_2 and the small curves round P_1 and P_2.

The sum of the parts contributed to $\int \Pi_{12}\, dw$ by the edges of all the cross-cuts a and b is, as in preceding instances,

$$\sum_{s=1}^{p} (A_s B_s' - A_s' B_s).$$

The direction of integration along P_1P_2 that is positive relative to the area in the resolved surface is indicated by the arrows; the portion of $\int \Pi_{12}\, dw$ along the two edges of the cut is

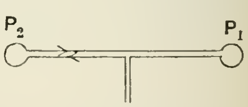

Fig. 83.

$$\int_{c_1}^{c_2} \Pi_{12}^{+}\, dw + \int_{c_2}^{c_1} \Pi_{12}^{-}\, dw$$

$$= \int_{c_1}^{c_2} (\Pi_{12}^{+} - \Pi_{12}^{-})\, dw = 2\pi i \int_{c_1}^{c_2} dw = 2\pi i \{w(c_2) - w(c_1)\}.$$

Lastly, the portion of the integral for the infinitesimal curve round P_1 is zero, by I. of § 24, because the limit of $(z - c_1) \Pi_{12} \frac{dw}{dz}$ for $z = c_1$ vanishes, P_1 being assumed not to be a branch-point; and similarly for the portion of the integral contributed by the infinitesimal curve round P_2.

As the integral $\int \Pi_{12}\, dw$ vanishes, we therefore have

$$\sum_{s=1}^{p} (A_s B_s' - A_s' B_s) + 2\pi i \{w(c_2) - w(c_1)\} = 0,$$

which is the relation required.

The most important instance is that in which Π_{12} is the normal elementary function of the third kind (and then $A_1, A_2, \ldots, A_p$ all vanish), and w is a normal function of the first kind, say W_r (and then

$$A_r' = \pi i,\ A_1' = A_2' = \ldots = A'_{r-1} = A'_{r+1} = \ldots = A_p' = 0).$$

Hence, if B_r be the modulus at b_r of the normal elementary integral ϖ_{12}, we have

$$B_r = 2\{W_r(c_2) - W_r(c_1)\},$$

so that *the moduli of the normal elementary function of the third kind can be expressed in terms of normal functions, of the first kind, of its logarithmic discontinuities.*

The important property of functions of the third kind, known as the *interchange of argument and parameter*, can be deduced by a similar process.

Let Π_{12} be an elementary function with logarithmic discontinuities at c_1 and c_2, with $2\pi i$ as its modulus for the cross-cut c_1c_2, and with

$$A_1, \ldots, A_p, B_1, \ldots, B_p$$

as its moduli for the cross-cuts $a_1, \ldots, a_p, b_1, \ldots, b_p$; and let Π_{34} be another elementary function with logarithmic discontinuities at c_3 and c_4, with $2\pi i$ as its modulus for the cross-cut c_3c_4, and with $A_1', \ldots, A_p', B_1', \ldots, B_p'$ as its moduli for the cross-cuts $a_1, \ldots, a_p, b_1, \ldots, b_p$. Then when the infinities are excluded and the surface is resolved so that both Π_{12} and Π_{34} are uniform finite and continuous throughout the whole surface, we have

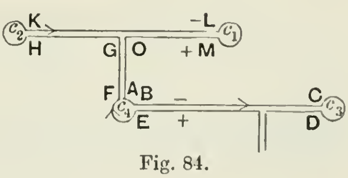

Fig. 84.

$$\frac{\partial \Pi_{12}}{\partial x}\frac{\partial \Pi_{34}}{\partial y} - \frac{\partial \Pi_{34}}{\partial x}\frac{\partial \Pi_{12}}{\partial y} = 0,$$

everywhere in the resolved surface; and therefore, as in the preceding instances, $\int \Pi_{12}\, d\Pi_{34}$ round the whole boundary vanishes.

The whole boundary is made up of the double edges of the cross-cuts a and the cross-cuts b, and of the configuration of cross-cuts and small curves round the points. The modulus of both Π_{12} and of Π_{34} for the cut AG is zero; the modulus of Π_{12} for the cut c_3c_4 is zero, and that of Π_{34} for the cut c_1c_2 is zero.

The part contributed to $\int \Pi_{12}\, d\Pi_{34}$ by the aggregate of the edges of the cross-cuts a and b is $\sum_{s=1}^{p} (A_s B_s' - A_s' B_s)$, as in preceding cases.

The part contributed by the small curve round c_1 is zero, because the limit, for $z = c_1$, of $(z - c_1)\, \Pi_{12} \dfrac{d\Pi_{34}}{dz}$ is zero; similarly the part contributed by the small curve round c_2 is zero.

The part contributed by the two edges of the cross-cut c_1c_2 is

$$\int_{c_2}^{c_1} \Pi_{12}^{-}\, d\Pi_{34} + \int_{c_1}^{c_2} \Pi_{12}^{+}\, d\Pi_{34}$$

$$= 2\pi i \int_{c_1}^{c_2} d\Pi_{34} = 2\pi i \left\{\Pi_{34}(c_2) - \Pi_{34}(c_1)\right\}.$$

The part contributed by the two edges of the cross-cut AO is

$$\left(\int_O^A + \int_F^G\right) \Pi_{12}\, d\Pi_{34}:$$

the subject of integration does not change in crossing from one edge to the other, and therefore this part is zero.

For points on the small curve round c_3, we have

$$d\Pi_{34} = -\frac{dz}{z - c_3} + p\,(z - c_3)\, dz,$$

where p is a converging series of integral powers of $z - c_3$: and therefore for points on that curve

$$\Pi_{12}\, d\Pi_{34} = -\frac{\Pi_{12}(c_3)}{z - c_3}\, dz + q\,(z - c_3)\, dz,$$

where $q\,(z - c_3)$ is a converging series of positive integral powers of $z - c_3$. Hence the part contributed to $\int \Pi_{12}\, d\Pi_{34}$ by the small curve round c_3 in the direction of the arrow, which is the negative direction for integration relative to c_3, is $2\pi i\, \Pi_{12}(c_3)$.

Again, for points on the small curve round c_4, we have

$$d\Pi_{34} = \frac{dz}{z - c_4} + p_1\,(z - c_4)\, dz;$$

proceeding as for c_3, we find the part contributed to $\int \Pi_{12}\, d\Pi_{34}$ by the small curve round c_4, which is negatively described, to be $-2\pi i\, \Pi_{12}(c_4)$.

Lastly, the sum of the parts contributed by the two edges of the cross-cut c_3c_4 is

$$\int_{c_3}^{c_4} \Pi_{12}\, d\Pi_{34}^{+} + \int_{c_4}^{c_3} \Pi_{12}\, d\Pi_{34}^{-}$$

$$= \int_{c_3}^{c_4} \Pi_{12} \left(\frac{d\Pi_{34}^{+}}{dz} - \frac{d\Pi_{34}^{-}}{dz}\right) dz.$$

But though Π_{34} has a modulus for the cross-cut c_3c_4, its derivative has not a modulus for that cross-cut: we have $d\Pi_{34}^{+}/dz = d\Pi_{34}^{-}/dz$, and therefore the last part contributed to $\int \Pi_{12}\, d\Pi_{34}$ vanishes.

The integral along the whole boundary vanishes; and therefore

$$\sum_{s=1}^{p} (A_s B_s' - A_s' B_s) + 2\pi i \{\Pi_{34}(c_2) - \Pi_{34}(c_1)\} + 2\pi i \Pi_{12}(c_3) - 2\pi i \Pi_{12}(c_4) = 0,$$

a relation between the moduli of two elementary functions of the third kind.

The most important case is that in which both of the functions are normal elementary functions. We have $A_1, \dots, A_p$ zero for ϖ_{12}, and $A_1', \dots, A_p'$ zero for ϖ_{34}; and the relation then is

$$\varpi_{34}(c_2) - \varpi_{34}(c_1) = \varpi_{12}(c_4) - \varpi_{12}(c_3),$$

which is often expressed in the form

$$\int_{c_1}^{c_2} d\varpi_{34} = \int_{c_3}^{c_4} d\varpi_{12},$$

the paths of integration in the unresolved surface being the directions of cross-cuts necessary to complete the resolution for the respective cases. Hence *the normal elementary integral of the third kind is unaltered in value by the interchange of its limits and its logarithmic infinities.*

239. From the simple examples, discussed in § 199 and elsewhere, it has appeared that when a function w is defined as the integral of some function of z, the integral being uniform except in regard to moduli of periodicity, a process of inversion is sometimes possible whereby z becomes a function of w, either uniform or multiform. But in all the cases, in which z thus proves to be a uniform function, the number of periods possessed by w is not greater than two; and it follows, from § 110, that, when w possesses more than two periods, z can no longer be regarded as a function of w. In fact, w then loses its property of being uniform by dependence upon a single variable.

A question therefore arises as to the form, if any, of functional inversion, when w has more than two independent periods and when there are more functions w than one.

Taking the most general case of a Riemann's surface of class p, let $w_1, w_2, \dots, w_p$ denote the p functions of the first kind. Let there be q independent variables $z_1, \dots, z_q$, where q is not, of initial necessity, equal to p; and, by means of any q of the functions of the first kind, say $w_1, \dots, w_q$, form q new functions, evidently also of the first kind and defined by the equations

$$v_r = w_r(z_1) + w_r(z_2) + \dots + w_r(z_q),$$

where $r = 1, 2, \dots, q$. We make the evident limitation that q is not greater than p, which is justifiable from the point of view of functional inversion. Then the functions v_r are multiform on the surface with constant moduli of periodicity; they have the same periods as w_r, say $\omega_{r,1}, \omega_{r,2}, \dots, \omega_{r,2p}$.

The various values of $w_r(z_m)$ differ by multiples of the periods: so that, if

$w_r(z_m)$ be the value for an exactly specified z_m-path (as in § 110), the value for any other z_m-path is

$$w_r(z_m) + n_{m,1}\,\omega_{r,1} + n_{m,2}\,\omega_{r,2} + \ldots + n_{m,2p}\,\omega_{r,2p}.$$

This being true for each of the integers $m = 1, 2, \ldots, q$, it follows that, if

$$m_s = \sum_{m=1}^{q} n_{m,s}, \quad (s = 1, 2, \ldots, 2p),$$

and if v_r be the value of $\sum_{m=1}^{q} w_r(z_m)$ for the exactly specified paths for $z_1, \ldots, z_q$, then the general value of v_r for any other set of paths for the variables is

$$v_r + m_1\omega_{r,1} + m_2\omega_{r,2} + \ldots + m_{2p}\omega_{r,2p},$$

holding for $r = 1, 2, \ldots, q$. The integers $n_{m,s}$, and therefore the integers m_s, are evidently the same for all the functions v.

The reason which, in the earlier case (§ 110), prevented the function w from being determinate as a function of z alone was, that integers could be determined so as to make the additive part of w, dependent upon the periods, an infinitesimal quantity. It is necessary to secure that this possibility be excluded.

Let $\omega_{\lambda,\mu} = \alpha_{\lambda,\mu} + i\beta_{\lambda,\mu}$, where the quantities α and β are real: then we have to prevent the possibility of the additive portions for all the functions v being infinitesimal. In order to reduce the additive part to an infinitesimal value for each of the functions v, it would be necessary to determine integers $m_1, m_2, \ldots, m_{2p}$ so that the $2q$ quantities

$$\left.\begin{array}{l} m_1\,\alpha_{r,1} + m_2\,\alpha_{r,2} + \ldots + m_{2p}\,\alpha_{r,2p} \\ m_1\beta_{r,1} + m_2\beta_{r,2} + \ldots + m_{2p}\beta_{r,2p} \end{array}\right\}$$

for $r = 1, \ldots, q$ all become infinitesimal.

If q be less than p, the $2p$ integers can be so determined. In that case, the general possibility of functional inversion between the q functions v and the q variables z would require that the quantities z are so dependent upon the quantities v that infinitesimal changes in the latter, carried out in an infinite variety of ways and capable of indefinite repetition, would leave the quantities z unchanged. The position, save that we have q variables instead of only one, is similar to that in § 110: we do not regard the functions v as having determinate values for assigned values of $z_1, \ldots, z_q$, but the values of $v_1, \ldots, v_q$ are determinate, only when the paths by which the independent variables acquire their values are specified. And, as before, the inversion is not possible.

If q be not less than p, so that it must in the present circumstances be equal to p, then the $2p$ integers cannot be determined so that the $2p$ quantities all become infinitesimal. They can be determined so as to make any $2p-1$ of the quantities become infinitesimal; but the remaining quantity is finite as, indeed, should be expected, because the determinant of the constants α and β is different from zero*.

If then there be p variables $z_1, \ldots, z_p$, and p functions $v_1, \ldots, v_p$ defined by the equations

$$v_r = w_r(z_1) + w_r(z_2) + \ldots + w_r(z_p),$$

for $r = 1, 2, \ldots, p$, then the values of the functions v for assigned values of the variables z, whatever be the paths by which the variables attain these values, are of the form

$$v_r + m_1\omega_{r,1} + m_2\omega_{r,2} + \ldots + m_{2p}\omega_{r,2p}$$

for $r = 1, 2, \ldots, p$; and it has been proved that the $2p$ integers m cannot be determined so that all the additive parts, dependent upon the periods, become infinitesimal. Hence the functions $v_1, \ldots, v_p$ are, except as to additive multiples of the periods (the numerical coefficients in these multiples being the same for all the functions), uniform functions of the variables $z_1, \ldots, z_p$; and they are finite for all values of the variables. Conversely, we may regard the quantities z as functions of the quantities $v_1, \ldots, v_p$, which have $2p$ sets of simultaneous periods $\omega_{1,s}, \omega_{2,s}, \ldots, \omega_{p,s}$ for $s = 1, 2, \ldots, 2p$: that is, the variables z are $2p$-ply periodic functions of p variables $v_1, \ldots, v_p$. This result is commonly called the *inversion-problem* for the Abelian transcendents.

In effecting the inversion of the equations

$$\left.\begin{array}{l} dv_1 = w_1'(z_1)\,dz_1 + w_1'(z_2)\,dz_2 + \ldots + w_1'(z_p)\,dz_p \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ dv_p = w_p'(z_1)\,dz_1 + w_p'(z_2)\,dz_2 + \ldots + w_p'(z_p)\,dz_p \end{array}\right\},$$

the actual form, which is adopted, expresses all symmetric functions of the quantities $z_1, \ldots, z_p$ as uniform functions of the variables, so that, if $z_1, z_2, \ldots, z_p$ be the roots of the equation

$$\phi(Z) = Z^p + P_1 Z^{p-1} + P_2 Z^{p-2} + \ldots + P_p = 0,$$

then† $P_1, \ldots, P_p$ are uniform multiply-periodic functions of the variables $v_1, \ldots, v_p$. Consequently, all rational symmetric functions of $z_1, \ldots, z_p$ are uniform multiply periodic functions of $v_1, \ldots, v_p$.

Frequent reference has been made to the functions determined by the equation

$$w^2 - R(z) = w^2 - (z - a_0)(z - a_1) \ldots (z - a_{2p}) = 0.$$

It has been proved that an integral of the form $\int \frac{U(z)}{w}\,dz$ is an integral of the first kind, provided $U(z)$ be an integral algebraical function of degree not higher than $p - 1$, and that the otherwise arbitrary character of $U(z)$ makes it possible to secure the necessary p integrals by allowing the suitable choice of the coefficients. Weierstrass takes the equations, which lead to the inversion, in the following form‡:—

* The $2p$ potential-functions, arising from the p functions w, would otherwise not be linearly independent.

† For further considerations see Clebsch und Gordan, *Theorie der Abel'schen Functionen*, Section vi.

‡ Equivalent to that given in *Crelle*, t. lii, (1856), pp. 285 et seq.; it is slightly different from the form adopted by him in *Crelle*, t. xlvii, (1854), p. 289.

The constants a are different from one another and can have any values: and it is convenient to take

$$P(x) = (x-a_1)(x-a_3)\dots(x-a_{2p-1}),$$
$$Q(x) = (x-a_0)(x-a_2)\dots(x-a_{2p-2})(x-a_{2p}),$$

so that $P(x)\,Q(x) = R(x)$. If the coefficients a be real, it is assumed that

$$a_0 > a_1 > a_2 > \dots > a_{2p}.$$

The equations which give the new variables are

$$\left.\begin{aligned} du_1 &= \frac{P(z_1)\,dz_1}{(z_1-a_1)\sqrt{R(z_1)}} + \frac{P(z_2)\,dz_2}{(z_2-a_1)\sqrt{R(z_2)}} + \dots\dots + \frac{P(z_p)\,dz_p}{(z_p-a_1)\sqrt{R(z_p)}} \\ du_2 &= \frac{P(z_1)\,dz_1}{(z_1-a_3)\sqrt{R(z_1)}} + \frac{P(z_2)\,dz_2}{(z_2-a_3)\sqrt{R(z_2)}} + \dots\dots + \frac{P(z_p)\,dz_p}{(z_p-a_3)\sqrt{R(z_p)}} \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ du_p &= \frac{P(z_1)\,dz_1}{(z_1-a_{2p-1})\sqrt{R(z_1)}} + \frac{P(z_2)\,dz_2}{(z_2-a_{2p-1})\sqrt{R(z_2)}} + \dots\dots + \frac{P(z_p)\,dz_p}{(z_p-a_{2p-1})\sqrt{R(z_p)}} \end{aligned}\right\};$$

and when integration takes place, the arbitrary constants are defined by the equations

$$u_1,\ u_2,\dots,\ u_p \equiv 0 \text{ (with periods for moduli)},$$

when $$z_1,\ z_2,\dots,\ z_p = a_1,\ a_3,\dots,\ a_{2p-1} \text{ respectively.}$$

The p variables z are the roots of an algebraical equation of degree p, the coefficients in which are (multiply-periodic) uniform functions of the variables u. The functions, arising out of the equations in this form, are discussed* in Weierstrass's two memoirs, just quoted.

Note 1. The results thus far established in this chapter are the basis of the theory of Abelian functions. The development of that theory is beyond the range of the present treatise.

So far as concerns the general theory, recourse must be had to the fundamental memoirs of Abel, Jacobi, Hermite, Riemann and Klein, and to treatises, in addition to those by Neumann and by Clebsch and Gordan already cited, by Prym, Krazer, Königsberger and Briot.

Moreover, as our propositions have for the most part dealt with functions of only a single variable, it is important in connection with the Abelian functions to take account of Weierstrass's memoir † on functions of several variables.

Note 2. We have discussed only very limited forms of integrals on the Riemann's surface: and any professedly complete discussion would include the theorem that $\int w'dz$, where w' is a general function of position on the surface, can be expressed as the sum of some or all of the following parts:—

(i) algebraical and logarithmic functions;

(ii) Abelian transcendents of the three kinds;

(iii) derivatives of these transcendents with regard to parameters;

but such a discussion is omitted as appertaining to the investigations relative to Abelian transcendents.

For the particular case in which the integral $\int w'dz$ is an algebraical function of z, see Briot et Bouquet, *Théorie des fonctions elliptiques*, (2^me^ éd.), pp. 218—221; Stickelberger, *Crelle*, t. lxxxii, (1877), pp. 45, 46; and Humbert, *Acta Math.*, t. x, (1887), pp. 281—298, by whom further references are given.

* Some of the results are obtained, somewhat differently, in a memoir by the author, *Phil. Trans.*, (1883), pp. 323—368.

† First published in 1886; *Abhandlungen aus der Functionenlehre*, pp. 105—164.

240. There are functions belonging to class (B) in § 229, other than those already considered. In particular, there are functions with assigned infinities on the surface and with the real parts of all their moduli of periodicity for the canonical system of cross-cuts equal to zero. But it does not therefore follow that all the moduli of periodicity vanish; in order that their imaginary parts may vanish, so as to make the moduli of periodicity zero, certain conditions would require to be satisfied.

We shall limit the ensuing discussion to some sets of these functions with zero moduli, and shall assign the conditions necessary to secure that the moduli shall be zero. We shall assume that all their infinities are algebraical; the functions are then uniform everywhere on the surface, and, except at a limited number of isolated points where they have only algebraical infinities, are finite and continuous. They are, in fact, algebraical functions of z.

Two classes of these functions are evidently simpler than any others. The first class consists of those which have a limited number, say m, of isolated accidental singularities each of the first order and which are not infinite at any of the branch-points; the other class consists of those which have no infinities except at the branch-points. These two classes will be briefly discussed in order.

Let w be a uniform function having accidental singularities, each of the first order, at the points $c_1, \dots, c_m$ and no other infinities; and let the normal function of the second kind, having c_r for its sole infinity, be Z_r. Then

$$\beta_1 Z_1 + \beta_2 Z_2 + \dots + \beta_m Z_m,$$

where $\beta_1, \dots, \beta_m$ are constants at our disposal, is a function, having infinities of the same class and at the same points as w has; the function is otherwise finite everywhere on the surface and therefore, by properly choosing the constants β, we have the function

$$w - (\beta_1 Z_1 + \dots + \beta_m Z_m)$$

finite everywhere on the surface, so that it is a function of the first kind.

Now because its modulus vanishes at each of the cross-cuts a in the resolved surface, it is a constant, so that

$$w = \beta_1 Z_1 + \dots + \beta_m Z_m + \beta_0.$$

The modulus of w is to vanish at each of the cross-cuts b_r. Let $\phi_r(z) = \dfrac{dW_r}{dz}$, so that $\phi_r(z)$ is an algebraical function on the surface: then assigning the condition that the modulus of w at the cross-cut b_r shall vanish, we have

$$\beta_1 \phi_r(c_1) + \beta_2 \phi_r(c_2) + \dots + \beta_m \phi_r(c_m) = 0,$$

an equation which must hold for all the values $r = 1, \dots, p$.

When the quantities c represent quite arbitrary points, *there must be at least* $p+1$ *of them;* otherwise, as the equations are independent of one another, they can be satisfied only by zero values of the constants β, a result which renders the uniform function evanescent. If $m > p$, the equations determine p of the coefficients β linearly in terms of the remaining $m-p$: when these values are substituted, the resulting expression for w contains $m-p+1$ constants, viz., the remaining $m-p$ constants β, and the constant β_0. The coefficient of each of the $m-p$ constants β is a function of z, which has $p+1$ accidental singularities of the first order, p of which are common to all the functions, so that w then is an arbitrary linear combination of constant multiples of $m-p$ functions, each of which possesses $p+1$ accidental singularities and can be expressed in the form

$$\Delta_r(z) = \begin{vmatrix} Z_1, & Z_2, \ldots\ldots, & Z_p, & Z_{p+r} \\ \phi_1(c_1), & \phi_1(c_2), \ldots\ldots, & \phi_1(c_p), & \phi_1(c_{p+r}) \\ \phi_2(c_1), & \phi_2(c_2), \ldots\ldots, & \phi_2(c_p), & \phi_2(c_{p+r}) \\ \cdots & \cdots & \cdots & \cdots \\ \phi_p(c_1), & \phi_p(c_2), \ldots\ldots, & \phi_p(c_p), & \phi_p(c_{p+r}) \end{vmatrix}.$$

When the quantities c are not completely arbitrary, but are such that relations among them can be satisfied so as no longer to permit the preceding forms to be definite, we proceed as follows.

The most general way in which the preceding forms cease to be definite is by the dependence of some of the equations

$$\beta_1\phi_r(c_1) + \beta_2\phi_r(c_2) + \ldots + \beta_m\phi_r(c_m) = 0$$

on the remainder. Let q of them, say those given by $r = 1, \ldots, q$, be dependent on the remaining $p-q$, so that $q > 0 < p$: then the conditions of dependence can be expressed by equations of the form

$$\phi_r(c_n) = A_{1,r}\phi_{q+1}(c_n) + A_{2,r}\phi_{q+2}(c_n) + \ldots + A_{p-q,r}\phi_p(c_n)$$

for $r = 1, 2, \ldots, q$ and $n = 1, 2, \ldots, m$.

The functions of the first kind W, through which the functions ϕ are derived, are a complete set of normal functions: when any number of them is replaced by the same number of independent linear combinations of some or all, the first derivatives are still algebraical functions. We therefore replace the functions $W_1, W_2, \ldots, W_q$ by $w_1, w_2, \ldots, w_q$, where

$$w_r = W_r - A_{1,r}W_{q+1} - A_{2,r}W_{q+2} - \ldots - A_{p-q,r}W_p$$

for $r = 1, 2, \ldots, q$, so that, for all values of z,

$$\Phi_r(z) = \phi_r(z) - A_{1,r}\phi_{q+1}(z) - A_{2,r}\phi_{q+2}(z) - \ldots - A_{p-q,r}\phi_p(z).$$

Hence the functions $\Phi_1, \Phi_2, \ldots, \Phi_q$ vanish at each of the points $c_1, c_2, \ldots, c_m$.

The original system of p equations in $\phi_1, \ldots, \phi_q, \phi_{q+1}, \ldots, \phi_p$, when made a system of equations in $\Phi_1, \ldots, \Phi_q, \phi_{q+1}, \ldots, \phi_p$ is equivalent to

$$\left.\begin{aligned} \beta_1\Phi_r(c_1) + \beta_2\Phi_r(c_2) + \ldots + \beta_m\Phi_r(c_m) = 0 \\ \beta_1\phi_s(c_1) + \beta_2\phi_s(c_2) + \ldots + \beta_m\phi_s(c_m) = 0 \end{aligned}\right\}$$

for $r = 1, \ldots, q$ and $s = q+1, \ldots, p$. The first q of these are evanescent; and therefore their form is the same as if we had initially assumed that each of the functions $\phi_1, \ldots, \phi_q$ vanished for each of the points $z = c_1, \ldots, c_m$, the two assumptions being in essence equivalent to one another on account of the property of linear combination characteristic of functions of the first kind.

Suppose, then, that q of the functions ϕ, derived through functions of the first kind, vanish at each of the points $c_1, \ldots, c_m$; the number of surviving equations of the form

$$\beta_1\phi_r(c_1) + \beta_2\phi_r(c_2) + \ldots + \beta_m\phi_r(c_m) = 0$$

is $p-q$, and they involve m arbitrary constants β. Hence they determine $p-q$ of these constants, linearly and homogeneously, in terms of the other $m-p+q$. When account is taken of the additive constant β_0, then* *the function w contains $m-p+q+1$ arbitrary constants; and it is a linear combination of arbitrary multiples of $m-p+q$ functions, each having $p-q+1$ accidental singularities of the first order, $p-q$ of which are common to all the functions in the combination.*

The functions under consideration, being linear combinations of normal functions Z of the second kind, have no infinities except at the accidental singularities; the branch-points of the surface are not infinities. And it appears, from the theorem just proved, that there are functions having only $p-q+1$ accidental singularities, each of the first order, so that the total number is less than $p+1$. A question therefore arises as to what is the inferior limit to the number of accidental singularities that can be possessed by a function which is uniform on the Riemann's surface and, except at these accidental singularities, is everywhere finite and continuous on the surface.

Let it be denoted by μ; then the p equations

$$\beta_1\phi_r(c_1) + \ldots + \beta_\mu\phi_r(c_\mu) = 0,$$

for $r = 1, 2, \ldots, p$, must determine $\mu - 1$ of the constants β in terms of the remaining constant β, say, B; and the function thence inferred contains two constants, viz., the surviving constant β and the additive constant, its form being

$$A + B \begin{vmatrix} Z_1, & Z_2, \ldots\ldots, & Z_\mu \\ \phi_1(c_1), & \phi_1(c_2), \ldots\ldots, & \phi_1(c_\mu) \\ \phi_2(c_1), & \phi_2(c_2), \ldots\ldots, & \phi_2(c_\mu) \\ \ldots\ldots & \vdots & \ldots\ldots \\ \phi_{\mu-1}(c_1), & \phi_{\mu-1}(c_2), \ldots\ldots, & \phi_{\mu-1}(c_\mu) \end{vmatrix}.$$

Among the points $c_1, c_2, \ldots, c_\mu$, the relations

$$\begin{vmatrix} \phi_{\mu+r}(c_1), & \phi_{\mu+r}(c_2), \ldots\ldots, & \phi_{\mu+r}(c_\mu) \\ \phi_1(c_1), & \phi_1(c_2), \ldots\ldots, & \phi_1(c_\mu) \\ \ldots\ldots & \vdots & \ldots\ldots \\ \phi_{\mu-1}(c_1), & \phi_{\mu-1}(c_2), \ldots\ldots, & \phi_{\mu-1}(c_\mu) \end{vmatrix} = 0$$

* This is usually known as Riemann-Roch's Theorem. It is due partly to Riemann and partly to Roch; see references in § 242.

for $r = 0, 1, \ldots, p - \mu$, must be satisfied, that is, $p - \mu + 1$ relations must be satisfied*.

Since there are μ points c among which $p - \mu + 1$ relations are satisfied it follows that the number of surviving arbitrary constants c is, in general, equal to $\mu - (p - \mu + 1)$, that is, to $2\mu - p - 1$. These occur as arbitrary constants in the inferred function, independently of the two constants A and B: so that the number of arbitrary constants, in the function with μ accidental singularities, is $2\mu - p - 1 + 2$, that is, $2\mu - p + 1$.

Again, the number of infinities of a uniform function of position on a Riemann's surface is equal to the number of its zeros (§ 194), and also to the number of points where it assumes an assigned value; and all these properties are possessed by any function, with which w is connected by any lineo-linear relation. If u be one such function, then another is

$$w = \frac{au + b}{u - d},$$

where a, b, d are arbitrary constants; and therefore w contains at least three arbitrary constants, when it is taken in the most general form that possesses the assigned properties.

But it has been shewn that the number of independent arbitrary constants in the general form of w is $2\mu - p + 1$. This number has just been proved to be at least three, and therefore

$$2\mu - p + 1 \geqslant 3,$$

or

$$\mu \geqslant 1 + \tfrac{1}{2}p.$$

Thus *the integer equal to, or next greater than,* $1 + \frac{1}{2}p$ *is the smallest number of isolated accidental singularities that an algebraical function can have on a Riemann's surface, on the supposition that it has no infinities at the branch-points*†.

241. The other simple class of uniform functions on a Riemann's surface consists of those which have no infinities except at the branch-points of the surface.

They will not be considered in any detail: we shall only briefly advert to those which consist of *the first derivatives of functions of the first kind.* This set is characterised by the theorem:—

These functions $\phi(z)$ *are infinite only at branch-points of the surface, and*

* This result implies that the relations are independent of one another, which is the case in general: but it is conceivable that special relations might exist among the branch-points, which would affect all these numbers.

† This result applies only to a completely general surface of class p. And, for special forms of surface of class p, a lower limit for μ can be obtained; thus, in the case of a two-sheeted surface, the limit is 2. (See Klein-Fricke, i, p. 556.)

the total number of infinities is $2p-2+2n$. For, let $w(z)$ be the most general integral of the first kind, and let

$$\frac{dw(z)}{dz}=\phi(z).$$

Near an ordinary point a on the surface we have

$$w(z)=w(a)+(z-a)\,P(z-a),$$

where P is a converging series that may, in general, be assumed not to vanish for $z=a$; hence

$$\phi(z)=P(z-a)+(z-a)\,P'(z-a);$$

that is, $\phi(z)$ is finite at an ordinary point.

Near $z=\infty$ (supposed not to be a branch-point) we have, if κ be the value of w there,

$$w-\kappa=\frac{1}{z}P\left(\frac{1}{z}\right),$$

where $P\left(\frac{1}{z}\right)$ may, in general, be assumed not to vanish for $z=\infty$; so that

$$\phi(z)=-\frac{1}{z^2}P\left(\frac{1}{z}\right)-\frac{1}{z^3}P'\left(\frac{1}{z}\right),$$

and therefore $\phi(z)$ has a zero of the second order at $z=\infty$.

Near a branch-point γ, where m sheets of the surface are connected, we have

$$w(z)-w(\gamma)=(z-\gamma)^{\frac{1}{m}}P\{(z-\gamma)^{\frac{1}{m}}\},$$

where P may, in general, be assumed not to vanish for $z=\gamma$: hence

$$\phi(z)=(z-\gamma)^{-\frac{m-1}{m}}\left[\frac{1}{m}P\{(z-\gamma)^{\frac{1}{m}}\}^{\frac{1}{m}}+\frac{1}{m}P'\{(z-\gamma)^{\frac{1}{m}}\}\right],$$

so that $\phi(z)$ is infinite at $z=\gamma$, and the infinity is of order $m-1$.

Hence the total number of infinities is $\Sigma(m-1)$, where m is the number of sheets connected at a branch-point and the summation extends over all the r branch-points. But $2p+1=\Sigma(m-1)-2n+3$, and therefore the number of infinities is $2p-2+2n$.

We can now prove that *the number of zeros of $\phi(z)$ in the finite part of the surface is $2p-2$, of which $p-1$ can be arbitrarily assigned.*

The total number of zeros is $2p-2+2n$, being equal to the number of infinities because $\phi(z)$ is an algebraical function. But $\phi(z)$ has been proved to have a zero of the second order when $z=\infty$ and this occurs in each of the n sheets, so that $2n$ (and no more) of the infinities of $\phi(z)$ are given by $z=\infty$. There thus remain $2p-2$ zeros, distributed in the finite part of the surface.

Moreover, the most general function $\phi(z)$ of the present kind is of the form

$$\phi(z)=C_1\phi_1(z)+C_2\phi_2(z)+\ldots+C_p\phi_p(z),$$

where $\phi_1(z), \ldots, \phi_p(z)$ are derived through the normal functions of the first kind. The $p-1$ ratios of the constants C can be chosen so as to make $\phi(z)$ vanish for $p-1$ arbitrarily assigned points. Hence, except as to a constant factor, *an algebraical function arising as the derivative of an integral of the first kind is determined, save as to a constant factor, by the assignment of* $p-1$ *of its zeros in the finite part of the plane.*

*Note**. It may happen that the assumptions as to the forms of the series in the vicinity of a particular point a, of ∞, and of γ are not justified.

If $\phi(a)$ vanish, we may regard a as one of the $2p-2$ zeros.

If $z=\infty$ on one sheet be a zero of $\phi(z)$ of order higher than two, say $2+s$, we may consider that s of the $2p-2$ zeros are removed from the finite part of the surface to coincide with $z=\infty$.

If $P\{(z-\gamma)^{\frac{1}{m}}\}$ vanish for $z=\gamma$, the order of the infinity for $\phi(z)$ is reduced from $m-1$ to, say, $m-s-1$; we may then consider that s of the $2p-2$ zeros coincide with the branch-point.

242. The existence of functions that are uniform on the surface and, except at points where they have assigned algebraical infinities, are finite and continuous, has now been proved; we proceed, as in § 99, to shew how algebraical functions imply the existence of a fundamental equation, now to be associated with the given surface.

The assigned algebraical infinities may be either at the branch-points, or at ordinary points which are singularities only of the branch associated with the sheet in which the ordinary points lie, or both at branch-points and at ordinary points.

Let the surface have n sheets; on the surface let the points $c_1, c_2, \ldots, c_m$ be ordinary infinities of orders $q_1, q_2, \ldots, q_m$ respectively—we shall restrict ourselves to the more special case in which $q_1, q_2, \ldots, q_m$ are finite integers, thus excluding (merely for the present purpose) the case of isolated essential singularities; and let the branch-points $a_1, a_2, \ldots$ be of orders $p_1, p_2, \ldots$ as infinities† and of orders $r_1-1, r_2-1, \ldots$ as winding-points.

Let $w_1, w_2, \ldots, w_n$ be the n values of the function for one and the same algebraical value of z; and consider the function $(w-w_1)(w-w_2)\ldots(w-w_n)$. The coefficients of w are symmetrical functions of the values $w_1, \ldots, w_n$ of the assigned function.

An ordinary point for all the branches w is an ordinary point for each of the coefficients.

* See Klein-Fricke, vol. i, p. 545.

† A branch-point a is said to be an infinity of order p and a winding-point of order $r-1$, when the affected branches in its vicinity can be expressed in the form $(z-a)^{-\frac{p}{r}} P\{(z-a)^{\frac{1}{r}}\}$, where P is finite when $z=a$.

An ordinary singularity of order q for any branch, which can occur only for one branch, is an ordinary singularity of the same order for each of the symmetric functions; and therefore, merely on the score of all the ordinary singularities, each of these symmetric functions can be expressed as a meromorphic function the denominator of which is the same rational integral algebraical function of degree $\sum_{s=1}^{m} q_s$ in z.

In the vicinity of the branch-point a_1, there are r_1 branches obtained from

$$(z-a_1)^{-\frac{p_1}{r_1}} P\{(z-a_1)^{\frac{1}{r_1}}\},$$

(where P is finite when $z=a_1$), by assigning to $(z-a_1)^{\frac{1}{r_1}}$ its r_1 various values. Then, as in § 99, the point a_1 is no longer a branch-point of any of the symmetric functions; and for some of the symmetric functions the point a_1 is an accidental singularity of order p_1, but for no one of them is it a singularity of higher order. Hence, merely on the score of the infinities at branch-points, each of the symmetric functions can be expressed as a meromorphic function the denominator of which is the same rational algebraical meromorphic function of degree Σp_1 in z.

No other points on the surface need be taken into account. If, then, $P(z)$ be the denominator of the coefficients arising through the isolated algebraical singularities, so that $P(z)$ is of degree $\sum_{s=1}^{m} q_s$ in z, and if $Q(z)$ be the denominator of the coefficients arising through the infinities at the branch-points, then

$$P(z)\, Q(z)\, (w-w_1)(w-w_2) \ldots (w-w_n)$$

is a rational integral uniform algebraical function of w and z: say $f(w, z)$, which is evidently of degree n in w and of degree $\sum_{s=1}^{m} q_s + \Sigma p$ in z.

Its only roots are $w=w_1, \ldots, w_n$; that is, the function w on the Riemann's surface is determined as the root of the equation $f(w, z)=0$; and therefore the equation $f(w, z)=0$ is a fundamental equation, to be associated with the surface.

Ex. 1. Shew that a fundamental equation for a three-sheeted surface, having $e^{\frac{1}{3}m\pi i}$ (for $m=0, 1, \ldots, 5$) for branch-points each of the first order, is

$$w^3 - 3wz^2 + 2 = 0;$$

and that a fundamental equation for a four-sheeted surface having the same branch-points each of the same order is

$$w^4 - (6+3\sqrt[3]{2}z^2)\, w^2 - 4\sqrt[3]{12}\,\sqrt[6]{2}\, wz = 3 + \sqrt[3]{2}z^2 - \tfrac{1}{4}9\sqrt[3]{4}z^4. \qquad \text{(Thomæ.)}$$

Every algebraical function on the surface requires its own fundamental equation; but, as the branch-points are the same for any surface, no fundamental equation can be regarded as unique. Having now obtained one fundamental equation for algebraical functions on the surface, all the investigations in chap. XVI. may be applied.

The preceding sketch, in §§ 240—242, of algebraical functions is intended only as an introduction; the developments are closely connected with the theory of Abelian functions and of curves. The propositions actually given are based upon

Riemann, *Theorie der Abel'schen Function*, *Ges. Werke*, pp. 100—102;

Roch, *Crelle*, t. lxiv, (1865), pp. 372—376;

Klein's *Vorlesungen über die Theorie der elliptischen Modulfunctionen*, (Fricke), vol. i, pp. 540—549;

for further information reference should be made to the following sources:—

Brill und Noether, *Math. Ann.*, t. vii, (1874), pp. 269—310;

Lindemann, *Untersuchungen über den Riemann-Roch'schen Satz*, (Leipzig, Teubner, 1879), 40 pp.;

Brill, *Math. Ann.*, t. xxxi, (1888), pp. 374—409; ib., t. xxxvi, (1890), pp. 321—360.

Ex. 2. Prove that the algebraical equation which subsists (§ 118) between two functions u and v of a variable z, doubly-periodic in the same periods, is of class either zero or unity; that it is of class unity, if only one incongruent value of z correspond to given values of u and v; and that it is of class zero, if more than one incongruent value of z correspond to given values of u and v. (Humbert, Günther.)

Ex. 3. If between two uniform analytical functions P and Q, which have an isolated point for their essential singularity, there exist an algebraical relation, then, when either is regarded as the independent variable, the connectivity of the Riemann's surface for the representation of the other is not greater than three. (Picard.)

243. We now pass to the consideration of another class of functions associated with a Riemann's surface.

The classes of pseudo-periodic functions, which have been discussed, originally occurred in connection with the functions that are doubly-periodic functions of the first kind; and it may, therefore, be expected that, in a discussion of functions which are multiply-periodic, similar pseudo-periodic functions will occur.

These functions, in particular such as are the generalisation of doubly-periodic functions of the second kind, have been considered in great detail by Appell*; they may be called *factorial functions*†.

But the essential difference between the former classes of functions and the present class is that now the argument of the function is a variable of position on the Riemann's surface and not, as before, an integral of the first kind. It is only in subsequent developments of the theory of these functions that the corresponding modification of argument takes place; and a factorial function then becomes a pseudo-periodic function of those integrals of the first kind.

* "Sur les intégrales des fonctions à multiplicateurs..." (Mém. Cour.), *Acta Math.*, t. xiii, (1890), 174 pp. This volume is prefaced by an interesting report, due to Hermite, on Appell's memoir.

They are also discussed in Neumann's *Abel'schen Functionen*, pp. 273—278; in Briot's *Théorie des fonctions Abéliennes*; in a memoir by Appell, *Liouville*, 3me Sér., t. ix, pp. 5—24; and they occur in a memoir by Prym, *Crelle*, t. lxx, (1869), pp. 354—362.

† *Fonctions à multiplicateurs*, by Appell.

We consider a Riemann's surface of connectivity $2p+1$, reduced to simple connectivity by $2p$ cross-cuts taken, as in § 181, to be a_1, b_1, c_2+a_2, b_2, ..., c_p+a_p, b_p. The functions already considered are such that their values at points on opposite edges of a cross-cut differ by additive constants, which are integral linear combinations of the cross-cut constants, necessarily zero for the portions c in the case of all the functions; the values of the constants for the cuts a and the cuts b depend upon the character of the functions and are simultaneously zero only when the function is a uniform function of position on the Riemann's surface, that is, is a rational function of w and z when the surface is associated with the fundamental equation

$$F(w, z) = 0.$$

A factorial function is defined as a uniform function of position on the resolved Riemann's surface, finite at the branch-points no one of which is at infinity; all its infinities are accidental singularities, so that it has no logarithmic infinities: and at two (practically coincident) points on opposite edges of a cross-cut the quotient of its values is independent of the point, being a factor (or multiplier) that is the same along the cut for all parts which can be reached without crossing another cut.

Then *for any portion c the factor is unity, for any cut a it is the same along its whole length, and for any cut b it is the same along its whole length.*

In order to consider the effect of passage over another cross-cut on the constant factor, we take the figures of §§ 196, 230. Where a_r and b_r intersect, we have

$$F(z_1) = m_r F(z_2), \quad F(z_4) = m_r' F(z_3);$$
$$F(z_4) = n_r' F(z_1), \quad F(z_3) = n_r F(z_2);$$

where m_r, m_r'; n_r, n_r' are the constants for the portions of the cuts a_r and b_r. From these equations it follows that

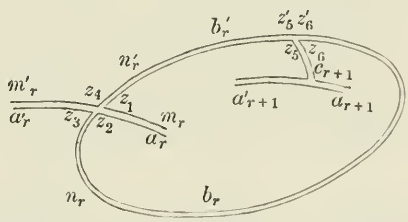

Fig. 85.

$$F(z_4) = n_r m_r' F(z_2),$$

and also $$= n_r' m_r F(z_2),$$

so that $$n_r m_r' = n_r' m_r.$$

Again, where c_{r+1} cuts b_r, we have

$$F(z_5') = n_r' F(z_5), \quad F(z_6') = n_r F(z_6),$$

so that, as $F(z_5') = F(z_6')$ when the points are infinitely close together, we have

$$F(z_6) = \frac{n_r'}{n_r} F(z_5),$$

or the multiplier l_r for c_{r+1} is $$l_{r+1} = \frac{n_r'}{n_r},$$

whence $$\frac{m_r'}{m_r} = \frac{n_r'}{n_r} = l_{r+1}.$$

Now a_1 is met only by b_1 and by no cut c: so that $m_1 = m_1'$. Hence $n_1 = n_1'$, and therefore $l_2 = 1$. Hence $m_2 = m_2'$; $n_2 = n_2'$ and therefore $l_3 = 1$; and so on, so that

$$l_{r+1} = 1, \quad m_r' = m_r, \quad n_r' = n_r,$$

the results necessary to establish the proposition.

We shall therefore take the factor along a_r to be m_r, and the factor along b_r to be n_r, for $r = 1, \ldots, p$: and, by reference to § 196, the function at the positive edge is equal to the function at the negative edge multiplied by the factor of the cut.

244. Before passing on to obtain expressions for factorial functions in terms of functions already known, we may shew that all factorial functions with assigned factors are of the form

$$\Phi(z) R(w, z),$$

where $\Phi(z)$ is a factorial function with the assigned factors and $R(w, z)$ is a function of w and z, uniform on the Riemann's surface. For if $\Psi(z)$ and $\Phi(z)$ be factorial functions with the same factors, then $\Psi(z) \div \Phi(z)$ has its factors unity at all the cross-cuts, so that it is a uniform function of position on the surface and is therefore* of the form $R(w, z)$. It is therefore sufficient at present to obtain some one factorial function with assigned factors

$$m_1, \ldots, m_p, n_1, \ldots, n_p.$$

Let $w_1(z), w_2(z), \ldots, w_p(z)$ be the p normal functions of the first kind connected with a Riemann's surface, with their periods as given in § 235.

Let $\pi_1(z)$, instead of ϖ_{12} of § 237, denote an elementary normal function of the third kind, having logarithmic infinities at α_1 and β_1 such that, in the vicinities of these points, the respective expressions for $\pi_1(z)$ are

$$-\log(z - \alpha_1) + P(z - \alpha_1),$$

and

$$+\log(z - \beta_1) + Q(z - \beta_1);$$

then the period of $\pi_1(z)$ for the cross-cut a_r is zero, and the period for the cross-cut b_r is

$$2\{w_r(\beta_1) - w_r(\alpha_1)\}$$

for $r = 1, 2, \ldots, p$. It therefore follows that $\Phi_1(z)$, where

$$\Phi_1(z) = e^{\pi_1(z)},$$

is uniform on the resolved Riemann's surface: it has a single zero (of the first order) at β_1 and a single accidental singularity (of the first order) at α_1; its factor for the cross-cut a_r is unity and its factor for the cross-cut b_r is

$$e^{2\{w_r(\beta_1) - w_r(\alpha_1)\}}.$$

* It may be pointed out that this result is an illustration of the remark, at the beginning of § 243, that the factorial functions have a uniform function of position on the surface for their argument and not the integrals of the first kind, of which that variable of position is a multiply-periodic function.

The function $\Phi_1(z)$ may therefore be regarded as an element for the representation of a factorial function.

Let $\Phi(z)$ be a factorial function on the Riemann's surface with given multipliers m and n; and let it have a number q of zeros $\beta_1, \beta_2, \dots, \beta_q$, each of the first order, and the same number q of simple accidental singularities $\alpha_1, \alpha_2, \dots, \alpha_q$, each of the first order, and no others. Then $\Phi'(z)/\Phi(z)$ has $2q$ accidental singularities; in the vicinity of the q points β, it is of the form

$$\frac{1}{z-\beta} + Q(z-\beta),$$

and in the vicinity of the q points α it is of the form

$$-\frac{1}{z-\alpha} + P(z-\alpha);$$

hence

$$\frac{\Phi'(z)}{\Phi(z)} - \sum_{s=1}^{q} \pi_s'(z)$$

is finite in the vicinity of all the singularities of $\dfrac{\Phi'(z)}{\Phi(z)}$. Thus

$$\log \Phi(z) - \sum_{s=1}^{q} \pi_s(z)$$

has no logarithmic infinities on the surface: neither $\log \Phi(z)$ nor any one of the functions $\pi(z)$ has infinities of any other kind; and therefore the foregoing function is finite everywhere on the surface. It is thus an integral of the first kind and is expressible in the form

$$2\lambda_1 w_1(z) + 2\lambda_2 w_2(z) + \dots + 2\lambda_p w_p(z) + \text{constant}.$$

Hence

$$\Phi(z) = A e^{\sum\limits_{s=1}^{q} \pi_s(z) + 2\sum\limits_{k=1}^{p} \lambda_k w_k(z)},$$

where A is a constant.

The function represented by the right-hand side evidently has the q points β as simple zeros and the q points α as simple accidental infinities, and no others. Higher order of a zero or an infinity is permitted by repetitions in the respective assigned series.

In order that it may acquire the factor m_r on passing from the negative edge to the positive edge of the cross-cut a_r, we have

$$m_r = e^{2\lambda_r \pi i};$$

and that it may acquire the factor n_r in passing from the negative edge to the positive edge of the cross-cut b_r, we have

$$n_r = e^{2\sum\limits_{s=1}^{q} \{w_r(\beta_s) - w_r(\alpha_s)\} + 2\sum\limits_{k=1}^{p} \lambda_k B_{kr}}.$$

The former equations determine the constants λ_r in the form

$$\lambda_r = \frac{1}{2\pi i} \log m_r$$

for $r = 1, 2, \ldots, p$; and then the latter equations give

$$\sum_{s=1}^{q} \{w_r(\beta_s) - w_r(\alpha_s)\} = \tfrac{1}{2}\log n_r - \frac{1}{2\pi i}\sum_{k=1}^{p}(B_{kr}\log m_k),$$

for $r = 1, 2, \ldots, p$.

Apparently, λ_r is determinate save as to an additive integer, say M_r; and the value of $\frac{1}{2}\log n_r$ is determinate save as to an additive quantity, say $N_r\pi i$, where N_r is an integer. The left-hand side of the derived set of equations being definite, these integers N_r and M_r must be subject to the equations

$$\pi i N_r = \sum_{k=1}^{p} M_k B_{kr}$$

for $r = 1, 2, \ldots, p$; and therefore, equating the real parts (§ 235), we have

$$\sum_{k=1}^{p} M_k \rho_{kr} = 0,$$

so that
$$\sum_{k=1}^{p}\sum_{r=1}^{p} M_k M_r \rho_{kr} = 0,$$

which, by § 235, can be satisfied only if all the integers M_r vanish and therefore also the integers N_r.

Hence when the foregoing equations connecting the quantities α, β, $\log n$, $\log m$ are satisfied, as they must be, for one set of values of $\log n$ and $\log m$, that set may be taken as the definite set of values; and the only way in which variation can enter is through the multiplicity in value of the functions $w_1, \ldots, w_p$, which may be supposed definitely assigned.

The expression for the function $\Phi(z)$ *is therefore*

$$Ae^{\sum\limits_{s=1}^{q}\pi_s(z) + \frac{1}{\pi i}\sum\limits_{k=1}^{p}\{w_k(z)\log m_k\}};$$

the q *zeros* β *and the* q *simple poles* α *being subject to the equations*

$$\sum_{s=1}^{q} \{w_r(\beta_s) - w_r(\alpha_s)\} = \tfrac{1}{2}\log n_r - \frac{1}{2\pi i}\sum_{k=1}^{p}(B_{kr}\log m_k).$$

Corollary I. The function $\Phi(z)$ is a rational function of position on the surface, that is, of w and z, if all the factors n and m be unity. Such a function has been proved (§ 194) to have as many infinities as zeros; and therefore *integers* $N_1', \ldots, N_p', M_1', \ldots, M_p'$ *exist such that, between the zeros and the infinities of a rational algebraical function of* w *and* z, *the* p *equations*

$$\sum_{s=1} \{w_r(\beta_s) - w_r(\alpha_s)\} = \pi i N_r' - \sum_{k=1}^{p} M_k' B_{kr},$$

for $r = 1, 2, \ldots, p$, *subsist**.

The function $\Phi(z)$ then corresponds to a rational algebraical function, when regarded as a product of simple factors, in the same way as the expression (§ 241) in terms of normal elementary functions of the second kind corresponds to the function, when regarded as a sum of simple fractions.

* Neumann, p. 275.

COROLLARY II. *Every factorial function has as many zeros as it has infinities.*

For if a special function $\Phi(z)$, with the given factors and possessing q zeros and q infinities, be formed, every other function with those factors is included in the form

$$F(z) = \Phi(z) R(w, z),$$

where $R(w, z)$ is a rational algebraical function of w and z. But $R(w, z)$ has as many zeros as it has infinities; and therefore the property holds of $F(z)$.

Further, it is easy to see that the equations of relation between the zeros, the infinities and the multipliers are satisfied for $F(z)$. For among the zeros and the infinities of $\Phi(z)$, the relations

$$\sum_{s=1}^{q} \{w_r(\beta_s) - w_r(\alpha_s)\} = \tfrac{1}{2}\log n_r - \frac{1}{2\pi i}\sum_{k=1}^{p} (B_{kr}\log m_k)$$

are satisfied; and among the zeros and the infinities of $R(w, z)$ the relations

$$\sum_{s=1} w_r(\beta_s') - w_r(\alpha_s') = \pi i N_r' - \sum_{k=1}^{p} (B_{kr} M_k')$$

are satisfied, where N_r' and the coefficients M' are integers. Hence, among the zeros and the infinities of $F(z)$, the relations

$$\Sigma\{w_r(\text{zero}) - w_r(\infty)\} = \tfrac{1}{2}(\log n_r + N_r' \, 2\pi i) - \frac{1}{2\pi i}\sum_{k=1}^{p}\{B_{kr}(\log m_k + 2M_k'\pi i)\}$$

are satisfied, giving the same multipliers n_r and m_r as for the special function $\Phi(z)$.

COROLLARY III. *It is possible to have factorial functions without zeros and therefore without infinities: but the multipliers cannot be arbitrarily assigned.*

Such a function is evidently given by

$$e^{2\Sigma\lambda_k w_k(z)}$$

derived from $\Phi(z)$ by dropping from the exponential the terms dependent upon the functions $\pi(z)$. The relations between the factors are easily obtained.

245. The effect of the p relations

$$\sum_{s=1}^{q} \{w_r(\beta_s) - w_r(\alpha_s)\} = \tfrac{1}{2}\log n_r - \frac{1}{2\pi i}\sum_{k=1}^{p} (B_{kr}\log m_k)$$

subsisting between the factors, the zeros and the infinities of the factorial function, varies according to the magnitude of q.

If q be equal to or be greater than p, it is evident that all the infinities α and $q-p$ of the zeros β can be assumed at will and that the above relations determine the p remaining zeros. The function therefore involves $2q-p$ arbitrary elements, in addition to the unessential constant A.

In particular, when q is equal to p, the infinities α can be chosen at will and the zeros β are then determined by the relations. It therefore appears that *a factorial function, which has only p infinities, is determined by its infinities and its cross-cut factors.*

When q is greater than p, say $= p + r$, then the q infinities and r zeros may be chosen at will. By assigning various sets of r zeros with a given set of infinities, various functions $\Phi_1(z)$, $\Phi_2(z)$, ... will be obtained all having the same infinities and the same cross-cut factors. Let s such functions have been obtained; consider the function

$$\Phi(z) = \mu_1\Phi_1(z) + \mu_2\Phi_2(z) + \ldots + \mu_s\Phi_s(z):$$

it will evidently have the assigned infinities and the assigned cross-cut factors. Then $s - 1$ ratios of the quantities μ can be chosen so as to cause $\Phi(z)$ to acquire $s - 1$ arbitrary zeros. The greatest number of arbitrary zeros that can be assigned to a function is r, which is therefore the greatest value of $s - 1$. Hence it follows that $r + 1$ *linearly independent factorial functions $\Phi_1(z), \ldots, \Phi_{r+1}(z)$ exist having assigned cross-cut factors and $p + r$ assigned infinities; and every other factorial function with those infinities and cross-cut factors can be expressed in the form*

$$\mu_1\Phi_1(z) + \mu_2\Phi_2(z) + \ldots + \mu_{r+1}\Phi_{r+1}(z),$$

where $\mu_1, \ldots, \mu_{r+1}$ are constants whose ratios can be used to assign r arbitrary zeros to the function.

These factorial functions are used by Appell to construct new classes of functions in a manner similar to that in which Riemann constructs the Abelian transcendents. Their properties are developed on the basis of algebraical functions; but as only the introduction to the theory can be given here, recourse must be had to Appell's interesting memoir, already cited.

246. Various examples of functions defined by differential equations of the first order have occurred, all the equations being of the form

$$F\left(w, \frac{dw}{dz}\right) = 0,$$

where F is a rational, integral, algebraical function of w and $\frac{dw}{dz}$. This is a special form of the more general equation

$$F\left(z, w, \frac{dw}{dz}\right) = 0$$

of the first order: the theorem, that such an equation determines a function, and the discussion of the characteristics of the function so determined, belong to the theory of differential equations. In this place we shall consider* the special form of differential equation, not in its generality but only in the limited instances in which *the function, determined by it, is a uniform function of z.*

* The following investigation has been placed here and not earlier, in order to avoid interrupting the development of the preceding theory.

Let the equation be of the mth degree in $\frac{dw}{dz}$, supposed irreducible; when arranged in powers of the derivative, it takes the form

$$\left(\frac{dw}{dz}\right)^m + \left(\frac{dw}{dz}\right)^{m-1} f_1(w) + \left(\frac{dw}{dz}\right)^{m-2} f_2(w) + \ldots = 0.$$

Because w is a uniform function of z, it has, quà function of z, no branch-points; and $\frac{dw}{dz}$ has, quà function of z, no branch-points. Hence infinities of w are infinities of $\frac{dw}{dz}$ and vice versa; and therefore $\frac{dw}{dz}$ cannot become infinite for a finite value of w. It follows that the coefficients $f_1(w), f_2(w), \ldots$ of the various powers of the derivative are integral functions of w; they are known, by the character of the equation, to be rational and algebraical.

Moreover all the general properties possessed by w are possessed by its reciprocal $u = \frac{1}{w}$. When u is made the dependent variable, we have

$$\left(\frac{du}{dz}\right)^m - \left(\frac{du}{dz}\right)^{m-1} u^2 f_1\left(\frac{1}{u}\right) + \left(\frac{du}{dz}\right)^{m-2} u^4 f_2\left(\frac{1}{u}\right) - \ldots = 0$$

as the equation determining u. Now $\frac{du}{dz}$ cannot become infinite except for infinite values of u, for u is a uniform function of z; hence the coefficients of powers of $\frac{du}{dz}$ must be rational integral algebraical functions of u. This condition can be satisfied only if $f_s(w)$ be of degree in w not higher than $2s$.

Hence, denoting $\frac{dw}{dz}$ by W and $\frac{du}{dz}$ by U, we have the theorem :—

I. *The differential equation*

$$F(W, w) = W^m + W^{m-1} f_1(w) + W^{m-2} f_2(w) + \ldots = 0$$

cannot determine w as a uniform function of z, unless the coefficients

$$f_1(w), \quad f_2(w), \quad f_3(w), \ldots$$

are rational integral algebraical functions of w of degrees not higher than 2, 4, 6, ... *respectively: and when this condition is satisfied, it is satisfied also for the equation*

$$U^m - U^{m-1} u^2 f_1\left(\frac{1}{u}\right) + U^{m-2} u^4 f_2\left(\frac{1}{u}\right) - \ldots = 0,$$

which determines u, the reciprocal of w.

247. The equation, in the first instance, determines W as a function of w; and values of w may be ordinary points or may be branch-points for W, quà function of w. In the vicinity of such points, it is necessary to secure that w, as depending upon z, shall be uniform.

First, consider finite values for w: let $w=\gamma$. For points in the immediate vicinity of that value, the values of W are not infinite: they may be

(i) distinct from one another, and no one of them zero at the point; or

(ii) distinct from one another and at least one of them zero at the point; or

(iii) not distinct from one another, so that $w=\gamma$ is then a branch-point of the function.

(i) Let any value Γ, a constant different from zero, be the value of W for $w=\gamma$. Then in the vicinity we have

$$\frac{dw}{dz}=\Gamma\{1+\lambda(w-\gamma)+\mu(w-\gamma)^2+\ldots\},$$

and therefore
$$\Gamma dz=\frac{dw}{1+\lambda(w-\gamma)+\mu(w-\gamma)^2+\ldots}$$
$$=\{1+2\lambda'(w-\gamma)+3\mu'(w-\gamma)^2+\ldots\}\,dw,$$

where λ', μ', ... are constants. Hence if z_0 be the value of z when $w=\gamma$, we have

$$\Gamma(z-z_0)=w-\gamma+\lambda'(w-\gamma)^2+\mu'(w-\gamma)^3+\ldots,$$

and the inversion of this equation gives

$$w-\gamma=\Gamma(z-z_0)+P(z-z_0),$$

that is, w is then a uniform function of z in the vicinity of z_0. No new condition, attaching to the original equation, arises.

(ii) Since the values are distinct from one another, and at least one of them is zero for $w=\gamma$, we must have

$$\frac{dw}{dz}=a(w-\gamma)^n\{1+b(w-\gamma)+c(w-\gamma)^2+\ldots\}$$

for at least one of the values of W, n being an integer. Now as γ is not a branch-point, it follows from § 97 that n is equal either to 1 or to 2.

First, if n be unity, we have

$$\frac{dw}{w-\gamma}\{1+b'(w-\gamma)+c'(w-\gamma)^2+\ldots\}=a\,dz,$$

so that
$$\log(w-\gamma)+P(w-\gamma)=az,$$

the constant of integration being absorbed in $P(w-\gamma)$. Thus

$$(w-\gamma)\,e^{P(w-\gamma)}=e^{az},$$

and therefore, inverting the functional relation,

$$w-\gamma=e^{az}Q(e^{az}),$$

that is, w is a uniform function in the vicinity of its own value γ, but it can acquire this value only for logarithmically infinite values of z. No new condition, attaching to the original equation, arises.

Secondly, if n be 2, so that

$$\frac{dw}{dz} = a(w-\gamma)^2 \{1 + b(w-\gamma) + c(w-\gamma)^2 + \ldots\},$$

then, proceeding as before, we have

$$-\frac{1}{w-\gamma} - b\log(w-\gamma) + Q(w-\gamma) = az.$$

If b be different from zero, then, as on pp. 474, 475, it can be proved that w is not uniform in the vicinity of $z = \infty$. Hence b must be zero, so that

$$w - \gamma = -\frac{1}{az} S\left(\frac{1}{az}\right),$$

giving w as a uniform function of z in the vicinity of its own value γ. In this case w can acquire the value γ only for algebraically infinite values of z. The new condition, attaching to the original equation, will be included in a subsequent case (III., § 248).

(iii) If $w = \gamma$ be a branch-point, then two cases arise according as W is not, or is, zero: it cannot be infinite, because γ is not infinite.

If W be not zero, we have the value of W in the form

$$W = a\{1 + b(w-\gamma)^{\frac{1}{p}} + c(w-\gamma)^{\frac{2}{p}} + \ldots\},$$

where p is a positive integer. The integral of this equation is of the form

$$(w-\gamma)\{1 + b'(w-\gamma)^{\frac{1}{p}} + c'(w-\gamma)^{\frac{2}{p}} + \ldots\} = a(z-\alpha),$$

and this makes w uniform in the vicinity of $z = \alpha$, only if powers of $w - \gamma$ with non-integral indices be absent from the last equation and therefore also from the former. When the fractional powers are absent from the former, the implication is that $w = \gamma$ is really not a branch-point for W, quà function of w, but only that more than one of its values are equal to a; then w is a uniform function of z, and therefore W is a uniform function of w, and vice versa.

If however W be zero at the branch-point, then its value in the vicinity takes the form

$$W = a(w-\gamma)^{\frac{q}{p}} + b(w-\gamma)^{\frac{q+1}{p}} + c(w-\gamma)^{\frac{q+2}{p}} + \ldots;$$

and, as W cannot be infinite for a finite value of w, the fraction q/p is positive. It may be less than 1, equal to 1, or greater than 1. Hence:—

II. *If any finite value γ of w be a branch-point of W regarded as a function of w, then, in order that w may be uniform, all the values of W affected by the point must be zero for $w = \gamma$.*

248. If $q/p < 1$, the integration of the equation leads to a relation of the form

$$z - \alpha = a' (w - \gamma)^{\frac{p-q}{p}} + b' (w - \gamma)^{\frac{p-q+1}{p}} + \ldots\ldots$$

in which all the indices are positive. The inversion of this relation makes w uniform in the vicinity of $z = \alpha$, only if $p - q$ be unity, that is, if *the zero of W as a function of w be of degree $1 - \frac{1}{p}$, when the degree is less than unity;* and the value of z is finite.

If $q/p = 1$, then we have

$$W = a (w - \gamma) + b (w - \gamma)^{1+\frac{1}{p}} + c (w - \gamma)^{1+\frac{2}{p}} + \ldots$$

and therefore $$a\, dz = \frac{dw}{w - \gamma} \{1 + a' (w - \gamma)^{\frac{1}{p}} + b' (w - \gamma)^{\frac{2}{p}} + \ldots\}$$

so that $$az = \log (w - \gamma) + a'' (w - \gamma)^{\frac{1}{p}} + b'' (w - \gamma)^{\frac{2}{p}} + \ldots.$$

Let $w - \gamma = v^p$, $Z = e^{\frac{az}{p}}$; then this equation becomes

$$p \log Z = p \log v + a''v + b''v^2 + \ldots,$$

that is, $$Z = ve^{\lambda v + \mu v^2 + \ldots} = vP (v);$$

whence, by inversion, we have a relation of the form

$$v = ZQ (Z),$$

so that $$w - \gamma = e^{az} Q (e^{\frac{az}{p}}),$$

shewing that w is uniform for values in the vicinity of $w = \gamma$: it is simply-periodic in that vicinity, the period being $\frac{2p\pi i}{a}$, and it can acquire the value γ only for (logarithmically) infinite values of z.

If $q/p > 1$, let $q = p + n$, where n and p are prime to one another; then we have

$$W = a (w - \gamma)^{1+\frac{n}{p}} + b (w - \gamma)^{1+\frac{n+1}{p}} + \ldots,$$

so that

$$a\, dz = \{(w - \gamma)^{-1-\frac{n}{p}} + b' (w - \gamma)^{-1-\frac{n-1}{p}} + c' (w - \gamma)^{-1-\frac{n-2}{p}} + \ldots\} dw,$$

or $$z = \alpha (w - \gamma)^{-\frac{n}{p}} + \beta (w - \gamma)^{-\frac{n-1}{p}} + \ldots$$
$$+ \delta (w - \gamma)^{-\frac{1}{p}} + \epsilon \log (w - \gamma) + P \{(w - \gamma)^{\frac{1}{p}}\}.$$

Hence w can acquire its value γ only for (algebraically) infinite values of z.

As a first condition for uniformity, the coefficient ϵ must vanish, that is, in the expansion of $\frac{dz}{dw}$ in powers of $(w - \gamma)^{\frac{1}{p}}$, there must be no term involving $(w - \gamma)^{-1}$. For let

$$z = Z^{-n}, \quad w - \gamma = v^p,$$

so that $$v^n = Z^n \{\alpha + \beta v + \ldots + \delta v^{n-1} + \epsilon v^n \log v + v^n P(v)\}.$$
Then, if $$v = uZ,$$
we have $$u^n = Q(uZ) + \epsilon u^n Z^n (\log u + \log Z),$$
where Q is a series of integral powers of uZ converging for sufficiently small values of $|uZ|$.

Since z is infinitely large for sufficiently small values of $|w-\gamma|$, we have Z infinitesimally small. When $Z=0$, the value of $Z^n \log Z$ is zero: but for values of Z that are not zero, the quantity has an infinite number of different values of the form

$$Z^n(\text{Log } Z + 2m\pi i),$$

and there will then be an infinite number of distinct equations determining u, one corresponding to each of the values of m. Hence u (and therefore v, and therefore also $w-\gamma$), in that case, has an infinite number of distinct branches in the vicinity of $Z=0$; then w is not uniform in the vicinity of $Z=0$. As a first condition for uniformity, we must therefore have $\epsilon=0$.

We take $\epsilon = 0$: then the equation between z and v, where $w-\gamma = v^p$, is

$$z = v^{-n}\{\alpha + \beta v + \gamma v^2 + \ldots\},$$

the inversion of which can give v (and therefore can give $w-\gamma$) as a uniform function of z, only if $n=1$. When $n=1$, we have $w-\gamma$ uniform; and w can obtain its value γ only for algebraically infinite values of z.

Combining these results, we have the theorem:

III. *If for a finite value γ of w, which is a branch-point of W, the equation in W has a zero for p branches, then, in order that w may be uniform, the degree of that zero is of one of the forms* $1-\dfrac{1}{p}$, 1, *and* $1+\dfrac{1}{p}$; *and if it be of the form** $1+\dfrac{1}{p}$, *the term in* $(w-\gamma)^{-1}$ *must be absent from the expression of* $\dfrac{dz}{dw}$ *in powers of* $w-\gamma$.

249. Only finite values of w have been considered. For the consideration of infinite values of w, we pass to the equation in u: and only zero values of u need be taken into account. If w be uniform, u also is uniform and vice versa; hence:—

IV. *In order that the function w may be uniform when its value tends to become infinitely large, the conditions in* II. *and* III. *must apply to the equation in u for the value $u=0$.*

The branch-points of W, regarded as a function of w, as well as points where the roots though equal are distinct as in II., are (in addition possibly to $u=0$) the common roots of the equations

$$f(W, w) = 0, \quad \frac{\partial f(W, w)}{\partial W} = 0.$$

* The case $p=1$ occurs in (ii), § 247: it will now be included in III.

If, then, the conditions in II. *and* III. *be satisfied for all these points, and if the conditions in* IV. *be satisfied for* $u=0$, *that is, for infinite values of* w, *then the integral of the equation*

$$\left(\frac{dw}{dz}\right)^m + f_1(w)\left(\frac{dw}{dz}\right)^{m-1} + \dots + f_{m-1}(w)\frac{dw}{dz} + f_m(w) = 0$$

is a uniform function of z.

250. The classes of uniform functions of z can be obtained as follows.

The function, inverse to w, is given by the equation

$$\frac{dz}{dw} = W^{-1},$$

and therefore

$$z = \int \frac{dw}{W}.$$

Let the Riemann's surface for the algebraical equation

$$f(W, w) = 0,$$

regarded as an equation between a dependent variable W and an independent variable w capable of assuming all values, be constructed; and let its connectivity be $2P+1$. Then $\int \frac{dw}{W}$ is the integral of a uniform function of position on the surface; and if w_0 be a value at any point, then all other values at that point differ from w_0 by integral multiples of

(i) the moduli of the integral at the $2P$ cross-cuts,

(ii) the moduli of the integral at such other cross-cuts as may be necessary on account of the expression of the subject of integration as a function of w.

Hence the argument of w, a uniform function of z, is of the form $z + \Sigma m\Omega$, where the coefficients m are integers and the quantities Ω are constant.

It has already been proved that uniform functions of z with more than two linearly independent periods cannot exist; hence there are at the utmost two moduli, and therefore, taking account of the results of §§ 235—242, it follows that *the uniform function of* z *is either*

(i) *a doubly-periodic function of* z; *or*

(ii) *a simply-periodic function of* z; *or*

(iii) *a rational function of* z.

Further*, *the class of the Riemann's surface for the equation* $f(W, w) = 0$ *is either unity or zero;* for in what precedes, the value of P is not greater than unity, when the limitations as to the possible number of periods are assigned.

It is now easy to assign the criteria determining the class of functions to

* This result is due to Hermite, and is stated by him in a letter to Cayley, *Lond. Math. Soc.*, t. iv, (1873), pp. 343—345. The limitation of the class to zero or unity is not, in itself, sufficient to ensure that w is a uniform function of z.

which w belongs, when it is known to be a uniform function of z satisfying the differential equation.

(i) Let w be a uniform doubly-periodic function. Take any parallelogram of periods in the finite part of the plane: all values of z within the parallelogram are finite, and all possible values of w are acquired within the parallelogram.

Let γ be a finite value of w for a point $z=c$; then, since the function is uniform, we have

$$w-\gamma=(z-c)^m P(z-c),$$

where m is an integer and $P(z-c)$ does not vanish for $z=c$: and, by inversion, we also have

$$z-c=(w-\gamma)^{\frac{1}{m}} Q\{(w-\gamma)^{\frac{1}{m}}\},$$

where Q is finite but does not vanish for $w=\gamma$.

Now
$$\frac{dw}{dz}=(z-c)^{m-1}\{mP(z-c)+(z-c)P'(z-c)\}$$
$$=(w-\gamma)^{1-\frac{1}{m}} Q_1\{(w-\gamma)^{\frac{1}{m}}\},$$

where Q_1 does not vanish for $w=\gamma$.

If $m=1$, then γ is an ordinary point for $\dfrac{dw}{dz}$.

If $m>1$, then γ is a zero branch-point for W, of index-degree equal to

$$1-\frac{1}{m}.$$

If, in the vicinity of $z=b$, w be infinitely large of order q, then $z=b$ is a zero of u of order q, so that we have

$$u=(z-b)^q P_1(z-b);$$

as in the first of these cases, it follows that

$$\frac{du}{dz}=u^{1-\frac{1}{q}} P_2(u^{\frac{1}{q}}),$$

where P_2 does not vanish for $u=0$.

Hence it follows that if, for finite or for infinite values of w, all the branch-points for W be zeros and each of them have its degree less than unity, the index of the degree being of the form $1-\dfrac{1}{p}$, where p is the number of branches of W affected, then the uniform function w is doubly-periodic.

(ii) Let w be a uniform simply-periodic function, of period ω; then it is known (§ 113) that w can be expressed in the form

$$w=f(e^{\frac{2\pi zi}{\omega}})=f(Z).$$

Take any strip in the z-plane as for a simply-periodic function, bounded by lines whose inclination to the axis of real quantity is $\frac{1}{2}\pi + \arg.\omega$, as in § 111: in this strip the function acquires all its values. The variable Z is finite in the strip except at the infinite limits; at one infinite limit we have $z = ki\omega$, where k is positive and infinitely great, and then $Z = e^{-2\pi k} = 0$, and at the other we can take $z = -ki\omega$ and then $Z = e^{2\pi k} = \infty$; so that $Z = 0$ and ∞ at the infinite limits.

Let γ be a finite value of w for a finite point $z = c$ and let $C = e^{\frac{2\pi ci}{\omega}}$: then we have

$$\begin{aligned} w - \gamma &= f(Z) - f(C) \\ &= (Z - C)^q g(Z - C), \end{aligned}$$

where $g(Z - C)$ does not vanish for $Z = C$ and q is a positive integer.

When $q = 1$, we have

$$Z - C = (w - \gamma) G(w - \gamma),$$

where G does not vanish for $w = \gamma$; and then

$$\begin{aligned} \frac{dw}{dz} &= \frac{2\pi i}{\omega} Z \{g(Z - C) + (Z - C) g'(Z - C)\} \\ &= H(w - \gamma), \end{aligned}$$

where H does not vanish for $w = \gamma$; the point $w = \gamma$ is an ordinary point for $\frac{dw}{dz}$.

When $q > 1$, we have

$$Z - C = (w - \gamma)^{\frac{1}{q}} G\{(w - \gamma)^{\frac{1}{q}}\},$$

where G does not vanish for $w = \gamma$; and then

$$\begin{aligned} \frac{dw}{dz} &= \frac{2\pi i}{\omega} Z (Z - C)^{q-1} \{qg(Z - C) + (Z - C) g'(Z - C)\} \\ &= (w - \gamma)^{1 - \frac{1}{q}} h\{(w - \gamma)^{\frac{1}{q}}\}, \end{aligned}$$

where h does not vanish for $w = \gamma$. Such a point is a branch-zero for q branches of W, and its index-degree is $1 - \frac{1}{q}$.

If the value of w be infinite for the finite point $z = c$, then we have

$$u = (Z - C)^q g(Z - C).$$

If $q = 1$, the point is an ordinary point for $\frac{du}{dz}$; if $q > 1$, it is a branch-zero for q branches of $\frac{du}{dz}$ and its index-degree is $1 - \frac{1}{q}$.

When $z = \infty$, then $Z = 0$ or $Z = \infty$. The value of the function w for infinite values of z is either finite or infinite.

Let w be a finite quantity γ, for infinitely large values of z. When Z is very small, we have

$$w - \gamma = Z^q f(Z),$$

where q is a positive integer and f does not vanish for $Z=0$; and then

$$Z=(w-\gamma)^{\frac{1}{q}} g\{(w-\gamma)^{\frac{1}{q}}\},$$

where g does not vanish for $w=\gamma$. Then

$$\frac{dw}{dz}=\frac{2\pi i}{\omega} Z Z^{q-1}\{qf(Z)+Zf'(Z)\}$$
$$=Z^q h(Z),$$

where h does not vanish when $Z=0$; and therefore

$$\frac{dw}{dz}=(w-\gamma)P_1\{(w-\gamma)^{\frac{1}{q}}\},$$

or the point $w=\gamma$ is a branch-zero of q branches of $\frac{dw}{dz}$ and its index-degree is unity. And when Z is very large, we have

$$w-\gamma=Z^{-q}f_1\left(\frac{1}{Z}\right),$$

where q is a positive integer and f_1 is finite and not zero for $Z=\infty$. As before, it is easy to see that

$$\frac{dw}{dz}=(w-\gamma)P_2\{(w-\gamma)^{\frac{1}{q}}\},$$

or the point $w=\gamma$ is a branch-zero of q branches of $\frac{dw}{dz}$ and its index-degree is unity.

If, however, the value of w be infinite for infinitely large values of z, then we have

$$u=Z^q f_1(Z)$$

when Z is very small, and $$u=Z^{-q}f_2\left(\frac{1}{Z}\right)$$

when Z is very large. As before, the point $u=0$ is then, in each case, a branch-zero of q branches $\frac{du}{dz}$, and its index-degree is unity.

Hence it follows that if all the branch-points of W be zeros, if one of them have its degree equal to unity, and if all the other branch-zeros are of index-degree less than unity, the index of the degree being of the form $1-\frac{1}{p}$, where p is the number of branches of W affected, then the uniform function w determined by the equation $f(W, w)=0$ is simply-periodic.

(iii) Let w be a rational function of z; then it can be expressed in the form

$$w=\frac{f_1(z)}{f_2(z)},$$

where f_1 and f_2 are rational, integral functions of z.

Finite values of w can arise from values of z in the vicinity of (a) a zero of $f_1(z)$, say $z=c$, or (b) an infinity of $f_2(z)$. For the former, we have, if γ denote the value of z,

$$w-\gamma=(z-c)^m F(z-c),$$

where F does not vanish for $z=c$: and then, inverting the functional relation,

$$z-c=(w-\gamma)^{\frac{1}{m}} P(w-\gamma),$$

where m is a positive integer which may be 1 or greater than 1.

Now $$\frac{dw}{dz}=(z-c)^{m-1}\{mF(z-c)+(z-c)F'(z-c)\},$$

so that, if $m=1$, we have $$\frac{dw}{dz}=Q(w-\gamma),$$

where Q does not vanish when $w=\gamma$; and, if $m>1$, we have

$$\frac{dw}{dz}=(w-\gamma)^{1-\frac{1}{m}} Q_1\{(w-\gamma)^{\frac{1}{m}}\},$$

where Q_1 does not vanish when $w=\gamma$. Hence $w=\gamma$ is either an ordinary point for W or a branch-point at which m branches vanish, the index-degree of the zero being $1-\dfrac{1}{m}$.

For an infinity of $f_2(z)$ we must have $z=\infty$; and therefore, for infinitely large values of z, we have

$$w-\gamma=z^{-\lambda} F\left(\frac{1}{z}\right),$$

where F does not vanish when $z=\infty$. Proceeding as before, we have

$$\frac{dw}{dz}=(w-\gamma)^{1+\frac{1}{\lambda}} F_1\{(w-\gamma)^{\frac{1}{\lambda}}\},$$

where F_1 does not vanish when $w=\gamma$. If $\lambda=1$, $w=\gamma$ is an ordinary point, a case which has been considered; if $\lambda>1$, $w=\gamma$ is a branch-point for W, at which λ branches vanish, and the index-degree of the zero is $1+\dfrac{1}{\lambda}$.

Infinite values of w can arise from values of z that are infinitely large—in connection with $f_1(z)$—or from values of z that are zeros of the denominator. For the former, we have

$$u=z^{-\lambda} F\left(\frac{1}{z}\right),$$

where λ is a positive integer and F does not vanish for $z=\infty$; and then proceeding as before, we have

$$\frac{du}{dz}=u^{1+\frac{1}{\lambda}} F(u^{\frac{1}{\lambda}}),$$

so that, if $\lambda = 1$, $u = 0$ is an ordinary point, a case of which account has already been taken; and if $\lambda > 1$, $u = 0$ (that is, $w = \infty$) is a branch-point for U at which λ branches vanish, and the index-degree of the zero is $1 + \frac{1}{\lambda}$.

Moreover, as w is a rational function, we do not have both $w = \gamma$ and $u = 0$ for infinite values of z, unless (possibly) $z = \infty$ is an essential singularity of the function.

It thus appears that, when w is a rational algebraical function, there is only one value of w which, being a branch-point for W, gives m branches vanishing, the index of the degree of the zero being $1 + \frac{1}{m}$; all other branch-points of W give zeros that are of degree-index less than unity, each being of the form $1 - \frac{1}{n}$, where n is the number of branches that vanish at the point.

251. The following is a summary of the results that have been obtained:—

I. In order that an irreducible differential equation of the first order may have a uniform function for its integral, it must be of the form

$$F(W, w) = \left(\frac{dw}{dz}\right)^m + \left(\frac{dw}{dz}\right)^{m-1} f_1(w) + \ldots + f_m(w) = 0,$$

where $f_1(w), f_2(w), \ldots, f_m(w)$ are rational, integral, algebraical functions of w of degrees not higher than $2, 4, 6, \ldots, 2m$ respectively: and this condition as to degree is then satisfied for the equation

$$G(U, u) = F\left(-\frac{1}{u^2} U, \frac{1}{u}\right)$$
$$= \left(\frac{du}{dz}\right)^m - \left(\frac{du}{dz}\right)^{m-1} u^2 f_1\left(\frac{1}{u}\right) + \ldots \pm u^{2m} f_m\left(\frac{1}{u}\right) = 0.$$

II. If any finite value of w be a branch-point of W when regarded as a function of w determined by the equation $F(W, w) = 0$, then all the affected values of W must be zero for that value of w; and likewise for the value $u = 0$ in connection with the equation

$$G(U, u) = 0.$$

III. If for a value of w, which is a branch-point of W when regarded as a function of w, there be a multiple root of $F(W, w) = 0$ which is zero for n branches, the index-degree for each of those branches is of one of the forms $1 - \frac{1}{n}$, 1, $1 + \frac{1}{n}$; and likewise for the value $u = 0$ in connection with the equation $G(U, u) = 0$.

IV. The class of the equation $F(W, w) = 0$, and therefore the class of the Riemann's surface associated with the equation, is either zero or unity.

V. If all the multiple zero-roots of W, for finite values or for an infinite value of w, be of index-degree less than unity, each of them being of the form $1-\frac{1}{n}$, then w is a uniform doubly-periodic function of z.

VI. If, for some value of w, there be a single set of m multiple zero-roots of index-degree equal to unity, and if, for finite values or for an infinite value of w, all the other sets of multiple zero-roots have their respective index-degrees less than unity and of the form $1-\frac{1}{n}$, then w is a uniform singly-periodic function of z.

VII. If, for some value of w, there be a single set of m multiple zero-roots the index-degree of which is equal to $1+\frac{1}{m}$, and if, for other values of w, all the other sets of multiple zero-roots have their respective index-degrees less than unity and of the forms $1-\frac{1}{n}$, then w is a rational algebraical function of z.

In all other cases the equation, supposed irreducible, cannot have a uniform function of z for its integral. If the equation have a uniform function of z for its integral, and the preceding conditions in V., VI. or VII., be not satisfied, the equation is reducible*, that is, it can be replaced by rational equations of lower degree to which the criteria apply.

Note. The preceding method may be considered as essentially due to Briot and Bouquet.

There is another method of proceeding, which leads to the same result. It is based upon Hermite's theorem (§ 250), proved independently; and its development will be found in memoirs by Fuchs† and Raffy‡. A reference to the memoirs which have been quoted shews that the equation $F(W, w)=0$, when it is satisfied by a uniform function of z, can be associated with the theory of unicursal curves and of bicursal curves.

252. The preceding general results will now be applied to the particular equation

$$\left(\frac{dw}{dz}\right)^s = f(w),$$

where f is a rational, integral, algebraical function of degree not greater than $2s$.

Let $$f(w) = \lambda^s (w-a)^l (w-b)^m \ldots,$$

* This investigation is based upon two memoirs by Briot et Bouquet, *Journ. de l'Éc. Polytechnique*, t. xxi, Cah. xxxvi, (1856), pp. 134—198, 199—254; and upon their *Traité des fonctions elliptiques*, pp. 341—350, 376—392. A memoir by Cayley, *Proc. Lond. Math. Soc.*, vol. xviii, (1887), pp. 314—324, may also be consulted.

† *Comptes Rendus*, t. xciii, (1881), pp. 1063—1065; *Sitzungsber. d. Akad. d. Wiss. zu Berlin*, 1884, (ii), pp. 709, 710.

‡ *Annales de l'Éc. Norm.*, 2me Sér., t. xii, (1883), pp. 105—190; ib., 3me Sér., t. ii, (1885), pp. 99—112.

where λ, a, b, ... are constants and l, m, ... are integers, and

$$l + m + \ldots \leqslant 2s.$$

The equation in $u\left(=\frac{1}{w}\right)$ and $\frac{du}{dz}$ is

$$(-1)^s \left(\frac{du}{dz}\right)^s = \lambda^s u^{2s-l-m-\ldots} (1-au)^l (1-bu)^m \ldots;$$

thus the values of $\frac{dw}{dz}$ and $\frac{du}{dz}$ are respectively

$$\frac{dw}{dz} = \lambda (w-a)^{\frac{l}{s}} (w-b)^{\frac{m}{s}} \ldots,$$

$$-\frac{du}{dz} = \lambda u^{2-\frac{l}{s}-\frac{m}{s}-\ldots} (1-au)^{\frac{l}{s}} (1-bu)^{\frac{m}{s}} \ldots.$$

Because the integral of the equation must be uniform, each of the indices $2-\frac{l}{s}-\frac{m}{s}-\ldots$, $\frac{l}{s}$, $\frac{m}{s}$, ... must be of one of the forms $1-\frac{1}{p}$, 1, or $1+\frac{1}{p}$; and p may be 1, but the point is then not a branch-point. Then the smallest value of p is 2 and the least index is therefore $\frac{1}{2}$; hence, as

$$\frac{l}{s} + \frac{m}{s} + \ldots \leqslant 2,$$

there cannot be more than *four* distinct (that is, non-repeated) factors in $f(w)$. Hence

(*a*) if one of the indices $\frac{l}{s}$, $\frac{m}{s}$, ..., be greater than 1, each of the other indices must be less than 1, unless it be 2 when all the others are zero;

(*b*) if one of the indices $\frac{l}{s}$, $\frac{m}{s}$, ..., be equal to 1, then either each of the other indices must be less than 1, or one other is equal to 1, and then there is no remaining index;

(*c*) if each of the indices $\frac{l}{s}$, $\frac{m}{s}$, ..., be less than 1, then $2-\frac{l}{s}-\frac{m}{s}-\ldots$ may be less than 1, or equal to 1, or greater than 1.

These cases, associated with the possible numbers of factors, will be taken in order.

I. Let there be a single factor; the equation is

$$\left(\frac{dw}{dz}\right)^s = \lambda^s (w-a)^l,$$

and therefore

$$\left(-\frac{du}{dz}\right)^s = \lambda^s u^{2s-l} (1-au)^l.$$

Now $\frac{l}{s}$, not being 2, is either $1-\frac{1}{s}$, 1, $1+\frac{1}{s}$; and these forms cover also the possible forms of $2-\frac{l}{s}$.

If $l = s - 1$, then one index (for $w = a$) is equal to $1 - \frac{1}{s}$, and the other (for $u = 0$) is equal to $1 + \frac{1}{s}$: the function w is rational and algebraical in z, and z is infinite only when $w = \infty$: hence the integral w is a rational, integral, *algebraical* function of z.

If $l = s + 1$, the reasoning is similar; and the integral is a rational, *algebraical*, meromorphic function of z.

If $l = s$, the indices are each equal to unity: the integral is a *simply-periodic* function of z. The equation is reducible.

If $l = 2s$, the equation is reducible; the integral is algebraical.

The equations in the respective cases are

$$\left(\frac{dw}{dz}\right)^s = \lambda^s (w - a)^{s-1} \text{ .... (A.)},$$

$$\left(\frac{dw}{dz}\right)^s = \lambda^s (w - a)^{s+1} \text{ .... (A.)},$$

$$\left(\frac{dw}{dz}\right) = \lambda \; (w - a) \text{ .... (S. P.)},$$

$$\frac{dw}{dz} = \lambda \; (w - a)^2 \text{ .... (A.)},$$

where (A.) implies that the uniform integral is an algebraical function of z, and (S. P.) implies that it is a simply-periodic function; the letters (D. P.) will be used to imply that the uniform integral is a doubly-periodic function.

II. Let there be two distinct factors; then the equation is

$$\left(\frac{dw}{dz}\right)^s = \lambda^s (w - a)^l (w - b)^m.$$

First, let one of the indices in the expression for $\frac{dw}{dz}$ be greater than 1, say $\frac{l}{s}$. It is not necessarily in its lowest terms; when reduced to its lowest terms, let

$$\frac{l}{s} = 1 + \frac{1}{\rho}.$$

Then $\frac{m}{s}$ must be less than 1; when reduced to its lowest terms, let

$$\frac{m}{s} = 1 - \frac{1}{\sigma},$$

which is the necessary form. And $2 - \frac{l}{s} - \frac{m}{s} - \dots$ must be less than 1, and it must be expressible in the form $1 - \frac{1}{\tau}$: hence

$$2 - \left(1 + \frac{1}{\rho}\right) - \left(1 - \frac{1}{\sigma}\right) = 1 - \frac{1}{\tau},$$

and therefore $$1+\frac{1}{\rho}=\frac{1}{\sigma}+\frac{1}{\tau},$$

where ρ and σ are each greater than unity. If $\tau>1$, the right-hand side is manifestly less than the left; and therefore we must have $\tau=1$, $\rho=\sigma$; and the common value of ρ and σ is s. The integral is then a rational algebraical function of z.

Secondly, let one of the indices in the expression for $\frac{dw}{dz}$ be equal to 1, say $l=s$. Then $\frac{m}{s}$ is either 1 or of the form $1-\frac{1}{\sigma}$.

If $\frac{m}{s}=1$, the exponent of u in the expression for $\frac{du}{dz}$ is zero: the equation is

$$\left(\frac{dw}{dz}\right)^s=\lambda^s(w-a)^s(w-b)^s,$$

which is reducible; it has a simply-periodic function for its integral.

If $\frac{m}{s}=1-\frac{1}{\sigma}$, the exponent of u in the expression for $\frac{du}{dz}$ is $\frac{1}{\sigma}$. This must be of the form $1-\frac{1}{\rho}$, so that

$$\frac{1}{\sigma}+\frac{1}{\rho}=1;$$

hence, as σ and ρ are each greater than 1, each must be 2. The equation is

$$\left(\frac{dw}{dz}\right)^s=\lambda^s(w-a)^s(w-b)^{\frac{1}{2}s},$$

which is reducible; and the integral is a simply-periodic function.

Thirdly, let each of the indices in the expression for $\frac{dw}{dz}$ be less than 1; as they are not necessarily in their lowest terms, let $\frac{l}{s}=1-\frac{1}{\rho}$, $\frac{m}{s}=1-\frac{1}{\sigma}$. Then the index of u in the expression for $\frac{du}{dz}$ is $\frac{1}{\rho}+\frac{1}{\sigma}$; because ρ and σ are each greater than 1, this index cannot be greater than 1.

If $\frac{1}{\rho}+\frac{1}{\sigma}=1$, the only possible values are $\rho=2$, $\sigma=2$; the equation is

$$\left(\frac{dw}{dz}\right)^s=\lambda^s(w-a)^{\frac{1}{2}s}(w-b)^{\frac{1}{2}s},$$

which is reducible; the integral is a simply-periodic function of z.

If $\frac{1}{\rho}+\frac{1}{\sigma}$ be less than 1, then, as it is the index of u in the expression for $\frac{du}{dz}$, it must be of the form $1-\frac{1}{\tau}$, where τ is greater than 1: thus

$$\frac{1}{\rho}+\frac{1}{\sigma}+\frac{1}{\tau}=1,$$

and then all the indices in the expressions for $\frac{dw}{dz}$ and $\frac{du}{dz}$ are less than 1. Hence for such equations as exist, the integrals will be doubly-periodic functions.

In this equation the interchange of ρ and σ gives no essentially new arrangement. We must have $\tau > 1$: the solutions for values of τ greater than 1 are :—

(a) $\tau = 2$; then $\frac{1}{\rho} + \frac{1}{\sigma} = \frac{1}{2}$, so that $\rho = 3,\ \sigma = 6$; $\rho = 4,\ \sigma = 4$.

(b) $\tau = 3$; then $\frac{1}{\rho} + \frac{1}{\sigma} = \frac{2}{3}$, so that $\rho = 2,\ \sigma = 6$; $\rho = 3,\ \sigma = 3$.

(c) $\tau = 4$; then $\frac{1}{\rho} + \frac{1}{\sigma} = \frac{3}{4}$, so that $\rho = 2,\ \sigma = 4$.

(d) $\tau = 5$ gives no solution.

(e) $\tau = 6$; then $\frac{1}{\rho} + \frac{1}{\sigma} = \frac{5}{6}$, so that $\rho = 2,\ \sigma = 3$.

And no higher value of τ gives solutions.

Hence the whole system of equations, satisfied by a uniform function of z and having two distinct factors in $f(w)$, is :—

$$\left(\frac{dw}{dz}\right)^s = \lambda^s (w-a)^{s-1} (w-b)^{s+1} \text{......... (A.)},$$

$$\left(\frac{dw}{dz}\right) = \lambda\ (w-a)\quad (w-b) \text{......... (S. P.)},$$

$$\left(\frac{dw}{dz}\right)^2 = \lambda^2 (w-a)^2 \quad (w-b) \text{......... (S. P.)},$$

$$\left(\frac{dw}{dz}\right)^2 = \lambda^2 (w-a) \quad (w-b) \text{......... (S. P.)},$$

$$\left(\frac{dw}{dz}\right)^6 = \lambda^6 (w-a)^4 \quad (w-b)^5 \text{......... (D. P.), (1)},$$

$$\left(\frac{dw}{dz}\right)^4 = \lambda^4 (w-a)^3 \quad (w-b)^3 \text{......... (D. P.), (2)},$$

$$\left(\frac{dw}{dz}\right)^6 = \lambda^6 (w-a)^3 \quad (w-b)^5 \text{......... (D. P.), (3)},$$

$$\left(\frac{dw}{dz}\right)^3 = \lambda^3 (w-a)^2 \quad (w-b)^2 \text{......... (D. P.), (4)},$$

$$\left(\frac{dw}{dz}\right)^4 = \lambda^4 (w-a)^2 \quad (w-b)^3 \text{......... (D. P.), (5)},$$

$$\left(\frac{dw}{dz}\right)^6 = \lambda^6 (w-a)^3 \quad (w-b)^4 \text{......... (D. P.), (6)}.$$

III. Let there be three distinct factors: then the equation is

$$\left(\frac{dw}{dz}\right)^s = \lambda^s (w-a)^l (w-b)^m (w-c)^n,$$

and therefore

$$\left(-\frac{du}{dz}\right)^s = \lambda^s u^{2s-l-m-n} (1-au)^l (1-bu)^m (1-cu)^n.$$

If one of the indices in the expression for $\frac{dw}{dz}$ be greater than 1, say $\frac{l}{s} = 1 + \frac{1}{\rho}$, then $\frac{m}{s}$, $\frac{n}{s}$ must be of the form $1 - \frac{1}{\sigma}$, $1 - \frac{1}{\tau}$, where σ and τ are each greater than 1.

The index of u in the expression for $\frac{du}{dz}$ is then $\frac{1}{\sigma} + \frac{1}{\tau} - \frac{1}{\rho} - 1$, a quantity which is necessarily negative, for ρ is finite; and the index should either be zero or be of a form $1 - \frac{1}{\mu}$. Hence no one of the indices $\frac{l}{s}$, $\frac{m}{s}$, $\frac{n}{s}$ can be greater than 1.

Secondly, let one of the indices in the expression for $\frac{dw}{dz}$ be equal to 1, say $l = s$. Then since $m + n \leqslant s$, only one of the indices is unity; and therefore $\frac{m}{s}$, $\frac{n}{s}$ are of the form $1 - \frac{1}{\rho}$, $1 - \frac{1}{\sigma}$, where ρ and σ are each greater than 1. The index of u in the expression for $\frac{du}{dz}$ is then $\frac{1}{\rho} + \frac{1}{\sigma} - 1$, and it cannot be negative; hence the only possible values are $\rho = 2 = \sigma$, and they make the index zero. There is thus one index equal to 1, and the others are less than 1: the integral of the equation is a simply-periodic function of z.

Thirdly, let all the indices in the expression for $\frac{dw}{dz}$ be less than 1: then they are of the forms $1 - \frac{1}{\rho}$, $1 - \frac{1}{\sigma}$, $1 - \frac{1}{\tau}$, where ρ, σ, τ are greater than 1; and the index of u in the expression for $\frac{du}{dz}$ is $\frac{1}{\rho} + \frac{1}{\sigma} + \frac{1}{\tau} - 1$. Because the smallest value of ρ, σ, τ is 2, this last index is not greater than $\frac{1}{2}$; hence it must be $1 - \frac{1}{\mu}$, where, because this quantity is the index of u, μ is equal to 1 or to 2. In either case, all the indices are less than 1; and therefore the integrals of the corresponding equations are doubly-periodic functions of z.

If $$\frac{1}{\rho} + \frac{1}{\sigma} + \frac{1}{\tau} - 1 = 1 - \frac{1}{2},$$

so that $\frac{1}{\rho} + \frac{1}{\sigma} + \frac{1}{\tau} = \frac{3}{2}$, the only possible solution is

$$\rho, \sigma, \tau = 2, 2, 2.$$

If $\frac{1}{\rho}+\frac{1}{\sigma}+\frac{1}{\tau}=1$, the only possible solutions are

$$\rho,\ \sigma,\ \tau = 2,\ 3,\ 6;$$
$$2,\ 4,\ 4;$$
$$3,\ 3,\ 3.$$

Hence the whole system of equations, satisfied by a uniform function of z and having three distinct factors in $f(w)$, is:—

$$\left(\frac{dw}{dz}\right)^2 = \lambda^2 (w-a)^2 (w-b)\ (w-c) \ \ldots\ldots\ldots \text{(S. P.)},$$

$$\left(\frac{dw}{dz}\right)^2 = \lambda^2 (w-a)\ (w-b)\ (w-c) \ \ldots\ldots\ldots \text{(D. P.), (7)},$$

$$\left(\frac{dw}{dz}\right)^6 = \lambda^6 (w-a)^3 (w-b)^4 (w-c)^5 \ldots\ldots\ldots \text{(D. P.), (8)},$$

$$\left(\frac{dw}{dz}\right)^4 = \lambda^4 (w-a)^2 (w-b)^3 (w-c)^3 \ldots\ldots\ldots \text{(D. P.), (9)},$$

$$\left(\frac{dw}{dz}\right)^3 = \lambda^3 (w-a)^2 (w-b)^2 (w-c)^2 \ldots\ldots\ldots \text{(D. P.), (10)}.$$

IV. Let there be four distinct factors; then the equation is

$$\left(\frac{dw}{dz}\right)^s = \lambda^s (w-a)^l (w-b)^m (w-c)^n (w-d)^p.$$

Since $\frac{l}{s}, \frac{m}{s}, \frac{n}{s}, \frac{p}{s}$ are each of a form $1-\frac{1}{\rho}$, and their sum is not greater than 2, it is easy to see that the only possible solution is given by $\frac{l}{s}=\frac{m}{s}=\frac{n}{s}=\frac{p}{s}=\frac{1}{2}$; each index is less than 1, and the integral is a doubly-periodic function.

Hence the single equation, satisfied by a uniform function of z and having four distinct factors in $f(w)$, is

$$\left(\frac{dw}{dz}\right)^2 = \lambda^2 (w-a)\ (w-b)\ (w-c)\ (w-d) \ \ldots\ldots\ldots \text{(D. P.), (11)}.$$

Those of the complete system of equations, which have their integrals either rational algebraic functions or simply-periodic functions of z, are easily integrated. The remainder, which have uniform doubly-periodic functions of z for their integrals, are most easily integrated by first determining the irreducible infinities of the functions and their orders: and then, by the results of Chapters X. and XI., the integral can be constructed.

The irreducible infinities can be determined as follows. In the equation for $\frac{du}{dz}$, let the index of u be $1-\frac{1}{\rho}$; and let $s=\sigma\rho$. Then the equation which determines u is

$$\left(\frac{du}{dz}\right)^s = \lambda^s u^{\sigma(\rho-1)} (1-au)^l \ldots,$$

so that for very small values of u, we have

$$\left\{u^{-1+\frac{1}{\rho}}+\ldots\right\} du = a\lambda dz,$$

where a is a primitive sth root of unity. Hence

$$a\lambda (z-c) = \rho u^{\frac{1}{\rho}} + \ldots,$$

and therefore

$$\frac{1}{w} = u = a^{\rho}\lambda^{\rho}(z-c)^{\rho} + \ldots.$$

It thus appears that the accidental singularity of w at $z=c$ is of order ρ; and, as there are σ distinct values of a^{ρ}, there are σ distinct accidental singularities to be associated with the respective values.

Applying these to the equations which, having doubly-periodic functions for the integrals, are numbered (1) to (11), we have the following results, where σ is the number of distinct irreducible accidental singularities and ρ is the order of each of these singularities:

number of equation	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
number of singularities $=\sigma$	3	2	2	1	1	1	1	6	4	3	2
order of singularity $=\rho$	2	2	3	3	4	6	2	1	1	1	1

All the binomial equations, which have uniform functions of z for their integrals, have been obtained. The general results, summarised in § 251, can be applied to other equations; the application to trinomial equations will be found in the treatise by Briot and Bouquet (cited p. 482, note).

Note. The binomial equations can be treated otherwise, by forming the equation

$$z-a=\int (w-a)^{-\frac{l}{s}}(w-b)^{-\frac{m}{s}}\ldots\ldots dw;$$

but, as indicated at the beginning of § 252, the method in the text is adopted in order to illustrate the general results of § 251. (See also Note, § 251.)

Ex. 1. Prove that the integral of the equation

$$\left(\frac{dw}{dz}\right)^3 - \left(\frac{dw}{dz}\right)^2 + 4w^3 - 27w^6 = 0$$

is a rational function of z; that the integral of

$$\left(\frac{dw}{dz}\right)^3 - \left(\frac{dw}{dz}\right)^2 - 4w^2 - 27w^4 = 0$$

is a simply-periodic function of z; and that the integral of

$$\left(\frac{dw}{dz}\right)^3 + 3\left(\frac{dw}{dz}\right)^2 + w^6 - 4 = 0$$

is a doubly-periodic function of z.

Find the infinities of each of the functions: and integrate the equations.

(Briot et Bouquet.)

Ex. 2. Shew that, if an irreducible trinomial equation of the form

$$\left(\frac{dw}{dz}\right)^m + \left(\frac{dw}{dz}\right)^{m-1} f_1(w) + f_m(w) = 0$$

have a uniform integral, then m may not be greater than 5; and that, if m be 4 or 5, the uniform integral is a doubly-periodic function.

Apply this result to the discussion of the equation

$$\left(\frac{dw}{dz}\right)^5 + \left(\frac{dw}{dz}\right)^4 (w^2-1) - \frac{4^4}{5^5} w^2 (w^2-1)^4 = 0.$$

(Briot et Bouquet.)

Ex. 3. Shew that the integral of the equation

$$\left(\frac{dw}{dz}\right)^6 = \lambda (w-a)^2 (w-b)^5 (w-c)^5$$

is a two-valued doubly-periodic function of z. (Schwarz.)

Ex. 4. Shew that, if a function w be determined by a differential equation

$$F\left(\frac{dw}{dz}, w\right) = 0,$$

where F is a rational integral algebraical function of w and $\frac{dw}{dz}$, of degree m in $\frac{dw}{dz}$, and does not contain z explicitly, then to each value of w there correspond m series of values of z, the terms in each series differing from one another by multiples of periods.

Prove further that, if the integral w have only a limited number of values for each value of z, then it is determined by an algebraical relation between w and u, where u may be z, or $e^{\frac{2\pi z i}{\omega}}$, or $\wp(z)$. (Briot et Bouquet.)

These results should be compared with the results obtained in Chapter XIII. relative to functions which possess an algebraical addition-theorem.

CHAPTER XIX.

Conformal Representation: Introductory.

253. In § 9 it was proved that a functional relation between two complex variables w and z can be represented geometrically as a copy of part of the z-plane made on part of the w-plane. At various stages in the theory of functions, particularly in connection with their developments in the vicinity of critical points, considerable use has been made of the geometrical representation of the analytical relation; but it has been used in such a way that, when the equations of transformation define multiform functions, the branches of the function used are uniform in the represented areas.

The characteristic property of the copy is that angles are preserved, and that no change is made in the relative positions and (save as to a uniform magnification) no change is made in the relative distances of points that lie in the immediate vicinity of a given point in the z-plane. The leading feature of this property is maintained over the whole copy for every small element of area: but the magnification, which is uniform for each element, is not uniform over the whole of the copy.

Two planes or parts of two planes, thus related, have been said to be *conformally represented*, each upon the other.

Now conformal representation of this character is essential to the constitution of a geographical map, made as perfect as possible: and a question is thus suggested whether the foregoing functional relation is substantially the only form that leads to what may be called geographical similarity. In this form, the question raises a converse more general than is implied by the converse of the functional relation, inasmuch as it implies the possibility that the property can be associated with curved surfaces and not merely with planes. But a little consideration will shew that the generalisation is a priori not unjustifiable, because, except at singular points, the elements of the curved surface can, in this regard, be treated as elements of successive planes. We therefore have* *to determine the most general form of analytical relation between parts of two surfaces which establishes the property of conformal similarity between the elements of the surfaces.*

* The following investigation is due to Gauss: for references, see p. 500, note.

Let x, y, z be the coordinates of a point R of one surface with t, u for its parameters, so that x, y, z can be expressed in terms of t, u; and let X, Y, Z be the coordinates of an associated point R' of the other surface with T, U for its parameters, so that X, Y, Z can be expressed in terms of T, U. Then the analytical problem presented is the determination of the most general relations which, by expressing T and U in terms of t and u, establish the conformal similarity of the surfaces.

Suppose that G and H are any points on the first surface in the immediate proximity of R, and that G' and H' are the corresponding points on the second surface in the immediate proximity of R': then the conformal similarity requires, and is established by, the conditions: (i), that the ratio of an arc RG to the corresponding arc $R'G'$ is the same for all infinitesimal arcs conterminous in R and R' respectively; and, (ii), that the inclination of any two directions RG and RH is the same as the inclination of the corresponding directions $R'G'$ and $R'H'$. Let the coordinates of G and of H relative to R be dx, dy, dz and $\delta x, \delta y, \delta z$ respectively; and those of G' and of H' relative to R' be dX, dY, dZ and $\delta X, \delta Y, \delta Z$ respectively. Let ds denote the length of RG and dS that of $R'G'$; let m be the magnification of ds into dS, so that

$$dS = mds,$$

a relation which holds for every corresponding pair of infinitesimal arcs at R and R'.

By the expressions of x, y, z in terms of t and u, we have equations of the form

$$dx = adt + a'du, \quad dy = bdt + b'du, \quad dz = cdt + c'du,$$

where the quantities a, b, c, a', b', c' are finite. Let there be some relations, which must evidently be equivalent to two independent algebraical equations, expressing T and U as functions of t and u; then we have equations of the form

$$dX = Adt + A'du, \quad dY = Bdt + B'du, \quad dZ = Cdt + C'du,$$

where the quantities A, B, C, A', B', C' are finite and are dependent partly upon the known equations of the surface and partly upon the unknown equations of relation between T, U and t, u. Then

$$ds^2 = (a^2 + b^2 + c^2)\,dt^2 + 2\,(aa' + bb' + cc')\,dtdu + (a'^2 + b'^2 + c'^2)\,du^2,$$

$$dS^2 = (A^2 + B^2 + C^2)\,dt^2 + 2\,(AA' + BB' + CC')\,dtdu + (A'^2 + B'^2 + C'^2)\,du^2.$$

Since the magnification is to be the same for all corresponding arcs, it must be independent of particular relations between dt and du; and therefore

$$\frac{A^2 + B^2 + C^2}{a^2 + b^2 + c^2} = \frac{AA' + BB' + CC'}{aa' + bb' + cc'} = \frac{A'^2 + B'^2 + C'^2}{a'^2 + b'^2 + c'^2},$$

each of these fractions being equal to m^2.

Again, since the inclinations of the two directions RG, RH; and RG', RH'; are given by

$$ds\,\delta s \cos GRH$$
$$= (a^2 + b^2 + c^2)\, dt\,\delta t + (aa' + bb' + cc')\,(dt\,\delta u + \delta t\, du) + (a'^2 + b'^2 + c'^2)\, du\,\delta u,$$

$$dS\,\delta S \cos G'R'H'$$
$$=(A^2+B^2+C^2)dt\delta t+(AA'+BB'+CC')(dt\delta u+\delta t du)+(A'^2+B'^2+C'^2)du\delta u,$$

we have, in consequence of the preceding relations,

$$m^2 ds\,\delta s \cos GRH = dS\,\delta S \cos G'R'H'.$$

But $dS = m ds$, $\delta S = m\delta s$; and therefore the angle GRH is equal to the angle $G'R'H'$. It thus appears that the two conditions, which make the magnification at R the same in all directions, are sufficient to make the inclinations of corresponding arcs the same; and therefore they are two equations to determine relations which establish the conformal similarity of the two surfaces.

These two equations are the conditions that the ratio dS/ds may be independent of relations between dt and du; it is therefore sufficient, for the present purpose, to assign the conditions that dS/ds be independent of values (or the ratio) of differential elements dt and du.

Now ds^2 is essentially positive and it is a real quadratic homogeneous function of these elements; hence, when resolved into factors linear in the differential elements, it takes the form

$$ds^2 = n\,(dp + idq)\,(dp - idq),$$

where n is a finite and real function of t and u, and dp, dq are real linear combinations of dt and du. Similarly, we have

$$dS^2 = N\,(dP + idQ)\,(dP - idQ),$$

where, again, N is a finite and real function of t and u or of T and U, and dP, dQ are real linear combinations of dt and du or of dT and dU. Thus

$$m^2 = \frac{N}{n}\,\frac{(dP + idQ)\,(dP - idQ)}{(dp + idq)\,(dp - idq)}.$$

It has been seen that the value of m is to be independent of the values and of the ratio of the differential elements.

Now taking

$$\theta = \frac{aa' + bb' + cc'}{a^2 + b^2 + c^2}, \qquad \phi = \frac{a'^2 + b'^2 + c'^2}{a^2 + b^2 + c^2},$$

so that θ and ϕ are, by the two equations of condition, the same for ds and dS, and denoting by ψ the real quantity $(\phi - \theta^2)^{\frac{1}{2}}$, we have

$$ds^2 = (a^2 + b^2 + c^2)\,\{dt + du\,(\theta + i\psi)\}\,\{dt + du\,(\theta - i\psi)\},$$

and

$$dS^2 = (A^2 + B^2 + C^2)\,\{dt + du\,(\theta + i\psi)\}\,\{dt + du\,(\theta - i\psi)\}.$$

Then, except as to factors which do not involve infinitesimals, the factors of ds^2 and of dS^2 are the same. Hence, except as to the former factors, the numerator of the fraction for m^2 is, quà function of the infinitesimal elements, substantially the same as the denominator; and therefore either

$$(\alpha) \qquad \frac{dP+idQ}{dp+idq} \text{ and } \frac{dP-idQ}{dp-idq} \text{ are finite quantities simultaneously;}$$

or

$$(\beta) \qquad \frac{dP+idQ}{dp-idq} \text{ and } \frac{dP-idQ}{dp+idq} \text{ are finite quantities simultaneously.}$$

Either of these pairs of conditions ensures the required form of m, and so ensures the conformal similarity of the surfaces.

Ex. Shew that both p and q satisfy the partial differential equation

$$\left\{\left(\theta\frac{\partial}{\partial t}-\frac{\partial}{\partial u}\right)^2+\left(\psi\frac{\partial}{\partial t}\right)^2\right\}f=0.$$

Consider (α) first. Since $(dP+idQ)/(dp+idq)$ is a finite quantity, the differentials $dP+idQ$ and $dp+idq$ vanish together and therefore the quantities $P+iQ$ and $p+iq$ are constant together. Now P and Q are functions of the variables which enter into the expressions for p and q; hence $P+iQ$ and $p+iq$, in themselves variable quantities, can be constant together only if

$$P+iQ=f(p+iq),$$

where f denotes some functional form. This equation implies two independent relations, because the real parts, and the coefficients of the imaginary parts, on the two sides of the equation must separately be equal to one another; and from these two relations we infer that

$$P-iQ=f_1(p-iq),$$

where $f_1(p-iq)$ is the function which results from changing i into $-i$ throughout $f(p+iq)$ and is equal to $f(p-iq)$, if i enter into f only through its occurrence in $p+iq$. From this equation, it follows that

$$\frac{dP-idQ}{dp-idq}$$

is finite, and therefore a necessary and sufficient condition for the satisfaction of (α) is that P, Q and p, q be connected by an equation of the form

$$P+iQ=f(p+iq).$$

Moreover, the function f is arbitrary so far as required by the preceding analysis; and so the conditions will be satisfied, either if special forms of f be assumed or if other (not inconsistent) conditions be assigned so as to determine the form of the function.

Next, consider (β). We easily see that similar reasoning leads to the conclusion that the conditions are satisfied, when P, Q and p, q are connected by an equation of the form

$$P+iQ=g(p-iq);$$

and similar inferences as to the use of the undetermined functional form of g may be drawn. Hence we have the theorem :—

Parts of two surfaces may be made to correspond, point by point, in such a way that their elements are similar to one another, by assigning any relation between their parameters, of either of the forms

$$P + iQ = f(p + iq), \quad P + iQ = g(p - iq);$$

and every such correspondence between two given surfaces is obtained by the assignment of the proper functional form in one or other of these equations.

254. Suppose now that there is a third surface, any point on which is determined by parameters λ and μ; then it will have conformal similarity to the first surface, if there be any functional relation of the form

$$\lambda + i\mu = h(p + iq).$$

But if h^{-1} be the inverse of the function h, then we have a relation

$$\begin{aligned} P + iQ &= f\{h^{-1}(\lambda + i\mu)\} \\ &= F(\lambda + i\mu), \end{aligned}$$

which is the necessary and sufficient condition for the conformal similarity of the second and the third surfaces.

This similarity to one another of two surfaces, each of which can be made to correspond to a third surface so as to be conformally similar to it, is an immediate inference from the geometry. It has an important bearing, in the following manner. If the third surface be one of simple form, so that its parameters are easily obtainable, there will be a convenience in making it correspond to one of the first two surfaces so as to have conformal similarity, and then in making the second of the given surfaces correspond, in conformal similarity, to the third surface which has already been made conformally similar to the first of them.

Now the simplest of all surfaces, from the point of view of parametric expression of points lying on it, is the plane: the parameters are taken to be the Cartesian coordinates of the point. Hence, in order to map out two surfaces so that they may be conformally similar, it is sufficient to map out a plane in conformal similarity to one of them and then to map out the other in conformal similarity to the mapped plane: that is to say, *we may, without loss of generality, make one of the surfaces a plane,* and all that is then necessary is the determination of a law of conformation.

We therefore take $P = X$, $Q = Y$, $N = 1$: and then

$$P + iQ = X + iY = Z,$$

where Z is the complex variable of a point in the plane; and the equations which establish the conformation of the surface with the plane are

$$\left.\begin{aligned} ds^2 &= n(dp^2 + dq^2) \\ X + iY &= f(p + iq) \\ m^2 n &= f'(p + iq) f_1'(p - iq) \end{aligned}\right\},$$

where $f_1(p-iq)$ is the form of $f(p+iq)$ when, in the latter, the sign of i is changed throughout.

As yet, only the form $P+iQ=f(p+iq)$ has been taken into account. It is sufficient for our present purpose, in regard to the alternative form $P+iQ=g(p-iq)$, to note that, by the introduction of a plane as an intermediate surface, there is no essential distinction between the cases*. For as $P=X$, $Q=Y$, we have

$$X+iY=g(p-iq),$$

and therefore

$$X-iY=g_1(p+iq),$$

which maps out the surface on the plane in a copy differing from the copy determined by

$$X+iY=g_1(p+iq)$$

only in being a reflexion of that former copy in the axis of X. It is therefore sufficient to consider only the general relation

$$X+iY=f(p+iq).$$

Ex. We have an immediate proof that the form of relation between two planes, as considered in § 9, is the most general form possible. For in the case in which the second surface is a plane, we have $ds^2=dx^2+dy^2$, so that $n=1$, $p=x$, $q=y$: hence the most general law is

$$X+iY=f(x+iy),$$

that is,

$$w=f(z)$$

in the earlier notation. Some illustrations arising out of particular forms of the function f will be considered later (§ 257).

255. In the case of a surface of revolution, it is convenient to take ϕ as the orientation of a meridian through any point, that is, the longitude of the point, σ as the distance along the meridian from the pole, and q as the perpendicular distance from the axis; there will then be some relation between σ and q, equivalent to the equation of the meridian curve. Then

$$\begin{aligned} ds^2 &= d\sigma^2+q^2\,d\phi^2 \\ &= q^2(d\phi^2+d\theta^2), \end{aligned}$$

where $d\theta=\dfrac{d\sigma}{q}$, so that θ is a function of only one variable, the parameter of the point regarded as a point on the meridian curve. Here $n=q^2$; and so the relation, which establishes the law of conformation between the plane and the surface in the most general form, is

$$x+iy=f(\phi+i\theta);$$

and the magnification m is given by

$$m^2q^2=f'(\phi+i\theta)f_1'(\phi-i\theta).$$

Evidently the lines on the plane, which correspond to meridians of

* A discussion is given by Gauss, *Ges. Werke*, t. iv, pp. 211—216, of the corresponding result when neither of the surfaces is plane.

longitude, are given by the elimination of θ, and the lines on the plane, which correspond to parallels of latitude, are given by the elimination of ϕ, between the equations

$$\left.\begin{aligned} 2x &= f(\phi + i\theta) + f_1(\phi - i\theta) \\ 2iy &= f(\phi + i\theta) - f_1(\phi - i\theta) \end{aligned}\right\}.$$

Ex. 1. On the surface of revolution, let

$$\psi = -4i \int m^2 q \, d\sigma,$$

where m, q, σ have the significations in the text; shew that ϕ and ψ satisfy the equation

$$\left(\frac{\partial u}{\partial z_1}\right)^2 \frac{\partial^2 u}{\partial z_2^2} + \left(\frac{\partial u}{\partial z_2}\right)^2 \frac{\partial^2 u}{\partial z_1^2} = 0,$$

where z_1, z_2 are the conjugate complexes $x \pm iy$ in the plane. (Korkine.)

Ex. 2. Prove that, in a plane map of a surface of revolution, the curvature of a meridian at a point θ is $\dfrac{\partial}{\partial\theta}\left(\dfrac{1}{mq}\right)$ and the curvature of a parallel of latitude at a point ϕ is $\dfrac{\partial}{\partial\phi}\left(\dfrac{1}{mq}\right)$. Hence shew that, if the meridians and the parallels of latitude become circles on the plane map given by

$$z = f(\phi + i\theta),$$

the function f and the conjugate function f_1 must satisfy the relation

$$\{f, \phi + i\theta\} = \{f_1, \phi - i\theta\},$$

where $\{f, \mu\}$ is the Schwarzian derivative. (Lagrange.)

Ex. 3. A plane map is made of a surface of revolution so that the meridians and the parallels of latitude are circles. Shew that, if (r, a) be the polar coordinates of a point on the map determined by the point (θ, ϕ) on the surface, then

$$\frac{\cos a}{r} = -2ac\{ae^{2c\theta}\cos 2(c\phi + g) + b\cos(g + h)\},$$

$$\frac{\sin a}{r} = 2ac\{ae^{2c\theta}\sin 2(c\phi + g) + b\sin(g + h)\},$$

where a, b, c, g, h are constants.

Prove also that the centres of all the meridians lie on one straight line and that the centres of all the parallels of latitude lie on a perpendicular straight line. (Lagrange.)

256. The surfaces of revolution which occur most frequently in this connection are the sphere and the prolate spheroid.

In the case of the sphere, the natural parameter of a point on a great-circle meridian is the latitude λ. We then have $d\sigma = a\,d\lambda$, where a is the radius; and $q = a\cos\lambda$, so that

$$\begin{aligned} ds^2 &= a^2\,d\lambda^2 + a^2\cos^2\lambda\,d\phi^2 \\ &= a^2\cos^2\lambda\,(d\phi^2 + d\vartheta^2), \end{aligned}$$

where $\operatorname{sech}\vartheta = \cos\lambda$. Hence we have

$$X + iY = f(\phi + i\vartheta);$$

and the magnification m is given by

$$ma\cos\lambda = \{f'(\phi + i\vartheta)\,f_1'(\phi - i\vartheta)\}^{\frac{1}{2}}.$$

There are two forms of f which are of special importance in representations of spherical surfaces.

First, let $f(\mu) = k\mu$, where k is a real constant; then

$$X + iY = k(\phi + i\vartheta),$$

and therefore $\qquad X = k\phi, \quad Y = k\vartheta = k\,\text{sech}^{-1}(\cos\lambda);$

that is, the meridians and the parallels of latitude are straight lines, necessarily perpendicular to each other, because angles are preserved. The meridians are equidistant from one another; the distance between two parallels of latitude, lying on the same side of the equator and having a given difference of latitude, increases from the equator. We have $f'(\phi + i\vartheta) = k = f_1'(\phi - i\vartheta)$; and therefore

$$m = \frac{k}{a}\sec\lambda,$$

or the map is uniformly magnified along a parallel of latitude with a magnification which increases very rapidly towards the pole. This map is known as *Mercator's Projection.*

Secondly, let $f(\mu) = ke^{ic\mu}$, where k and c are real constants; then

$$X + iY = ke^{ic(\phi + i\vartheta)} = ke^{-c\vartheta}(\cos c\phi + i\sin c\phi),$$

and therefore $\qquad X = ke^{-c\vartheta}\cos c\phi \;\text{ and }\; Y = ke^{-c\vartheta}\sin c\phi.$

For the magnification, we have

$$f'(\phi + i\vartheta) = icke^{ic(\phi + i\vartheta)} \;\text{ and }\; f_1'(\phi - i\vartheta) = -icke^{-ic(\phi - i\vartheta)},$$

so that

$$ma\cos\lambda = cke^{-c\vartheta},$$

or

$$m = \frac{ck}{a}e^{-c\vartheta}\sec\lambda = \frac{ck}{a}\frac{(1 - \sin\lambda)^{\frac{1}{2}(c-1)}}{(1 + \sin\lambda)^{\frac{1}{2}(c+1)}}.$$

The most frequent case is that in which $c = 1$. Then the meridians are represented by the concurrent straight lines

$$Y = X\tan\phi;$$

the parallels of latitude are represented by the concentric circles

$$X^2 + Y^2 = k^2e^{-2\vartheta} = k^2\frac{1 - \sin\lambda}{1 + \sin\lambda},$$

the common centre of the circles being the point of concurrence of the lines; and the magnification is

$$m = \frac{k}{a(1 + \sin\lambda)}.$$

This map is known as the *stereographic* projection: the South pole being the pole of projection.

It is convenient to take the equatorial plane for the plane of z: the direction which, in that plane, is usually positive for the measurement of longitude, is negative for ordinary measurement of trigonometrical angles. If we project on the equatorial plane, we have

$$Z = ke^{i(\phi+i\vartheta)}$$
$$= ke^{-\vartheta+i\phi},$$

which gives a stereographic projection.

Ex. 1. Prove that, if x, y, z be the coordinates of any point on a sphere of radius a and centre the origin, every plane representation of the sphere is included in the equation

$$X+iY=f\left(\frac{x+iy}{a+z}\right),$$

for varying forms of the function f.

Ex. 2. Shew that rhumb-lines (loxodromes) on a sphere become straight lines in Mercator's projection and equiangular spirals in a stereographic projection.

Ex. 3. A great circle cuts the meridian of reference ($\phi=0$) in latitude a at an angle α; shew that the corresponding curve in the stereographic projection is the circle

$$(X+k\tan a)^2+(Y+k\cot\alpha\sec a)^2=k^2\sec^2 a\operatorname{cosec}^2\alpha.$$

Ex. 4. A small circle of angular radius r on the sphere has its centre in latitude c and longitude α; shew that the corresponding curve in the stereographic projection is the circle

$$\left(X+\frac{k\cos c\cos\alpha}{\cos r+\sin c}\right)^2+\left(Y+\frac{k\cos c\sin\alpha}{\cos r+\sin c}\right)^2=\frac{k^2\sin^2 r}{(\cos r+\sin c)^2}.$$

The less frequent case is that in which the constant c is allowed to remain in the function for the purpose of satisfying some useful condition. One such condition is assigned by making the magnification the same at the points of highest and of lowest latitude on the map. If these latitudes be λ_1, λ_2, then

$$\frac{(1-\sin\lambda_1)^{\frac{1}{2}(c-1)}}{(1+\sin\lambda_1)^{\frac{1}{2}(c+1)}}=\frac{(1-\sin\lambda_2)^{\frac{1}{2}(c-1)}}{(1+\sin\lambda_2)^{\frac{1}{2}(c+1)}},$$

so that

$$c=\frac{\log\left(\frac{1-\sin\lambda_1}{1-\sin\lambda_2}\right)+\log\left(\frac{1+\sin\lambda_1}{1+\sin\lambda_2}\right)}{\log\left(\frac{1-\sin\lambda_1}{1-\sin\lambda_2}\right)-\log\left(\frac{1+\sin\lambda_1}{1+\sin\lambda_2}\right)}.$$

This representation is used for star-maps: it has the advantage of leaving the magnification almost symmetrical with respect to the centre of the map.

Ex. Prove that the magnification is a minimum at points in latitude arc sin c.

Shew that, if the map be that of a belt between latitudes 30° and 60°, the magnification is a minimum in latitude 45° 40′ 50″; and find the ratio of the greatest and the least magnifications.

Note. Of the memoirs which treat of the construction of maps of surfaces as a special question, the most important are those of Lagrange* and

* *Nouv. Mém. de l'Acad. Roy. de Berlin* (1779). There are two memoirs: they occur in his collected works, t. iv, pp. 635—692.

Gauss*. Lagrange, after stating the contributions of Lambert and of Euler, obtains a solution, which can be applied to any surface of revolution; and he makes important applications to the sphere and the spheroid. Gauss discusses the question in a more general manner and solves the question for the conformal representation of any two surfaces upon each other, but without giving a single reference to Lagrange's work: the solution is worked out for some particular problems and it is applied, in subsequent memoirs†, to geodesy. Other papers which may be consulted are those of Bonnet‡, Jacobi§, Korkine||, and Von der Mühll¶; and there is also a treatise by Herz**.

But after the appearance of Riemann's dissertation††, the question ceased to have the special application originally assigned to it; it has gradually become a part of the theory of functions. The general development will be discussed in the next chapter, the remainder of the present chapter being devoted to some special instances of functional relations between w and z and their geometrical representations.

The following three examples give the conformal representation of three surfaces upon a plane.

Ex. 1. A point on an oblate spheroid is determined by its longitude l and its geographical latitude μ. Shew that the surface will be conformally represented upon a plane by the equation

$$X+iY=f\left[l+i\left\{\phi+\tfrac{1}{2}e\log\left(\frac{1-e\sin\mu}{1+e\sin\mu}\right)\right\}\right]$$

for any form of the function f; where $\operatorname{sech}\phi=\cos\mu$, and e is the eccentricity of the meridian.

Also shew that, if the function f be taken in the form $f(u)=ke^{icu}$, the meridians in the map are concurrent straight lines, and the parallels of latitude concentric circles; and that the magnification is stationary at points in geographical latitude arc sin c. (Gauss.)

Ex. 2. Let the semi-axes of an ellipsoid be denoted by ρ, $(\rho^2-b^2)^{\frac{1}{2}}$, $(\rho^2-c^2)^{\frac{1}{2}}$ in descending order of magnitude. Shew that the surface will be conformally represented upon a plane by the equation

$$X+iY=f\left\{h(u+iv)+\tfrac{1}{2}\log\frac{\Theta(u+a)\,\Theta(iv+a)}{\Theta(u-a)\,\Theta(iv-a)}\right\}$$

for any form of the function f; where u and v are expressed in terms of the elliptic coordinates ρ_1, ρ_2 of a point on the surface by the equations

$$\frac{c^2(\rho_1{}^2-b^2)}{\rho_1{}^2(c^2-b^2)}=\operatorname{sn}^2 u,\quad \frac{c^2(\rho_2{}^2-b^2)}{\rho_2{}^2(c^2-b^2)}=\operatorname{sn}^2 iv\,;$$

* Schumacher's *Astr. Abh.* (1825); *Ges. Werke*, t. iv, pp. 189—216.

† *Gött. Abh.*, t. ii, (1844), ib., t. iii, (1847); *Ges. Werke*, t. iv, pp. 259—340.

‡ *Liouville*, t. xvii, (1852), pp. 301—340.

§ *Crelle*, t. lix, (1861), pp. 74—88; *Ges. Werke*, t. ii, pp. 399—416.

|| *Math. Ann.*, t. xxxv, (1890), pp. 588—604.

¶ *Crelle*, t. lxix, (1868), pp. 264—285.

** *Lehrbuch der Landkartenprojectionen*, (Leipzig, Teubner, 1885).

†† "Grundlagen für eine allgemeine Theorie der Functionen einer veränderlichen complexen Grösse," Göttingen, 1851; *Ges. Werke*, pp. 3—45, especially § 21.

the modulus is $\frac{\rho}{c}\left(\frac{c^2-b^2}{\rho^2-b^2}\right)^{\frac{1}{2}}$, the constant a is given by

$$b = c \operatorname{dn} a,$$

and the value of the constant h is $\operatorname{tn} a \operatorname{dn} a - Z(a)$. (Jacobi.)

Ex. 3. The circular section of an anchor-ring by a plane through the axis subtends an angle $\pi - 2\epsilon$ at the centre of the ring, and the position of any point on such a section is determined by l, the longitude of the section, and by λ, the angle between the radius from the centre of the section to the point and the line from the centre of the section to the centre of the ring.

Shew that, by means of the equations

$$l = 2\pi x,$$

$$\tan \tfrac{1}{2}\lambda = \tan \tfrac{1}{2}\epsilon \tan (\pi y \tan \epsilon),$$

the surface of the anchor-ring is conformally represented on the area of a rectangle whose sides are 1 and $\cot \epsilon$. (Klein.)

257. It was pointed out that the conformation of surfaces is obtained by a relation

$$P + iQ = f(p + iq),$$

and therefore that the conformation of planes is obtained by a relation

$$w = f(z),$$

whatever be the form of the function f, or by a relation

$$\phi(w, z) = 0,$$

whatever be the form of the function ϕ. Some examples of this conformal representation of planes will now be considered; in each of them the representation is such that one point of one area corresponds to one (and only one) point of the other.

Ex. 1. Consider the correspondence of the two planes represented by

$$(a-b)w^2 - 2zw + (a+b) = 0,$$

that is,

$$2z = (a-b)w + \frac{a+b}{w}.$$

Let r and θ be the coordinates of any point in the w-plane: and x, y the coordinates of any point in the z-plane: then

$$2x = \left[(a-b)r + \frac{a+b}{r}\right]\cos\theta, \qquad 2y = \left[(a-b)r - \frac{a+b}{r}\right]\sin\theta.$$

Hence the z-curves, corresponding to circles in the w-plane having the origin for their common centre, are confocal ellipses, $2c$ being the distance between the foci, where $c^2 = a^2 - b^2$: and the z-curves, corresponding to straight lines in the w-plane passing through the origin, are the confocal hyperbolas, a result to be expected, because the orthogonal intersections must be maintained.

Evidently the interior of a w-circle, of radius unity and centre the origin, is, by the above relation, transformed into the part of the z-plane which lies outside the ellipse $x^2/a^2 + y^2/b^2 = 1$, the w-circumference being transformed into the z-ellipse.

Ex. 2. Consider the correspondence implied by the relation

$$k^{-\frac{1}{2}} w = \operatorname{sn}\left(\frac{2K}{\pi} z\right) = \operatorname{sn} z', \text{ where } x' + iy' = z' = \frac{2K}{\pi} z,$$

with the usual notation of elliptic functions. Taking $w=X+iY$, we have

$$k^{-\frac{1}{2}}(X+iY)=\text{sn}(x'+iy')$$
$$=\frac{\text{sn}\,x'\,\text{cn}\,iy'\,\text{dn}\,iy'+\text{sn}\,iy'\,\text{cn}\,x'\,\text{dn}\,x'}{1-k^2\,\text{sn}^2 x'\,\text{sn}^2 iy'}.$$

Let $y'=\pm\frac{1}{2}K'$: then $\text{sn}\,iy'=\pm\frac{i}{\sqrt{k}}$, $\text{cn}\,iy'=\sqrt{\frac{1+k}{k}}$, $\text{dn}\,iy'=\sqrt{1+k}$, so that

$$k^{-\frac{1}{2}}(X+iY)=\frac{1+k}{k^{\frac{1}{2}}}\frac{\text{sn}\,x'}{1+k\,\text{sn}^2 x'}\pm\frac{i}{k^{\frac{1}{2}}}\frac{\text{cn}\,x'\,\text{dn}\,x'}{1+k\,\text{sn}^2 x'};$$

whence
$$X=\frac{(1+k)\,\text{sn}\,x'}{1+k\,\text{sn}^2 x'},\quad Y=\pm\frac{\text{cn}\,x'\,\text{dn}\,x'}{1+k\,\text{sn}^2 x'},$$

and therefore
$$X^2+Y^2=1,$$

which is the curve in the w-plane corresponding to the lines $y'=\pm\frac{1}{2}K'$ in the z'-plane, that is, to the lines $y=\pm\frac{\pi K'}{4K}$ in the z-plane.

When $y=+\frac{\pi K'}{4K}$ and x' lies between K and $-K$, that is, x lies between $\frac{1}{2}\pi$ and $-\frac{1}{2}\pi$, then Y is positive and X varies from 1 to -1; so that the actual curve corresponding to the line $y=\frac{\pi K'}{4K}$ is the half of the circumference on the positive side of the axis of X. Similarly the actual curve corresponding to the line $y=-\frac{\pi K'}{4K}$ is the half of the circumference on the negative side of that axis.

The curve hereby suggested for the z-plane is a rectangle, with sides $x=\pm\frac{1}{2}\pi$, $y=\pm\frac{\pi}{4}\frac{K'}{K}$. To obtain the w-curve corresponding to $x=\frac{1}{2}\pi$, that is, to $x'=K$, we have

$$k^{-\frac{1}{2}}(X+iY)=\frac{\text{cn}\,iy'}{\text{dn}\,iy'},$$

so that
$$Y=0\quad\text{and}\quad X=k^{\frac{1}{2}}\frac{\text{cn}\,iy'}{\text{dn}\,iy'}.$$

Now y' varies from $\frac{1}{2}K'$ through 0 to $-\frac{1}{2}K'$: hence X varies from 1 to $k^{\frac{1}{2}}$ and back from $k^{\frac{1}{2}}$ to 1. Similarly, the curve corresponding to $x=-\frac{1}{2}\pi$, that is, to $x'=-K$, is part of the axis of X repeated from -1 to $-k^{\frac{1}{2}}$ and back from $-k^{\frac{1}{2}}$ to -1.

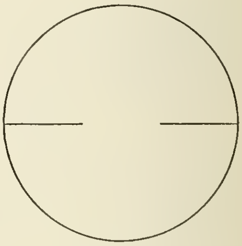

Fig. 86.

Hence the area in the w-plane, corresponding to the rectangle in the z-plane, is a circle of radius unity with two diametral slits from the circumference cut inwards, each to a distance $k^{\frac{1}{2}}$ from the centre.

The boundary of this simply connected area is the homologue of the boundary of the z-rectangle given by $x=\pm\frac{1}{2}\pi$, $y=\pm\frac{\pi K'}{4K}$: the analysis shews that the two interiors correspond*. And the sudden change in the direction of motion of the w-point at the inner extremity of each slit, while z moves continuously along a side of the rectangle, is due to the fact that dw/dz vanishes there, so that the inference of § 9 cannot be made at this point. (See also Ex. 10.)

* For details of corresponding curves in the interiors of the two areas, see Siebeck, *Crelle*, t. lvii, (1860), pp. 359—370; ib., t. lix, (1861), pp. 173—184: Holzmüller, treatise cited (p. 2, *note*), pp. 256—263: Cayley, *Camb. Phil. Trans.*, vol. xiv, (1889), pp. 484—494.

Corollary. We pass at once from the rectangle to a square, by assuming $K'=2K$; then $k=(\sqrt{2}-1)^2$, and the corresponding modifications are easily made.

Ex. 3. Shew that, if $z=\text{sn}^2(\frac{1}{2}w, k)$ where $w=u+iv$, then the curves $u=$constant, $v=$constant, are confocal Cartesian ovals whose equations may be written in the form

$$r_1-r\,\text{dn}\,(u, k)=\text{cn}\,(u, k), \quad r_1+r\,\text{dn}\,(vi, k')=\text{cn}\,(vi, k'),$$

where r and r_1 denote the distances from the foci $z=0$ and $z=1$.

If r_2 denote the distance of a point from the third focus $z=\frac{1}{k}$, find the corresponding equations connecting r, r_2; and r_1, r_2.

Shew that the curves $u=K$, $v=K'$ are circles, and that the outer and the inner branches of an oval are given by u and $2K-u$, or by v and $2K'-v$. (Math. Trip. Part II, 1891.)

Ex. 4. The w-plane is conformally represented on the z-plane by the equation

$$\frac{z}{c}=\left(\frac{1+w}{1-w}\right)^{\frac{2h}{\pi i}},$$

where h and c are real positive constants.

Shew that, if an area be chosen in the w-plane included within a circle, centre the origin and radius unity, and otherwise bounded by two circles centres 1 and -1 (so that its whole boundary consists of four circular arcs), then the corresponding area in the z-plane is a portion of a ring, bounded by two circles, of radii ce^h and ce^{-h} and centre the origin, and by two lines each passing from one circle to the other.

Prove that, when the semi-circles in the w-plane are very small, so as merely to exclude the points 1 and -1 from the circular area and boundary, the corresponding z-figure is the ring with a single slit along the axis of real quantities*.

Ex. 5. Consider the correspondence implied by the relation

$$z=c\sin w.$$

Taking $w=X+iY$, we have

$$x+iy=c\sin(X+iY)$$
$$=c\sin X\cosh Y+ic\cos X\sinh Y,$$

so that

$$x=c\sin X\cosh Y, \quad y=c\cos X\sinh Y.$$

When Y is constant, then z describes the curves

$$\frac{x^2}{c^2\cosh^2 Y}+\frac{y^2}{c^2\sinh^2 Y}=1,$$

which, for different values of Y, are confocal ellipses.

Now take a rectangle lying between $X=\pm\frac{1}{2}\pi$, $Y=\pm\lambda$. For all values of X, $\cos X$ is positive: hence when $Y=+\lambda$, y is positive and x varies from $c\cosh\lambda$ to $-c\cosh\lambda$, that is, the half of the ellipse on the positive side of the axis of y is covered.

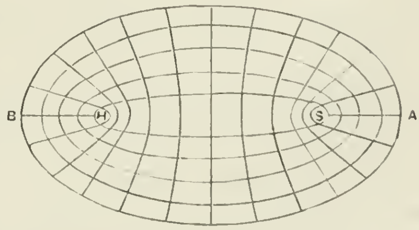

Fig. 87.

Let $X=-\frac{1}{2}\pi$: then

$$y=0 \text{ and } x=-c\cosh Y.$$

As Y varies from $+\lambda$ through 0 to $-\lambda$ along the side of the rectangle, x passes from B to H (the focus) and back from H to B.

* See reference, p. 431, note.

When $Y=-\lambda$, then z describes the half of the ellipse on the negative side of the axis of y: when $X=+\frac{1}{2}\pi$, then $y=0$, $x=c\cosh Y$, so that z passes from A to S (a focus) and back from S to A.

Hence the z-curve corresponding to the contour of the w-rectangle is the ellipse with two slits from the extremities of the major axis each to the nearer focus: the analytical relations shew that the two interiors correspond.

Ex. 6. Consider the correspondence implied by the relation

$$k^{-\frac{1}{2}}w=\operatorname{sn}\left(\frac{2K}{\pi}\sin^{-1}\frac{z}{c}\right)=\operatorname{sn}\left(\frac{2K}{\pi}\zeta\right).$$

From Ex. 2, it follows that the interior of a w-circle, centre the origin and radius unity, corresponds to the interior of the ζ-rectangle bounded by $x=\pm\frac{1}{2}\pi$, $y=\pm\frac{\pi K'}{4K}$, provided two diametral slits be made in the w-circle along the axis of x to distances $1-k^{\frac{1}{2}}$ from the circumference; and, from Ex. 5, it follows that the same ζ-rectangle is transformed into the interior of the z-ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

where $a=c\cosh\frac{\pi K'}{4K}$ and $b=c\sinh\frac{\pi K'}{4K}$, provided two slits be made in the elliptical area along the major axis from the curve each to the nearer focus.

Thus, by means of the rectangle, the interiors of the slit w-circle and the slit z-ellipse are shewn to be conformal areas.

But the lines of the two slits are conformally equivalent by the above equation. For the elliptical slit on the positive side of the axis of x extends from $x=c$ to $x=c\cosh\lambda$, where $\lambda=\frac{\pi K'}{4K}$, and it has been described in both directions: we thus have

$$z=c\cosh\beta,$$

where β passes from 0 to λ and back from λ to 0. Hence

$$\sin^{-1}\frac{z}{c}=\sin^{-1}(\cosh\beta)=\tfrac{1}{2}\pi+i\beta,$$

so that the corresponding w-curve is given by

$$k^{-\frac{1}{2}}w=\operatorname{sn}\left(K+\frac{2K\beta i}{\pi}\right)=\frac{\operatorname{cn}\left(\frac{2K\beta i}{\pi}\right)}{\operatorname{dn}\left(\frac{2K\beta i}{\pi}\right)}.$$

Then, when β assumes its values, w passes from 1 to $k^{\frac{1}{2}}$ and back from $k^{\frac{1}{2}}$ to 1, that is, w describes the circular slit on the positive side of the axis of X.

Similarly for the two slits on the negative side of the axis of real quantities. Thus the two slits may be obliterated: and the whole interior of the w-circle can be represented on the interior of the z-ellipse.

From the equations defining a and b, it follows that

$$\left(\frac{a-b}{a+b}\right)^2=e^{-\frac{\pi K'}{K}}=q,$$

in the Jacobian notation; and $c^2=a^2-b^2$.

Combining the results of Ex. 1 and Ex. 6 we have the theorem* :—

The part of the z-plane, which lies outside the ellipse $x^2/a^2+y^2/b^2=1$, *is transformed into the interior of a w-circle, of radius unity and centre the origin, by the relation*

$$(a-b)\,w^2-2zw+(a+b)=0\,;$$

and the part of the z-plane, which lies inside the same ellipse, is transformed into the interior of the same w-circle by the relation

$$k^{-\frac{1}{2}}w=\operatorname{sn}\left[\frac{2K}{\pi}\sin^{-1}\{z\,(a^2-b^2)^{-\frac{1}{2}}\}\right],$$

where the Jacobian constant q which determines the constants of the elliptic functions, is given by

$$q=\left(\frac{a-b}{a+b}\right)^2.$$

Ex. 7. Consider the correspondence implied by the relation

$$(w+1)^2 z=4.$$

When w describes a circle, of radius unity and centre the origin, then $w=e^{\phi i}$: so that, if r and θ be the coordinates of z, we have

$$\frac{4}{r}(\cos\theta-i\sin\theta)=(1+e^{\phi i})^2,$$

or
$$\frac{2}{\sqrt{r}}\left(\cos\frac{\theta}{2}-i\sin\frac{\theta}{2}\right)=1+e^{\phi i}=1+\cos\phi+i\sin\phi.$$

Hence
$$\left(\frac{2}{\sqrt{r}}\cos\frac{\theta}{2}-1\right)^2+\frac{4}{r}\sin^2\frac{\theta}{2}=1,$$

that is,
$$r\cos^2\frac{\theta}{2}=1,$$

shewing that z then describes a parabola, having its focus at the origin and its latus rectum equal to 4.

Take curves outside the parabola given by

$$r=\mu^2\sec^2\frac{\theta}{2},$$

where μ is a constant $\geqslant 1$. Then

$$\frac{1}{\sqrt{r}}=\frac{1}{\mu}\cos\tfrac{1}{2}\theta,$$

so that
$$w+1=\frac{2}{\sqrt{r}}e^{-\frac{1}{2}\theta i}=\frac{2}{\mu}e^{-\frac{1}{2}\theta}\cos\tfrac{1}{2}\theta\,;$$

therefore
$$X+1=\frac{2}{\mu}\cos^2\tfrac{1}{2}\theta=\frac{1}{\mu}(1+\cos\theta),$$

$$Y=-\frac{1}{\mu}\sin\theta,$$

so that
$$\left(X+1-\frac{1}{\mu}\right)^2+Y^2=\frac{1}{\mu^2},$$

a series of circles touching at the point $X=-1$, $Y=0$, and (for μ varying from 1 to ∞) covering the whole of the interior of the w-circle, centre the origin and radius unity.

* Schwarz, *Ges. Werke*, t. ii, pp. 77, 78, 102—107, 141.

Hence, by means of the relation $(w+1)^2 z=4$, the exterior of the z-space bounded by the parabola is transformed into the interior of the w-space bounded by the circle.

Ex. 8. Consider the correspondence implied by the relation

$$w=\tan^2(\tfrac{1}{4}\pi z^{\frac{1}{2}}).$$

We have
$$\frac{1-w}{1+w}=\cos(\tfrac{1}{2}\pi z^{\frac{1}{2}})=\cos(\tfrac{1}{2}\pi r^{\frac{1}{2}}e^{\frac{1}{2}\theta i}),$$

so that, if $w+1=Re^{\Theta i}$, $u=\frac{1}{2}\pi r^{\frac{1}{2}}\cos\frac{1}{2}\theta$, $v=\frac{1}{2}\pi r^{\frac{1}{2}}\sin\frac{1}{2}\theta$, then

$$2R^{-1}\cos\Theta-1=\cos u\cosh v,$$
$$2R^{-1}\sin\Theta \quad =\sin u\sinh v.$$

The w-curves, corresponding to the confocal parabolas in the z-plane, are

$$\frac{(2\cos\Theta-R)^2}{\cos^2 u}-\frac{4\sin^2\Theta}{\sin^2 u}=R^2.$$

If $u<\frac{1}{2}\pi$, then $2R^{-1}\cos\Theta>1$, that is, $R<2\cos\Theta$; while, if $u>\frac{1}{2}\pi$, we have $R>2\cos\Theta$.

It thus appears that the z-space lying within the parabola $u=\frac{1}{2}\pi$, that is, $r\cos^2\frac{1}{2}\theta=1$, is transformed into the interior of a w-circle, centre the origin and radius unity, by means of the relation

$$w=\tan^2(\tfrac{1}{4}\pi z^{\frac{1}{2}}).$$

By the two relations* in Ex. 7 and Ex. 8, the spaces within and without the parabola are conformally represented on the interior of a circle.

Ex. 9. Consider the relation

$$z=\frac{i-w}{i+w};$$

then, if $z=x+iy$ and $w=X+iY$, we have

$$x+iy=\frac{1-X^2-Y^2+i\,2X}{X^2+(1+Y)^2}.$$

When w describes the whole of the axis of X from $-\infty$ to $+\infty$, so that we can take $X=\tan\phi$, $Y=0$, where ϕ varies from $-\dfrac{\pi}{2}$ to $+\dfrac{\pi}{2}$, we have $x=\cos 2\phi$, $y=\sin 2\phi$; and z describes the whole circumference of a circle, centre the origin and radius 1. For internal points of this circle $1-x^2-y^2$ is positive: it is equal to $4Y\div\{X^2+(1+Y)^2\}$, and therefore the positive half of the w-plane is the area conformal with the interior of the circle, of radius unity and centre the origin, in the z-plane.

Ex. 10. Again, consider a relation

$$w=\left(\frac{z-ic}{z+ic}\right)^2.$$

We have
$$X+iY=\frac{(x^2+y^2-c^2)^2-4c^2x^2+4icx(c^2-x^2-y^2)}{\{x^2+(y+c)^2\}^2},$$

so that
$$X=\frac{(x^2+y^2-2cx-c^2)(x^2+y^2+2cx-c^2)}{\{x^2+(y+c)^2\}^2},$$

$$Y=\frac{4cx(c^2-x^2-y^2)}{\{x^2+(y+c)^2\}^2}.$$

Let $x=0$, so that $Y=0$; then

$$X=\frac{(y^2-c^2)^2}{(y+c)^4}=\left(\frac{y-c}{y+c}\right)^2.$$

* Schwarz, *Ges. Werke*, t. ii, p. 146.

As z passes from A to B (where $OA=OB=c$), then y changes from $-c$ to $+c$, and X changes continuously from $+\infty$ to 0.

Let $x^2+y^2-c^2=0$, so that $Y=0$; then

$$X=\frac{-4x^2}{(2c+2y)^2}=-\frac{c-y}{c+y}=-\tan^2\tfrac{1}{2}\theta,$$

where $y=c\cos\theta$. Hence, as z describes the semi-circular arc BCA, the angle θ varies from 0 to π and X changes from 0 to $-\infty$.

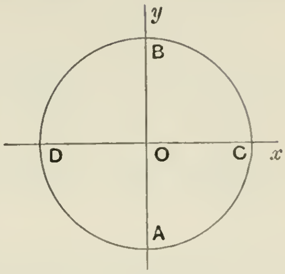

Fig. 88.

(The whole axis of X is the equivalent of $AOBCA$; and at the w-origin, corresponding to B, there is no sudden change of direction through $\frac{1}{2}\pi$. The result is apparently in contradiction to § 9: the explanation is due to the fact that $\frac{dw}{dz}=0$ at B, and the inference of § 9 cannot be made. Similarly for A where $\frac{dw}{dz}$ is infinite. See also Ex. 2.)

For any point lying within the z-semi-circle, both x and $c^2-x^2-y^2$ are positive, so that Y is positive. Hence by the relation

$$w=\left(\frac{z-ic}{z+ic}\right)^2,$$

the interior of the z-semi-circle is conformally represented on the positive half of the w-plane.

It is easy to infer that the positive half of the w-plane is the conformal equivalent of

(i) the interior of the semi-circle $ACBA$ by the relation $w=\left(\frac{z-ic}{z+ic}\right)^2$:

(ii) ....$CBDC$........................$w=\left(\frac{z+c}{z-c}\right)^2$:

(iii) ....$BDAB$ .... $w=\left(\frac{z+ic}{z-ic}\right)^2$:

(iv) ....$DACD$........................$w=\left(\frac{z-c}{z+c}\right)^2$.

And, by combination with the result of Ex. 9, it follows that the relation

$$w=\frac{i-\left(\frac{z-ic}{z+ic}\right)^2}{i+\left(\frac{z-ic}{z+ic}\right)^2}=i\,\frac{z^2-c^2+2cz}{z^2-c^2-2cz}$$

conformally represents the interior of the z-semi-circle $ACBA$ on the interior of the w-circle, radius unity and centre the origin.

Similarly for the other cases.

Ex. 11. Find a figure in the z-plane, the area of which is conformally represented on the positive half of the w-plane by

(i) $w=z^n$, (ii) $w=\left(\frac{z-ic}{z+ic}\right)^n$.

Ex. 12. Consider the relation

$$w=ae^{iz}:$$

then $$X=ae^{-y}\cos x,\quad Y=ae^{-y}\sin x.$$

The curves corresponding to y=constant are concentric circumferences; those corresponding to x=constant are concurrent straight lines.

As x ranges from 0 to $\frac{1}{2}\pi$, both X and Y are positive; for a given value of x between these limits, each of them ranges from 0 to ∞, as y ranges from ∞ to $-\infty$. As x ranges from $\frac{1}{2}\pi$ to π, X is negative and Y is positive; for a given value of x between these limits, $-X$ and Y range from 0 to ∞, as y ranges from ∞ to $-\infty$.

Hence the portion of the z-plane lying between $y=-\infty$, $y=\infty$, $x=0$, $x=\pi$, that is, a rectangular strip of finite breadth and infinite length, is conformally represented by the relation

$$w = ae^{iz}$$

on the positive half of the w-plane. Combining this result with that in Ex. 9, we see that the same strip is conformally represented on the area of a w-circle, centre the origin and radius a, by means of the relation

$$\frac{w-1}{w+1} = aie^{iz}.$$

Note. It may be convenient to restate the various instances of areas in the z-plane, bounded by simple curves, which can be conformally represented on the area of a circle in the w-plane:

(i) The positive half of the z-plane; Ex. 9.

(ii) An infinite strip of finite breadth; Ex. 9, Ex. 12.

(iii) Area without an ellipse; Ex. 1.

(iv) Area within an ellipse; Ex. 6.

(v) Area without a parabola; Ex. 7.

(vi) Area within a parabola; Ex. 8.

(vii) Area within a rectangle; Ex. 2.

(viii) As will be seen, in § 258, any circle changes into itself by a proper homographic relation.

Ex. 13. Consider the correspondence implied by the relation

$$z = \left(\frac{1-w^3}{1+w^3}\right)^2.$$

Then we have two values of w^3, say w_1^3, w_2^3, where

$$w_1^3 = \frac{1-z^{\frac{1}{2}}}{1+z^{\frac{1}{2}}}, \qquad w_2^3 = \frac{1+z^{\frac{1}{2}}}{1-z^{\frac{1}{2}}}.$$

Let z describe the axis of x, so that $z=x$.

When $0<x<1$, then w_1^3 is real and less than unity and w_2^3 is real and greater than unity. Hence drawing a circle in the w-plane, centre the origin and radius 1, and six lines as diameters making angles of $\frac{1}{3}\pi$ with one another, and denoting a cube root of 1 by α, then, as z passes from 0 to 1 along the axis of x,

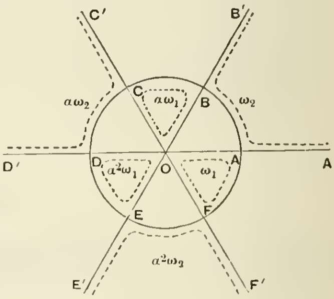

Fig. 89.

w_1 passes from A to O,

w_2 .... A to A' (at infinity),

αw_1 .... C to O,

αw_2 .... C to C' (at infinity),

$\alpha^2 w_1$ .... E to O,

$\alpha^2 w_2$ .... E to E' (at infinity).

When $1<x<\infty$, then $w_1{}^3$ is a real quantity changing continuously from 0 to -1, and $w_2{}^3$ is a real quantity changing continuously from $-\infty$ to -1. As z passes from 1 to ∞ along the positive part of axis of X,

w_1 passes from O to F,
w_2 .... B' (at infinity) to B,
aw_1 .... O to B,
aw_2 .... D' (at infinity) to D,
a^2w_1 .... O to D,
a^2w_2 .... F' (at infinity) to F.

Hence, as z describes the whole of the positive part of the axis of x, the branches of w describe the whole of the three lines $A'D'$, $B'E'$, $C'F'$.

When x is negative, we can take $x=-\tan^2\phi$, so that ϕ varies from 0 to $\frac{1}{2}\pi$. Then

$$w_1{}^3=\frac{1-i\tan\phi}{1+i\tan\phi}=e^{-2\phi i};$$

so that, as z passes from 0 to $-\infty$, w_1 describes the arc of the circle from A to F, aw_1 the arc from C to B, and a^2w_1 the arc from E to D. And then

$$w_2{}^3=e^{2\phi i},$$

so that w_2 describes the arc of the circle from A to B, aw_2 the arc from C to D, and a^2w_2 the arc from E to F. Hence, as z describes the whole of the negative part of the axis of x, the branches of w describe the whole of the circumference.

As z describes a line parallel to the axis of x and very near it on the positive side, the paths traced by the branches are the dotted lines in the figures; the six divisions, in which the symbols are placed, are the conformal representations by the six branches of w of the positive half of the z-plane*.

Ex. 14. When the variables are connected† by a relation

$$w=\frac{c^{m+1}\phi(z)}{z^m\phi_0\left(\frac{c^2}{z}\right)},$$

where ϕ_0 is the function which in coefficients is conjugate to ϕ, then the z-circumference, centre the origin and radius c, is transformed into the w-circumference, centre the origin and radius c.

Taking w_0 and z_0 as the conjugate variables, we have

$$w_0=\frac{c^{m+1}\phi_0(z_0)}{z_0{}^m\phi\left(\frac{c^2}{z_0}\right)},$$

so that

$$ww_0=\frac{c^{2m+2}}{z_0{}^mz^m}\frac{\phi(z)}{\phi\left(\frac{c^2}{z_0}\right)}\frac{\phi_0(z_0)}{\phi_0\left(\frac{c^2}{z}\right)}.$$

Now if z describe the circumference of a circle, centre the origin and radius c, we have

$$z=ce^{\theta i},\quad z_0=ce^{-\theta i},\quad zz_0=c^2,$$

so that

$$ww_0=c^2,$$

shewing that w describes the circumference of a circle, centre the origin and radius c.

* Cayley, *Camb. Phil. Trans.*, vol. xiii, (1880), pp. 30, 31.

† Cayley, *Crelle*, t. cvii, (1891), pp. 262—277.

To determine whether the internal area of the z-circumference corresponds to the internal area of the w-circumference, we take $zz_0=c^2-\epsilon$, where ϵ is small. Then

$$\phi\left(\frac{c^2}{z_0}\right)=\phi\left(z+\frac{\epsilon}{z_0}\right)=\phi(z)+\frac{\epsilon}{z_0}\phi'(z),$$

$$\phi_0\left(\frac{c^2}{z}\right)=\phi_0\left(z_0+\frac{\epsilon}{z}\right)=\phi_0(z_0)+\frac{\epsilon}{z}\phi_0'(z_0);$$

therefore

$$ww_0=c^2\left(1+\frac{m\epsilon}{c^2}\right)\left\{1-\frac{\epsilon}{z_0}\frac{\phi'(z)}{\phi(z)}\right\}\left\{1-\frac{\epsilon}{z}\frac{\phi_0'(z_0)}{\phi_0(z_0)}\right\}$$

$$=c^2+c^2\epsilon\left\{\frac{m}{c^2}-\frac{1}{z_0}\frac{\phi'(z)}{\phi(z)}-\frac{1}{z}\frac{\phi_0'(z)}{\phi(z)}\right\}$$

$$=c^2+\epsilon\left\{m-z\frac{\phi'(z)}{\phi(z)}-z_0\frac{\phi'(z_0)}{\phi(z_0)}\right\},$$

so that the interior of the z-circumference finds its conformal correspondent in the interior or in the exterior of the w-circumference according as

$$m < \text{or} > z\frac{\phi'(z)}{\phi(z)}+z_0\frac{\phi'(z_0)}{\phi(z_0)},$$

taken along the circumference.

The simplest case is that in which $\phi(z)$ is of degree m, so that it can be resolved into m factors, say $\phi(z)=A(z-\alpha)(z-\beta)\ldots(z-\theta)$: then

$$\phi_0\left(\frac{c^2}{z}\right)=A_0\left(\frac{c^2}{z}-\alpha_0\right)\left(\frac{c^2}{z}-\beta_0\right)\ldots\left(\frac{c^2}{z}-\theta_0\right),$$

and

$$w=\frac{A}{A_0c^{m-1}}\frac{(z-\alpha)(z-\beta)\ldots(z-\theta)}{\left(1-\frac{\alpha_0}{c^2}z\right)\left(1-\frac{\beta_0}{c^2}z\right)\ldots\left(1-\frac{\theta_0}{c^2}z\right)}.$$

But the converse of the result obtained—that to the w-circumference there corresponds the z-circumference—is not complete unless the correspondence is (1, 1). Other curves which are real—they may be, but are not necessarily, circles—and imaginary curves enter into the complete analytical representation on the z-plane corresponding to the w-circumference, of centre the origin and radius c on the w-plane.

Ex. 15. Discuss the z-curves corresponding to $|w|=1$, determined by

$$w=\frac{z(z-\sqrt{2})}{1-\sqrt{2}z}. \qquad \text{(Cayley.)}$$

Ex. 16. Consider the relation

$$w=\frac{4}{27}\frac{(z^2-z+1)^3}{(z^2-z)^2}.$$

We have

$$w-w_0=\frac{4}{27}\left\{\frac{(z^2-z+1)^3}{(z^2-z)^2}-\frac{(z_0^2-z_0+1)^3}{(z_0^2-z_0)^2}\right\}.$$

The function on the right-hand side, being connected with the expressions for the six anharmonic ratios of four points in terms of any one ratio, vanishes for

$$z=z_0,\ \frac{1}{z_0},\ 1-z_0,\ \frac{1}{1-z_0},\ \frac{z_0}{z_0-1},\ \frac{z_0-1}{z_0},$$

so that

$$w-w_0=\frac{4}{27}\frac{(z-z_0)(zz_0-1)(z+z_0-1)\{z(z_0-1)-z_0\}(zz_0-z_0+1)\{z(z_0-1)+1\}}{(z^2-z)^2(z_0^2-z_0)^2}.$$

Hence, taking

$$w=X+iY,\quad z=x+iy,$$

we have

$$2iY=\frac{4}{27}\frac{2iy(x^2+y^2-1)(2x-1)(x^2+y^2-2x)\{(x^2+y^2-x+1)^2+y^2\}}{(x^2+y^2)^2(x^2+y^2-2x+1)^2}.$$

Hence it appears that, when $Y=0$, so that w traces the axis of real quantities in its own plane, the z-variable traces the curves

$$y=0, \quad x^2+y^2-1=0,$$
$$2x-1=0, \quad x^2+y^2-2x=0,$$

that is, two straight lines and two circles in its own plane.

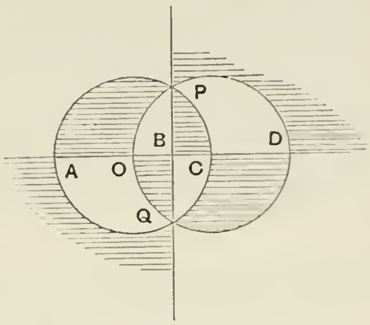

Fig. 90.

In order to determine the parts of the z-plane that correspond to the positive part of the w-plane, it is sufficient to take Y equal to a small positive quantity and determine the corresponding sign of y. Let

$$y=\mu Y,$$

where Y (and therefore y) is small: then, to a first approximation,

$$\mu=\frac{27}{4}\frac{x^3(x-1)^3}{(2x-1)(x+1)(x-2)(x^2-x+1)^2},$$

and the sign of μ determines whether the part on the positive or negative side of the axis of x is to be taken.

When $x<-1$, μ is negative; z lies below the axis of x. When x is in AO, so that $x>-1<0$, μ is positive; z lies above. When x is in OB, so that $x>0<\frac{1}{2}$, μ is negative; z lies below. When x is in BC, so that $x>\frac{1}{2}<1$, μ is positive; z lies above. When x is in CD, so that $x>1<2$, μ is negative; z lies below. And, lastly, when x is beyond D, so that $x>2$, μ is positive and z lies above the axis of real quantities. The parts are indicated by the shading in fig. 90.

It is easy to see that $w=0$, for $z=P, Q$; that $w=1$, for $z=A, B, D$; and that $w=\infty$, for $z=O, C$. The zero value of w is of triple occurrence for each of the points P and Q; the unit-value and the infinite value are of double occurrence for their respective points*.

Note. It is easy to see that figures 89 and 90 are two different stereographic projections of the same configuration of lines on a sphere (§ 277, I, $n=3$), so that the relations in Ex. 13 and Ex. 16 may be regarded as equivalent.

Ex. 17. Find, in the same way, the curves in the z-plane, which are the conformal representation of the axis of X in the w-plane by the relation†

$$w=\frac{(z^4+z^{-4}+14)^3}{108(z^4+z^{-4}-2)^2}.$$

Ex. 18. Shew that, by the relation

$$w^2=1+e^z,$$

the lines, $x=$constant in the z-plane, are transformed into a series of confocal lemniscates in the w-plane; and that, by the relation

$$z^2(c^2+w^2-1)=cw^2,$$

where c is a real positive constant greater than unity, the interior of a z-circle, centre the origin and radius unity, is transformed into the interior of the lemniscate $RR'=c^2$ in the w-plane, where R and R' are the distances of a point from the foci (1, 0) and (−1, 0). (Weber.)

* See Klein-Fricke, vol. i, p. 70. † See Klein-Fricke, vol. i, p. 75.

258. The preceding examples* may be sufficient to indicate the kind of correlation between two planes or assigned portions of two planes that is provided in the conformal representation determined by a relation $\phi(w, z) = 0$ connecting the complex variables of the planes. We shall consider only one more instance; it is at once the simplest and functionally the most important of all†. The equation, which characterises it, is linear in both variables; and so it can be brought into the form

$$w = \frac{az + b}{cz + d},$$

where a, b, c, d are constants: it is called a *homographic transformation*, sometimes a *homographic* or a *linear substitution*.

Taking first the more limited form

$$w = \frac{\mu}{z},$$

and writing $w = Re^{i\Theta}$, $z = re^{i\theta}$, $\mu = k^2 e^{2\gamma i}$, we have

$$Rr = k^2,\quad \Theta + \theta = 2\gamma, \text{ that is, } \Theta - \gamma = \gamma - \theta,$$

and therefore the new w-locus will be obtained from the old z-locus by turning the plane through two right angles round the line γ through the origin, and inverting the displaced locus relative to the origin. The first of these processes is a reflexion in the line γ; and therefore the geometrical change represented by $wz = \mu$ is a combination of reflexion and inversion.

A straight line not through the origin and a circle through the origin are corresponding inverses; a circle not through the origin inverts into another circle not through the origin and it may invert into itself; and so on.

Taking now the general form, we have

$$w - \frac{a}{c} = -\frac{ad - bc}{c^2\left(z + \dfrac{d}{c}\right)},$$

or transforming the origins to the points $\frac{a}{c}$ and $-\frac{d}{c}$ in the w- and the z-planes respectively, and denoting $-\frac{ad - bc}{c^2}$ by μ, we have $WZ = \mu$, that is, the former case. Hence, to find the w-locus which is obtained through the transformation of a z-locus by the general relation, we must transfer the origin to $-\frac{d}{c}$, turn the plane through two right angles round a line through the new origin

* Many others will be found in Holzmüller's treatise, already cited, which contains ample references to the literature of the subject.

† For the succeeding properties, see Klein, *Math. Ann.*, t. xiv, pp. 120—124, ib., t. xxi, pp. 170—173; Poincaré, *Acta Math.*, t. i, pp. 1—6; Klein-Fricke, *Elliptische Modulfunctionen*, vol. i, pp. 163 et seq. They are developed geometrically by Möbius, *Ges. Werke*, t. ii, pp. 189—204, 205—217, 243—314.

whose angular coordinate is $\frac{1}{2}$ arg. $\left(\frac{bc-ad}{c^2}\right)$, invert the locus in the displaced position with a constant of inversion equal to $\left|\frac{bc-ad}{c^2}\right|$, and then displace the origin to the point $-\frac{a}{c}$. Hence a circle will be changed into a circle by a homographic transformation unless it be changed into a straight line; and a straight line will be changed into a circle by a homographic transformation unless it be changed into a straight line.

The result can also be obtained analytically as follows; the formulæ relating to the circle will be useful subsequently.

A circle, whose centre is the point (α, β) and whose radius is r, can be expressed in the form

$$(z-\alpha-\beta i)(z_0-\alpha+\beta i)=r^2,$$

or

$$zz_0+\theta z+\theta_0 z_0+\gamma=0,$$

where $-\theta=\alpha-\beta i$, $-\theta_0=\alpha+\beta i$, $\gamma=\theta\theta_0-r^2$. Conversely, this equation represents a circle, when θ and θ_0 are conjugate imaginaries and γ is real; its centre is at the point $-\frac{1}{2}(\theta+\theta_0)$, $\frac{1}{2}i(\theta-\theta_0)$, and its radius is $(\theta\theta_0-\gamma)^{\frac{1}{2}}$.

When the circle is subjected to the homographic transformation

$$w=\frac{az+b}{cz+d},$$

we have

$$z=\frac{-dw+b}{cw-a} \text{ and therefore } z_0=\frac{-d_0w_0+b_0}{c_0w_0-a_0}.$$

Substituting these values, the relation between w and w_0 is

$$\delta' ww_0+\theta' w+\theta_0' w_0+\gamma'=0,$$

where

$$\begin{aligned}
\delta' &= \quad dd_0-\theta dc_0-\theta_0 cd_0+\gamma cc_0,\\
\theta' &= -b_0 d+\theta a_0 d+\theta_0 cb_0-\gamma ca_0,\\
\theta_0' &= -bd_0+\theta c_0 b+\theta_0 ad_0-\gamma c_0 a,\\
\gamma' &= \quad bb_0-\theta a_0 b-\theta_0 ab_0+\gamma aa_0:
\end{aligned}$$

here δ' and γ' are real, and θ' and θ_0' are conjugate imaginaries; therefore the equation between w and w_0 represents a circle.

Ex. A circle, of radius r and centre at the point (e, f), in the z-plane is transformed into a circle in the w-plane, by the homographic substitution

$$w=\frac{az+b}{cz+d};$$

shew that the radius of the new circle is

$$\frac{r}{\Delta}\left|\frac{ad-bc}{c^2}\right|,$$

where

$$\Delta=(\sigma\cos\beta+e)^2+(\sigma\sin\beta+f)^2-r^2,$$

and σ, β are the modulus and the argument respectively of $\frac{d}{c}$. Find the coordinates of the centre of the w-circle.

Moreover, since there are three independent constants in the general homographic transformation, they may be chosen so as to transform any three assigned z-points into any three assigned w-points. And three points on a circle uniquely determine a circle: hence *any circle can be transformed into any other circle (or into itself) by a properly chosen homographic transformation.* The choice of transformation can be made in an infinite number of ways: for three points on the circle can be chosen in an infinite number of ways.

A relation which changes the three points z_1, z_2, z_3 into the three points w_1, w_2, w_3 is evidently

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_2)(w_1-w_3)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_2)(z_1-z_3)}.$$

Hence this equation, or any one of the other five forms of changing the three points z_1, z_2, z_3 into the three points w_1, w_2, w_3 in any order of correspondence, is a homographic transformation changing the circle through z_1, z_2, z_3 into the circle through w_1, w_2, w_3.

It has been seen that a transformation of the form $w = f(z)$ does not affect angles: so that two circles cutting at any angle are transformed by $w = \frac{az+b}{cz+d}$ into two others cutting at the same angle. Hence* *a plane crescent, of any angle, can be transformed into any other crescent, of the same angle.*

The expression of homographic transformations can be modified, so as to exhibit a form which is important for such transformations as are periodic.

If we assume that w and z are two points in the same plane, then there will in general be two different points which are unaltered by the transformation; they are called the *fixed* (or *double*) points of the transformation. These fixed points are evidently given by the quadratic equation

$$u = \frac{au+b}{cu+d},$$

that is,
$$cu^2 - (a-d)u - b = 0.$$

Let the points be α and β, and let M denote $(d-a)^2 + 4bc$; then

$$2c\alpha = a - d + M^{\frac{1}{2}}, \quad 2c\beta = a - d - M^{\frac{1}{2}}.$$

If, then, the points be distinct, we have

$$\frac{w-\alpha}{w-\beta} = \frac{(z-\alpha)(a-c\alpha)}{(z-\beta)(a-c\beta)} = K\frac{z-\alpha}{z-\beta},$$

* Kirchhoff, *Vorlesungen über mathematische Physik*, i, p. 286.

where
$$K = \frac{a - c\alpha}{a - c\beta} = \frac{a + d - M^{\frac{1}{2}}}{a + d + M^{\frac{1}{2}}},$$

and therefore
$$\left(\sqrt{K} + \frac{1}{\sqrt{K}}\right)^2 = \frac{(a+d)^2}{ad - bc}.$$

The quantity K is called the *multiplier* of the substitution.

If there be a z-curve in the plane passing through α, the w-curve which arises from it through the linear substitution also passes through α. To find the angle at which the z-curve and the w-curve intersect, we have $w = \alpha + \delta w$, $z = \alpha + \delta z$: and then
$$\delta w = K \delta z,$$
so that the inclination of the tangents at the point is the argument of K. Similarly, if a z-curve pass through β, the angle between the tangents to the w curve and the z-curve is supplementary to the argument of K.

The form of the substitution now obtained evidently admits of reapplication; if z_n be the variable after the substitution has been applied n times, (so that $z_0 = z$, $z_1 = w$), we have
$$\frac{z_n - \alpha}{z_n - \beta} = K^n \frac{z - \alpha}{z - \beta}.$$

The condition that the transformation should be *periodic* of the nth order is that $z_n = z$ and therefore that $K^n = 1$; hence
$$(a + d)^2 = 4\,(ad - bc) \cos^2 \frac{s\pi}{n},$$
where s is any integer different from zero and prime to n; K cannot be purely real, and, in general, M is not a real positive quantity. The various substitutions that arise through different values of s are so related that, if points $z_1, z_2, \ldots, z_n$ be given by the continued application of one substitution through its period, the same points are given in a different cyclical order by the continued application of the other substitution through its period.

Ex. 1. The value of z_n has been given by Cayley in the form
$$\frac{(K^{n+1} - 1)(az + b) + (K^n - K)(-dz + b)}{(K^{n+1} - 1)(cz + d) + (K^n - K)(cz - a)}:$$
obtain this expression.

Ex. 2. Periodic substitutions can be applied, in connection with Kirchhoff's result that a plane crescent can be transformed into another plane crescent of the same angle; the plane can be divided into a limited number of regions when the angle of the crescent is commensurable with π.

Let $ACBDA$ be a circle of radius unity, having its centre at the origin: draw the diameter AB along the axis of y. Then the semi-circle ACB can be regarded as a plane crescent, of angle $\frac{1}{2}\pi$; and the semi-circle ABD as another, of the same angle. Hence they can be transformed into one another.

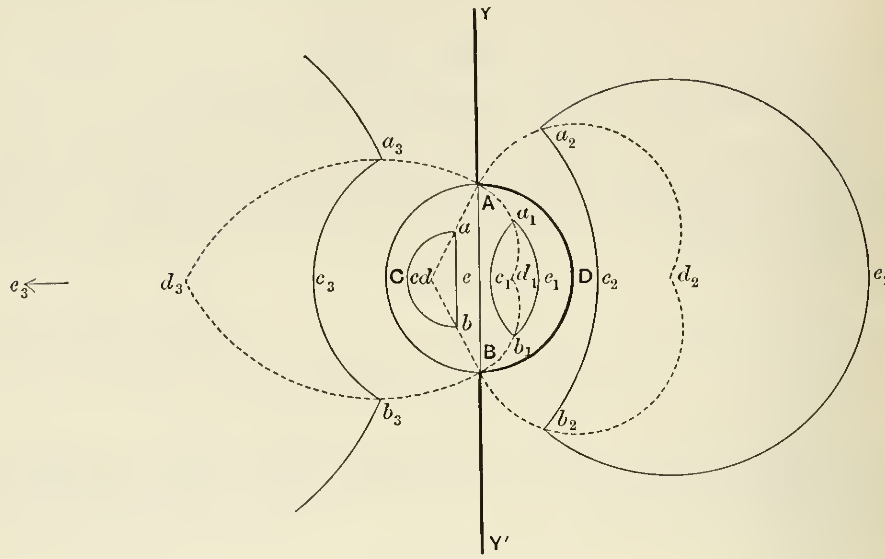

Fig. 91.

We can effect the transformation most simply by taking A $(=i)$ and $B(=-i)$ as the fixed points of the substitution, which then has the form

$$\frac{w-i}{w+i}=K\frac{z-i}{z+i}.$$

The line AB for the w-curve is transformed from the z-circular arc ACB: these curves cut at an angle $\frac{1}{2}\pi$, which is therefore the argument of K. Considerations of symmetry shew that the z-point C on the axis of x can be transformed into the w-origin, so that

$$-1=K\frac{-1-i}{-1+i},$$

whence $K=i$, so that the substitution is

$$\frac{w-i}{w+i}=i\frac{z-i}{z+i}.$$

It is periodic of order 4, as might be expected: when simplified, it takes the form

$$w=\frac{1+z}{1-z}.$$

The above figure shews the effect of repeated application of the substitution through a period. The first application changes the interior of $ACBA$ into the interior of $ABDA$: by a second application, the latter area is transformed into the area on the positive side of the axis of y lying without the semi-circle ADB; by a third application, the latter area is transformed into the area on the negative side of the axis of y lying without the semi-circle ACB; and by a fourth application, completing the period, the latter area is transformed into the interior of $ACBA$, the initial area.

The other lines in the figure correspond in the respective areas.

Ex. 3. Shew that, if the plane crescent of the preceding example have an angle of $\frac{1}{n}\pi$ instead of $\frac{1}{2}\pi$ but still have $+i$ and $-i$ for its angular points, then the substitution

$$w=\frac{z+t}{1-tz},$$

where t denotes $\tan\frac{\pi}{2n}$, is a periodic substitution of order $2n$ which, by repeated application through a period to the area of the crescent, divides the plane into $2n$ regions, all but two of which must be crescent in form. Under what circumstances will all the $2n$ regions be crescent in form?

Note. The formula in the text may be regarded as giving the nth *power* of a substitution. The form of the substitution obtained is equally effective for giving the nth *root* of a substitution: all that is necessary is to express K in the form $\rho e^{\theta i}$, and the nth root is then

$$\frac{z_{\frac{1}{n}}-a}{z_{\frac{1}{n}}-\beta}=\rho^{\frac{1}{n}}e^{\frac{1}{n}\theta i}\frac{z-a}{z-\beta}.$$

259. Homographic substitutions are divided into various classes, according to the fixed points and the value of the multiplier. As the quantities a, b, c, d can be modified, by the association of an arbitrary factor with each of them without altering the substitution, we may assume that $ad-bc=1$; we shall suppose that all substitutions are taken in such a form that their coefficients satisfy this relation. Figures which, by them, are transformed into one another are called *congruent* figures.

If the fixed points of the substitution coincide, it is called* a *parabolic* substitution.

There are three classes of substitutions, which have distinct fixed points.

If the multiplier be a real positive quantity, the substitution is called *hyperbolic.*

If the multiplier have its modulus equal to unity and its argument different from zero, it is called *elliptic.*

If the multiplier have its modulus different from unity and its argument different from zero, it is called *loxodromic.*

These definitions apply to all substitutions, whether their coefficients be real or be complex constants; when we consider only those substitutions, which have real coefficients, only the first three classes occur. Such substitutions are often called *real.*

The quadratic equation, which determines the common points of a real substitution, has its coefficients real; according as the roots of the quadratic are imaginary, equal, or real, the real substitution will be proved to be *elliptic, parabolic,* or *hyperbolic* respectively. For all of these, we take

$$z+\frac{d}{c}=x+iy,\qquad w-\frac{a}{c}=X+iY,$$

* All these names are due to Klein: l. c., p. 512, *note.*

which imply a transference of the respective origins along the respective axes of real quantity; and then

$$X+iY=-\frac{ad-bc}{c^2}\frac{1}{x+iy}$$

$$=-\frac{x-iy}{c^2(x^2+y^2)},$$

so that

$$\frac{Y}{y}=\frac{1}{c^2(x^2+y^2)}.$$

The axes of x and of X have been unaltered by any of the changes made in the substitution; and Y, y have the same sign and vanish together; hence the effect of a real transformation is to conserve the axis of real quantities, by transforming the half of the z-plane above the axis of x into the half of the w-plane above the axis of X.

A real transformation, which changes z into w, also changes z_0 into w_0 (these being conjugate complexes). A circle, having its centre on the axis of x and passing through α, β, passes through α_0, β_0 also: hence a transformation, which changes a circle through α, β with its centre on the axis of x into one through γ, δ with its centre on the axis of X, is

$$\frac{z-\alpha}{z-\alpha_0}\cdot\frac{\beta-\alpha_0}{\beta-\alpha}=\frac{w-\gamma}{w-\gamma_0}\cdot\frac{\delta-\gamma_0}{\delta-\gamma}.$$

Ex. 1. Shew that, if this circle, through a, β, a_0, β_0, cut the axis of x in h and k, where h lies in $\beta\beta_0$ and k in aa_0, and if $[a\beta]$ denote $\frac{a-h}{a-k}\cdot\frac{\beta-k}{\beta-h}$, a real quantity greater than 1, then

$$\frac{(a-a_0)(\beta-\beta_0)}{(a-\beta_0)(\beta-a_0)}=\frac{4[a\beta]}{\{1+[a\beta]\}^2}. \qquad \text{(Poincaré.)}$$

Ex. 2. Prove that the magnification at any point, by a real substitution, is Y/y. (Poincaré.)

Ex. 3. Any z-circle, having its centre on the axis of x, is transformed by a real substitution into a w-circle, having its centre on the axis of X.

Let the classes of real substitutions be considered in order.

(i) For real *parabolic* substitutions, the quadratic has equal roots: let their common value be α, necessarily a real quantity, so that the fixed points of the substitution coalesce into one on the axis of x. The quantity M is then zero, so that $(d+a)^2=4$. We may, without loss of generality, take $d+a=2$. If both origins be removed to the point α, then, in the new form, zero is a repeated root of the quadratic, so that $b=0$, and $a-d=0$. Hence $a=d=1$, and the real substitution is

$$w=\frac{z}{cz+1},$$

that is,*

$$\frac{1}{w}=\frac{1}{z}+c.$$

* If the origins be not removed to the point α, the form is $\frac{1}{w-\alpha}=\frac{1}{z-\alpha}+c$.

The equations of transformation of real coordinates are

$$\frac{x}{X - c(X^2 + Y^2)} = \frac{y}{Y} = \frac{x^2 + y^2}{X^2 + Y^2}$$
$$= \frac{1}{(1 - cX)^2 + c^2 Y^2}.$$

Ex. 1. A z-circle passing through the origin is transformed, by a real parabolic substitution having the origin for its common point, into a w-circle, passing through the origin and touching the z-circle: and a z-circle, touching the axis of x at the origin, is transformed into itself.

Ex. 2. Let A be a circle touching the axis of x at the origin: and let c_0 be the extremity of its diameter through the origin. Let a real parabolic substitution, having the origin for its common point, transform c_0 into c_1, c_1 into c_2, c_2 into c_3, and so on: all these points being on the circumference of A.

Prove that the radii of the successive circles, which have their centres on the axis of x and pass through the origin and c_1, the origin and c_2, ... respectively, are in harmonic progression, and that, if these circles be denoted by C_1, C_2, ..., then C_k is the locus of all points c_k arising through different initial circumferences A.

Ex. 3. What is the effect of the inverse substitution, applied as in Ex. 2?

Ex. 4. Shew that, if a curve of finite length be drawn so as to be nowhere infinitesimally near the axis of x, it can cut only a finite number of the circles C in Ex. 2.

(*Note.* All these results are due to Poincaré.)

(ii) For real *elliptic* substitutions, α and β are conjugate complexes; hence M is negative, so that

$$(d - a)^2 + 4bc < 0,$$

or
$$(d + a)^2 < 4(ad - bc) < 4.$$

The value of K, by using the relation $ad - bc = 1$, is

$$K = \tfrac{1}{2}\left[(a + d)^2 - 2 - i(a + d)\{4 - (a + d)^2\}^{\frac{1}{2}}\right].$$

It is easy to see that $|K| = 1$ and that its argument is $\cos^{-1}\{\frac{1}{2}(a + d)^2 - 1\}$, so that, if this angle be denoted by σ, we have

$$K = e^{\sigma i},$$

shewing that the substitution is elliptic.

It is evident that, if z describe a circle through α and β, its centre being therefore on the axis of x, then w also describes a circle through α and β cutting the z-circle at an angle σ. The two curves together make a plane crescent of angle σ having α, β for its angular points.

Ex. Shew that a real elliptic substitution transforms into itself any circumference, which has its centre on $\alpha\beta$ produced and cuts the line $\alpha\beta$ harmonically. (Poincaré.)

(iii) For real *hyperbolic* substitutions, the roots of the quadratic are real and different; hence the fixed points of the substitution are two (different) points on the axis of x. The quantity M is positive, so that

$$(a + d)^2 > 4:$$

we may evidently take $a + d > 2$. Moreover K is real and positive, shewing that the substitution is hyperbolic.

Taking one of the fixed points for origin and denoting by f the distance of the other, we have 0 and f as the roots of

$$u = \frac{au + b}{cu + d},$$

with the conditions $ad - bc = 1$, $a + d > 2$. Hence $b = 0$, $a - d = cf$, $ad = 1$, $K = \frac{a}{d}$; then K is greater or is less than 1 according as cf is positive or is negative. We shall take $K > 1$ as the normal case; and then the substitution is

$$w = \frac{az}{cz + d},$$

with $a > 1 > d$, $a + d > 2$, $ad = 1$.

Ex. 1. A z-curve is drawn through either of the fixed points of a real hyperbolic substitution: shew that the w-curve, into which it is changed by the substitution, touches the z-curve. Hence shew that any z-circle through the two fixed points of the substitution is transformed into itself.

Ex. 2. Let A be a circle through the origin and the point f; and let c_0 be the other extremity of its diameter through f. Let a real hyperbolic substitution, having the origin and f for its fixed points, transform c_0 into c_1, c_1 into c_2, c_2 into c_3, and so on: all these points being on the circumference of A.

Shew that the radius of a circle C_n, having its centre on the axis of x and passing through c_n and the origin, is

$$\tfrac{1}{2} f \frac{a^n}{a^n - d^n},$$

so that C_n is the locus of all the points c_n arising through different initial circumferences A. What is the limit towards which C_n tends as n becomes infinitely great?

Ex. 3. Apply the inverse substitution, as in Ex. 2, to obtain the corresponding result and the corresponding limit.

Ex. 4. Prove that a curve of finite length will meet an infinite number, or only a finite number, of the circles C_n, according as it meets or does not meet the circle having the line joining the common points of the substitution for diameter.

(*Note.* All these results are due to Poincaré.)

It follows from what precedes that no real substitution can be loxodromic; for, when the multiplier of a real substitution is not real, its modulus is unity.

It is not difficult to prove that when a substitution, with complex coefficients a, b, c, d, is parabolic, elliptic, or hyperbolic, then $a + d$ is either purely real or purely imaginary. In all other cases, the substitution is loxodromic.

Any loxodromic substitution can be expressed in the form

$$\frac{w-\alpha}{w-\beta} = K\frac{z-\alpha}{z-\beta}:$$

the coefficients of the quadratic determining α and β are generally not real, and the multiplier K, defined by

$$2K = (a+d)^2 - 2 - (a+d)\{(a+d)^2-4\}^{\frac{1}{2}},$$

is a complex quantity such that, if

$$K = \rho e^{i\omega},$$

where ρ and ω are real, then ρ is not equal to unity and ω is not zero.

260. Further, it is important to notice one property, possessed by elliptic substitutions and not by those of the other classes: viz. *an elliptic substitution is either periodic or infinitesimal.*

Any elliptic substitution of which α and β are the distinct fixed points, (they are conjugate imaginaries), can be put into the form

$$\frac{w-\alpha}{w-\beta} = K\frac{z-\alpha}{z-\beta},$$

where $|K| = 1$: let $K = e^{\theta i}$. Then the mth power of the substitution is

$$\frac{w_m-\alpha}{w_m-\beta} = \frac{z-\alpha}{z-\beta}e^{m\theta i}.$$

Now if θ be commensurable with 2π, so that

$$\theta/2\pi = \lambda/\mu,$$

then, taking $m = \mu$, we have

$$\frac{w_\mu-\alpha}{w_\mu-\beta} = \frac{z-\alpha}{z-\beta},$$

that is, $$w_\mu = z,$$

or the substitution is periodic.

But if θ be not commensurable with 2π, then, by proper choice of m, the argument $m\theta$ can be made to differ from an integral multiple of 2π by a very small quantity. For we expand $\theta/2\pi$ as an infinite continued fraction: let p/q, p'/q' be two consecutive convergents, so that $p'q - pq' = \pm 1$. We have

$$\frac{\theta}{2\pi} = \frac{p}{q} + \lambda\left(\frac{p'}{q'} - \frac{p}{q}\right), \text{ where } 0 < \lambda < 1,$$

$$= \frac{p}{q} + \frac{\eta}{q^2},$$

where $|\eta| < 1$, that is, $$q\theta - 2p\pi = 2\pi\eta\frac{1}{q},$$

where η, being real, is numerically less than 1. Hence, taking $m = q$, we have

$$\frac{w_q - \alpha}{w_q - \beta} = \frac{z - \alpha}{z - \beta} e^{\frac{2\pi\eta i}{q}} = \frac{z - \alpha}{z - \beta} \left[1 + \frac{2\pi\eta}{q} i + \dots \right],$$

where, by making q large, we can neglect all terms of the expansion after the second. Then

$$w_q - z = \frac{(z - \alpha)(z - \beta)}{\alpha - \beta} \frac{2\pi\eta}{q} i,$$

that is, by taking a series of values of q sufficiently large, we can, for *every value* of z find a value of w differing only by an infinitesimal amount from the value of z. Such a substitution is called *infinitesimal*; and thus the proposition is established.

But no parabolic and no hyperbolic substitution is infinitesimal in the sense of the definition. For in the case of a parabolic substitution we have

$$\frac{1}{w_q - \alpha} = \frac{1}{z - \alpha} + qc,$$

which does not, by a proper choice of q, give w_q nearly equal to z for every value of z: and a parabolic substitution is not substitutionally periodic, that is, it does not reproduce the variable after a certain number of applications. But it may lead to periodic functions of variables: thus $(z, z + \omega)$ is a parabolic substitution. And in the case of a hyperbolic substitution, we have

$$\frac{w_q - \alpha}{w_q - \beta} = \lambda^q \frac{z - \alpha}{z - \beta},$$

where λ is a real quantity which differs from 1. No value of q gives w_q nearly equal to z for every value of z: hence the substitution is not infinitesimal. And it is not substitutionally periodic.

Similarly, a loxodromic substitution is not periodic, and is not infinitesimal.

Hence it follows that, in dealing with groups of substitutions of the kind above indicated, viz. discontinuous, *all the elliptic transformations which occur must be substitutionally periodic:* for all other elliptic transformations are infinitesimal. It is easy to see, from the above equations, that the effect of an unlimited repetition of a parabolic substitution is to make the variable ultimately coincide with the fixed point of the substitution; and that the effect of an unlimited repetition of a hyperbolic substitution is to make the variable ultimately coincide with one of the fixed points of the substitution. These common points are called the essential singularities of the respective substitutions.

261. It has been proved (§ 258) that a linear relation between two variables can be geometrically represented as an inversion with regard to a circle, followed by a reflexion at a straight line. The linear relation can be associated with a double inversion by the following proposition*, due to Poincaré:—

When the inverse of a point P with regard to a circle is inverted with regard to another circle into a point Q, the complex variables of P and Q are connected by a lineo-linear relation.

Let z be the variable of P, u that of its inverse with regard to the first circle of centre f and radius r; let w be the variable of Q, and let the second circle have its centre at g and its radius s. Then, since inversion leaves the vectorial angles unaltered, we have

$$(u-f)(z_0-f_0)=r^2$$

for the first inversion, and

$$(w-g)(u_0-g_0)=s^2$$

for the second. From the former, it follows that

$$\frac{r^2}{z-f}=u_0-f_0,$$

and therefore
$$-\frac{r^2}{z-f}+\frac{s^2}{w-g}=f_0-g_0,$$

leading to
$$w=\frac{\alpha z+\beta}{\gamma z+\delta},$$

where, when $\alpha\delta-\beta\gamma=1$, we have

$$\begin{aligned} rs\alpha &= g(f_0-g_0)+s^2, \\ rs\beta &= gr^2-fs^2-fg(f_0-g_0), \\ rs\gamma &= f_0-g_0, \\ rs\delta &= -f(f_0-g_0)+r^2. \end{aligned}$$

This proves the proposition.

Moreover, as the quantities f, g, r, s are limited by no relations, and as, on account of the relation $\alpha\delta-\beta\gamma=1$, there are substantially only three equations to determine them in terms of α, β, γ, δ, it follows at once that *the lineo-linear relation can be obtained in an infinite number of ways by a pair of inversions, and therefore in an infinite number of ways by an even number of inversions.*

Again, taking the two circles used in the above proof, we have

$$\begin{aligned} rs(\alpha+\delta\pm 2) &= (r\pm s)^2-(f-g)(f_0-g_0) \\ &= (r\pm s)^2-d^2, \end{aligned}$$

* *Acta Math.*, t. iii, (1883), p. 51.

where d is the distance between the centres of the circles. Hence $\alpha+\delta$ is real, and the substitution cannot be loxodromic. Moreover, if the circles touch, the substitution is parabolic; if they intersect, it is elliptic; if they do not intersect, it is hyperbolic.

Eliminating r and s between the equations which determine α, β, γ, δ, we find

$$g=\frac{\alpha f+\beta}{\gamma f+\delta},$$

so that, when one centre is chosen arbitrarily, the other centre is connected with it by the linear substitution*.

Ex. 1. Shew that, if f and g lie on the axis of real quantities, so that the substitution is real, then

$$r^2=(f-\lambda)(f-\mu), \qquad s^2=(g-\lambda)(g-\mu),$$

where λ and μ are the fixed points of the substitution.

Hence prove that, if two real substitutions be given, it is generally possible to determine three circles 1, 2, 3 such that the substitutions are equivalent to successive inversions at 1 and 2 and at 1 and 3 respectively. Discuss the reality of these circles.

(Burnside.)

Ex. 2. Shew that, if a loxodromic substitution be represented in the preceding geometrical manner, at least four inversions are necessary. (Burnside.)

This geometrical aspect of the lineo-linear relation as a double inversion will be found convenient, when the relation is generalised from a connection between the variables of two points in a plane into a connection between the variables of two points in space.

* Burnside, *Mess. of Math.*, vol. xx, (1891), pp. 163—166.

CHAPTER XX.

Conformal Representation: General Theory.

262. In Gauss's solution of the problem of the conformal representation of surfaces, there is a want of determinateness. On the one hand, there is an element arbitrary in character, viz., the form of the function; on the other hand, no limitation to any part of either surface, as an area to be represented, has been assigned. And when, in particular, the solution is applied to two planes, then, corresponding to any curve given in one of the planes, a curve or curves in the other can be obtained, partially dependent on the form of functional relation assumed, different curves being obtained for different forms of functional relation.

But now a converse question suggests itself. Suppose a curve given in the second plane: can a function be determined, so that this curve corresponds to the given curve in the first plane and at the same time the conformal similarity of the bounded areas is preserved, with unique correspondence of points within the respective areas? in fact, does the conformal correspondence of two arbitrarily assigned areas lead to conditions which can be satisfied by the possibilities contained in the arbitrariness of a functional relation? And, if the solution be possible, how far is it determinate?

An initial simplification can be made. If the areas in the planes, conformally similar, be T and R, and if there be an area S in a third plane conformally similar to T, then S and R are also conformally similar to one another, whatever S may be. Hence, choosing some form for S, it will be sufficient to investigate the question for T and that chosen form. The simplest of closed curves is the circle, which will therefore be taken as S: and the natural point within a circle to be taken as a point of reference is its centre.

Two further limitations will be made. It will be assumed that the plane surfaces are simply connected* and one-sheeted. And it will be assumed

* The conformal representation of multiply connected plane surfaces is considered by Schottky, *Crelle*, t. lxxxiii, (1877), pp. 300—351.

that the boundary of the area T is either an analytical curve* or is made up of portions of a finite number of analytical curves—a limitation that arises in connection with the proof of the existence-theorem. This limitation, initially assumed by Schwarz in his early investigations† on conformal representation of plane surfaces, is not necessary: and Schwarz himself has shewn‡ that the problem can be solved when the boundary of the area T is any closed convex curve in one sheet. The question is, however, sufficiently general for our purpose in the form adopted.

Then, with these limitations and assumptions, Riemann's theorem§ on the conformation of a given curve with some other curve is effectively as follows:—

Any simply connected part of a plane bounded by a curve T can always be conformally represented on the area of a circle, the two areas having their elements similar to one another; the centre of the circle can be made the homologue of any point O_0 within T, and any point on the circumference of the circle can be made the homologue of any point O' on the boundary of T; the conformal representation is then uniquely and completely determinate.

263. We may evidently take the radius of the circle to be unity, for a circle of any other radius can be obtained with similar properties merely by constant magnification. Let w be the variable for the plane of the circle, z the variable for the plane of the curve T; and let

$$\log w = t = m + ni.$$

Evidently n will be determined by m (save as to an additive constant), for $m + ni$ is a function of z: and therefore we need only to consider m.

At the centre of the circle the modulus of w is zero, that is, e^m is zero: hence *m must be $-\infty$ for the centre of the circle, that is, for (say) $z = z_0$ in T.*

At the boundary of the circle the modulus of w is unity, that is, e^m is unity: hence *m must be 0 along the circumference of the circle, that is, along the boundary of T.*

Moreover, the correspondence of points is, by hypothesis, unique for the areas considered: and therefore, as e^m and n are the polar coordinates of the point in the copy and as m is entirely real, m is a one-valued function, which within T is to be everywhere finite and continuous except only at the point z_0. Hence, so far as concerns m, the conditions are:—

(i) m must be the real part of some function of z:

(ii) m must be $-\infty$ at some arbitrary point z_0:

* A curve is said to be an analytical curve (§ 265) when the coordinates of any point on it can be expressed as an analytical function (§ 34) of a real parameter.

† *Crelle*, t. lxx, (1869), pp. 105—120.

‡ *Ges. Werke*, t. ii, pp. 108—132.

§ *Ges. Werke*, p. 40.

(iii) m must be 0 along the boundary of T:

(iv) for all points, except z_0, within T, m must be one-valued, finite and continuous.

Now since $m + ni = \log w = \log R + i\Theta$, the negatively infinite value of m at z_0 arises through the logarithm of a vanishing quantity; and therefore, in the vicinity of z_0, the condition (ii) will be satisfied by having some constant multiple of $\log (z - z_0)$ as the most important term in $m + ni$; and the rest of the expansion in the vicinity of z_0 can be expressed in the form $p(z - z_0)$, an integral rational series of positive powers of $z - z_0$, because m is to be finite and continuous. Hence, in the vicinity of z_0, we have

$$\log w = m + ni = \frac{1}{\lambda}\log (z - z_0) + p(z - z_0),$$

where λ is some constant. This includes the most general form: for the form of any other function for $m + ni$ is

$$\frac{1}{\lambda}\log \{(z - z_0)\, g(z - z_0)\} + P(z - z_0),$$

where g is any function not vanishing when $z = z_0$: and this form is easily expressed in the form adopted. Hence

$$w = (z - z_0)^{\frac{1}{\lambda}} e^{p(z-z_0)}.$$

Since w is one-valued, we must have λ the reciprocal of an integer; and since the area bounded by T is simply connected and one-sheeted we must have $z - z_0$ a one-valued function of w. Hence $\lambda = 1$; and therefore, in the vicinity of z_0,

$$w = (z - z_0)\, e^{p(z-z_0)},$$

a form which is not necessarily valid beyond the immediate vicinity of z_0, for $p(z - z_0)$ might be a diverging series at the boundary. Thus, assuming that $p(z - z_0)$ is 1 when $z = z_0$, we have, in the immediate vicinity of z_0,

$$m + ni = \log (z - z_0),$$

a form which satisfies the second of the above conditions.

It now appears that the quantity m must be determined by the conditions:

(i) it must be the real part of a function of z, that is, it must satisfy the equation $\nabla^2 m = 0$:

(ii) along the boundary of the curve T, it must have the value zero:

(iii) at all points, except z_0, in the area bounded by T, m must be uniform, finite and continuous: and, for points z in the immediate vicinity of z_0, it must be of the form $\log r$, where r is the distance from z to z_0.

When m is obtained, subject to these conditions, the variable w is thence determinate, being dependent on z in such a way as to make the area bounded by T conformally represented on the circle in the w-plane.

264. The investigations, connected with the proof of the existence-theorem, shewed that a function exists for any simply connected bounded area, if it satisfy the conditions, (1) of acquiring assigned values along the boundary, (2) of acquiring assigned infinities at specified points within the area, (3) of being everywhere, except at these specified points, uniform, finite, and continuous, together with its differential coefficients of the first and the second order, (4) of satisfying $\nabla^2 u = 0$ everywhere in the interior, except at the infinities. Such a function is uniquely determinate.

But the preceding conditions assigned to m are precisely the conditions which determine uniquely the existence of the function: hence the function m exists and is uniquely determinate. And thence the function w is determinate.

It thus appears that *any simply connected bounded area can be conformally represented on the area of a circle, with a unique correspondence of points in the areas, so that the centre of the circle can be made the homologue of an internal point of the bounded area.*

An assumption was made, in passing from the equation

$$w = (z - z_0)\, e^{p(z-z_0)}$$

to the equation which determines the infinity of m, viz. that, when $z = z_0$, the value of $p(z - z_0)$ is 1. If the value of $p(z - z_0)$ when $z = z_0$ be some other constant, then there is no substantial change in the conditions: instead of having the infinity of m actually equal to $\log|z - z_0|$, the new condition is that m is infinite in the same way as $\log|z - z_0|$, and then a constant factor must be associated with w. A constant factor may also arise through the circumstance that n is determined by m, save as to an additive constant, say γ: hence the form of $w = e^{m+ni}$ will be

$$w = A'e^{\gamma i}u = Au.$$

Since displacement in the plane makes no essential change, we may take a form $w = Au + B$, where now the conformal transformation given by w is over any circle in its plane, the one given by u being over a particular circle, centre the origin and radius unity.

The conformation for w is derived from that for u by three operations:

(i) displacement of the origin to the point $-B/A$:

(ii) magnification equal to A':

(iii) rotation of the circle round its centre through an angle γ:

these operations evidently make no essential change in the conformation. If the limitation to the particular circle, centre the origin and radius 1, be made, evidently $B=0$, $A'=1$, but γ is left arbitrary. This constant can be determined by assigning a condition that, as the curve C has its homologue in the circle, one particular point of C has one particular point of the circumference for its homologue: the equation of transformation is then completely determined.

This determination of A', B, γ is a determination by very special conditions, which are not of the essence of the conformal representation: and therefore the apparent generality for the present case should arise in the analysis. Now, if $w=Au+B$, we have

$$\frac{d}{dz}\left\{\log\left(\frac{dw}{dz}\right)\right\}=\frac{d}{dz}\left\{\log\left(\frac{du}{dz}\right)\right\},$$

which is the same for the two forms; and therefore *the function to be sought is*

$$\frac{d}{dz}\left\{\log\left(\frac{dw}{dz}\right)\right\},$$

when the area included by C is to be represented on a circle so that a given point internal to C shall have the centre of the circle as its homologue. The arbitrary constants, that arise when w is thence determined, are given by special conditions as above.

Again, if the conformation be merely desired as a representation of the z-area bounded by the analytical curve C on the area of a circle in the w-plane (without the specification of an internal point being the homologue of the centre), there will be a further apparent generality in the form of the function. From what was proved in § 258, a circle in the u-plane is transformed into a circle in the w-plane by a substitution of the form

$$w=\frac{Au+B}{Cu+D},$$

so that, if u be a special function, w will be the more general function giving a desired conformal representation; and, without loss of this generality, we may assume $AD-BC=1$. Using $\{w, z\}$ to denote

$$\frac{d^2}{dz^2}\left(\log\frac{dw}{dz}\right)-\tfrac{1}{2}\left[\frac{d}{dz}\left(\log\frac{dw}{dz}\right)\right]^2,$$

that is,

$$\frac{w'''}{w'}-\tfrac{3}{2}\left(\frac{w''}{w'}\right)^2,$$

called the Schwarzian derivative by Cayley*, we have

$$\{w, z\}=\{u, z\},$$

* *Camb. Phil. Trans.*, vol. xiii, (1879), p. 5; for its properties, see Cayley's memoir just quoted, pp. 8, 9, and my *Treatise on Differential Equations*, pp. 92, 93.

which is the same for the two forms: and therefore *the function to be sought is*

$$\{w, z\},$$

when the area included by the analytical curve C is to be conformally represented on a circle. The (three) arbitrary constants, that arise when w is thence determined, are obtained by special conditions.

These two remarks will be useful when the transforming equation is being derived for particular cases, because they indicate the character of the initial equation to be obtained: but the importance of the investigation is the general inference that the conformal representation of an area bounded by an analytical curve on the area of a circle is possible, though, as the proof depends on the existence-theorem, no indication is given of the form of the function that secures the representation.

Further, it may be remarked that it is often convenient to represent a z-area on a w-half-plane instead of on a w-circle as the space of reference. This is, of course, justifiable, because there is an equation of unique transformation between the circular area and the half-plane; it has been given (Ex. 9, § 257). Moreover, a further change, given by $u' = \frac{au+b}{cu+d}$, is still possible: for, when a, b, c, d are real, this transformation changes the half-plane into itself, and these real constants can be obtained by making points p, q, r on the axis change into three points, say $0, 1, \infty$, respectively—the transformation then being

$$u' = \frac{u-p}{u-r}\,\frac{q-r}{q-p}.$$

265. Before discussing the particular forms just indicated, we shall indicate a method for the derivation of a relation that secures conformal representation of an area bounded by a given curve C.

Let* the curve C be an analytical curve, in the sense that the coordinates x and y can be expressed as functions of a real parameter, say of u, so that we have $x = p(u)$, $y = q(u)$; then

$$z = x + iy = p + iq = \phi(u).$$

If for u we substitute $w = u + iv$, we have

$$z = \phi(w);$$

and the curve C is described by z, when w moves along the axis of real quantities in its plane.

When the equation $x + iy = \phi(u + iv)$ is resolved into two equations involving real quantities only, of the form $x = \lambda(u, v)$, $y = \mu(u, v)$, then the eliminations of v and of u respectively lead to curves of the form

$$\psi(x, y, u) = 0, \quad \chi(x, y, v) = 0,$$

* Beltrami, *Ann. di Mat.*, 2[da] Ser., t. i, (1867), pp. 329—366; Cayley, *Quart. Journ. Math.*, vol. xxv, (1891), pp. 203—226; Schwarz, *Ges. Werke*, t. ii, p. 150.

which are orthogonal trajectories of one another when u and v are treated as parameters. Evidently $\chi(x, y, 0)=0$ is the equation of C: also

$$\lambda(u, 0)=p, \quad \mu(u, 0)=q.$$

So far as the representation of the area bounded by C on a half-plane is concerned, we can replace w by an arbitrary function of $Z(=X+iY)$ with real coefficients: for then, when $Y=0$, we have $w=f(X)$ and

$$x=p\{f(X)\}, \quad y=q\{f(X)\},$$

which lead to the equation of C as before, for all values of f. This arbitrariness in character is merely a repetition of the arbitrariness left in Gauss's solution of the original problem.

Now let the w-plane be divided into infinitesimal squares with sides parallel and perpendicular to the axis of real quantities. Then the area bounded by C is similarly divided, though, as the magnification is not everywhere the same, the squares into which the area is divided are not equal to one another. The successive lines parallel to the axis of u are homologous with successive curves in the area, the one nearest to that axis being the curve consecutive to C. Similarly, if the Z-plane be divided.

Conversely, if a curve consecutive to C, say C', be arbitrarily chosen, then the space of infinitesimal breadth between C and C' can be divided up into infinitesimal squares. Suppose the normal to C at a point L meet C' in L': along C take $LM=LL'$, and let the normal to C at M meet C' in M'; along C take $MN=MM'$, and let the normal to C at N meet C' in N': and so on. Proceeding from C' with $L'M'$, $M'N'$, ... as sides of infinitesimal squares, we can obtain the next consecutive curve C'', and so on; the whole area bounded by C may then be divided up into an infinitude of squares. It thus appears that the arbitrary choice of a curve consecutive to C completely determines the division of the whole area into infinitesimal squares, that is, it is a geometrical equivalent of the analytical assumption of a functional form which, once made, determines the whole division.

Next, we shall shew how the form f of the function can be determined so as to make the curve consecutive to C a *given* curve. As above, the curve C is given by the elimination of a (real) parameter between

$$x=p(u), \quad y=q(u);$$

and the representation is obtained by taking

$$x+iy=z=p(w)+iq(w)=p\{f(Z)\}+iq\{f(Z)\}.$$

Let the arbitrarily assumed curve C', consecutive to C, be given by the elimination of a (real) parameter θ between

$$x=p+\epsilon P, \quad y=q+\epsilon Q,$$

where p, P, q, Q are functions with real coefficients, and ϵ is an infinitesimal constant: the form of f has to be determined so that the curve corresponding to an infinitesimal value of Y is the curve C'. Taking $u=f(X)$, where u and X are real, we have, for the infinitesimal value of Y,

$$x+iy=p\{f(Z)\}+iq\{f(Z)\}$$
$$=\left\{p(u)+iY\frac{du}{dX}p'(u)\right\}+i\left\{q(u)+iY\frac{du}{dX}q'(u)\right\},$$

so that
$$x=p-Y\frac{du}{dX}q', \qquad y=q+Y\frac{du}{dX}p',$$

dashes denoting differentiation with regard to u. This is to be the same as the curve C', given by the equations

$$x=p+\epsilon P, \quad y=q+\epsilon Q.$$

Hence the (real) parameter θ in the latter differs from u only by an infinitesimal quantity: let it be $u-\mu$, so that we have

$$x=p-\mu p'+\epsilon P, \quad y=q-\mu q'+\epsilon Q,$$

the terms involving products of ϵ and μ being neglected, because they are of at least the second order. Hence

$$-\mu p'+\epsilon P=-Y\frac{du}{dX}q', \qquad -\mu q'+\epsilon Q=Y\frac{du}{dX}p';$$

whence
$$\mu(p'^2+q'^2)=\epsilon(Pp'+Qq'),$$

and
$$\epsilon(p'Q-q'P)=Y\frac{du}{dX}(p'^2+q'^2)^*.$$

Now ϵ is a real infinitesimal constant, as is also Y for the present purpose: so that we may take $\epsilon=AY$, where A is a finite real constant: and A may have any value assigned to it, because variations in the assumed value merely correspond to constant magnification of the Z-plane, which makes no difference to the division of the area bounded by C. Thus

$$A(p'Q-q'P)=\frac{du}{dX}(p'^2+q'^2),$$

and therefore
$$AX=\int\frac{p'^2+q'^2}{p'Q-q'P}du,$$

the inversion of which gives $u=f(X)$ and therefore $w=f(Z)$, the form required.

Also we have
$$\mu=AY\frac{Pp'+Qq'}{p'^2+q'^2},$$

shewing that, if the point $x=p+\epsilon P$, $y=q+\epsilon Q$ on C' lie on the normal to C at $x=p$, $y=q$, the parameters in the two pairs of equations are the same; the more general case is, of course, that in which the typical point on C' is in

* Beltrami obtains this result more directly from the geometry by assigning as a condition that the normal distance between the curves is equal to the arc given by du: l.c., (p. 530, *note*), p. 343.

the vicinity of C. And it is easy to prove that the normal distance between the curves at the point in consideration is

$$Y\frac{ds}{dX},$$

where ds is an arc measured along the curve C.

Ex. 1. As an illustration*, let C be an ellipse $x^2/a^2+y^2/b^2=1$ and let C' be an interior confocal ellipse of semi-axes $a-\alpha$, $b-\beta$, where α and β are infinitesimally small; so that, since

$$(a-\alpha)^2-(b-\beta)^2=a^2-b^2=c^2,$$

we have $a\alpha=b\beta=c\epsilon$ say; then the semi-axes of C' are $a-\frac{c}{a}\epsilon$, $b-\frac{c}{b}\epsilon$. We have

$$p=a\cos u, \qquad q=b\sin u,$$

$$P=-\frac{c}{a}\cos u, \quad Q=-\frac{c}{b}\sin u,$$

so that

$$AX=\int\frac{ab}{c}\,du=\frac{ab}{c}u,$$

or, taking $A=\frac{ab}{c}$, we have $X=u$ and therefore $Z=w$. Hence the equation of transformation is

$$z=x+iy=a\cos Z+ib\sin Z;$$

or, if $a=c\cosh Y_0$, $b=c\sinh Y_0$, and if Y' denote Y_0-Y, the equation is

$$z=c\cos(X+iY')=c\cos Z'.$$

The curves, corresponding to parallels to the axes, are the double system of confocal conics.

Ex. 2. When the curve C is a parabola, with the origin as focus and the axis of real quantities as its axis, and C' is an external confocal coaxial parabola, the relation is

$$z=a(Z+i)^2;$$

substantially the same relation as in Ex. 7, § 257.

Ex. 3. When C is a circle with its centre on the axis of real quantities and C' is an interior circle, having its centre also on the axis but not coinciding with that of C, the circles being such that the axis of imaginary quantities is their radical axis, the relation can be taken in the form

$$z=c\tan Z. \qquad \text{(Beltrami; Cayley.)}$$

Note. Although, in the examples just considered, the successive curves C ultimately converge to a curve of zero area (either a point or a line), so that the whole of the included area is transformed, yet this convergence is not always a possibility, when a consecutive to C is assigned arbitrarily. There will then be a limit to the ultimate curve of the series, so that the representation ceases to be effective beyond that limit. The limitation may arise, either through the occurrence of zero or of infinite values of $\frac{dz}{dZ}$ for areas and not merely for isolated points, or through the occurrence of branch-points for the transforming function. In either case, the uniqueness of the representation ceases.

* Beltrami, l.c., (p. 530, *note*), p. 344; Cayley, (ib.), p. 206.

Ex. 4. Consider the area, bounded by the cardioid

$$r=2a(1+\cos\theta);$$

then we can take

$$x=p=2a(1+\cos u)\cos u, \qquad y=q=2a(1+\cos u)\sin u,$$

where evidently $u=\theta$ along the curve. Let the consecutive curve be given by

$$x=-a\epsilon+2a(1+\epsilon)(1+\cos u')\cos u', \qquad y=2a(1+\epsilon)(1+\cos u')\sin u',$$

so that, to determine X, we assume $P=-a+2a(1+\cos u)\cos u$, $Q=2a(1+\cos u)\sin u$, for $u'-u=-\mu$ a small quantity.

We have
$$p'^2+q'^2=16a^2\cos^2\tfrac{1}{2}u,$$
$$q'P-p'Q=12a^2\cos^2\tfrac{1}{2}u,$$
$$p'P+q'Q=-2a^2\sin u;$$

and then, proceeding as before and choosing A of the text as equal to $-\frac{4}{3}$, (which implies that ϵ is negative and therefore that the interior area is taken), we find

$$X=u,$$
$$\mu=\tfrac{1}{3}Y\tan\tfrac{1}{2}u;$$

therefore $Z=w$. Thus the cardioid itself and the consecutive curves are given by

$$z=p+iq=2a(1+\cos Z)e^{iZ}.$$

To trace the curves, corresponding to lines parallel to the axes of X and Y, we have

$$\left(\frac{z}{a}\right)^{\frac{1}{2}}=2\cos\tfrac{1}{2}Ze^{\frac{1}{2}iZ},$$
$$\left(\frac{z_0}{a}\right)^{\frac{1}{2}}=2\cos\tfrac{1}{2}Z_0e^{-\frac{1}{2}iZ_0}.$$

Hence, multiplying, we have

$$r=4ae^{-Y}(\cos\tfrac{1}{2}Z\cos\tfrac{1}{2}Z_0)$$
$$=2ae^{-Y}(\cosh Y+\cos X);$$

and, dividing, we have

$$e^{i\theta}=e^{iX}\frac{\cos\frac{1}{2}Z}{\cos\frac{1}{2}Z_0},$$

that is,
$$e^{i(X-\theta)}=\frac{\cos\frac{1}{2}X\cosh\frac{1}{2}Y+i\sin\frac{1}{2}X\sinh\frac{1}{2}Y}{\cos\frac{1}{2}X\cosh\frac{1}{2}Y-i\sin\frac{1}{2}X\sinh\frac{1}{2}Y},$$

and therefore
$$\tan\tfrac{1}{2}(X-\theta)=\tan\tfrac{1}{2}X\tanh\tfrac{1}{2}Y.$$

Moreover, we have
$$\frac{dz}{dZ}=2aie^{iZ}(1+e^{iZ}),$$

which vanishes when $Z=\pi(2n+1)$, that is, at the point $X=(2n+1)\pi$, $Y=0$; whence the cusp of the cardioid is a singularity in the representation.

When $Y=0$, then $X=\theta$ and $r=2a(1+\cos\theta)$, which is the cardioid; when Y is very small and is expressed in circular measure, then

$$\tan\tfrac{1}{2}(X-\theta)=\tfrac{1}{2}Y\tan\tfrac{1}{2}X,$$

or
$$X=\theta+Y\tan\tfrac{1}{2}\theta,$$

so that
$$r=2a(1+\cos\theta)-4aY.$$

It is easy to verify that
$$\theta=u'+\tfrac{1}{3}Y\tan\tfrac{1}{2}u',$$

agreeing with the former result.

The relation may be taken in the form

$$(z/a)^{\frac{1}{2}}=2\cos\tfrac{1}{2}Ze^{\frac{1}{2}Zi}=e^{iZ}+1,$$

which shews that $z=a$ is a branch-point for Z. Two different paths from any point to a point P, which together enclose a, give different values of Z at P. Hence the representation ceases to be effective for any area that includes the point a.

Consider a strip of the Z-plane between the lines $Y=0$, $Y=+\infty$, $X=-\frac{1}{2}\pi$, $X=+\frac{1}{2}\pi$.

First, when $Z=\frac{1}{2}\pi+iY$, we have $X=\frac{1}{2}\pi$, so that

$$\tan(\tfrac{1}{4}\pi-\tfrac{1}{2}\theta)=\tanh\tfrac{1}{2}Y;$$

and therefore
$$\tan\tfrac{1}{2}\theta=e^{-Y},$$

whence
$$r=\frac{2a}{1+\cos\theta},$$

a part of a parabola. And when Y varies from ∞ to 0, θ varies from 0 to $\frac{1}{2}\pi$.

Secondly, when $Z=X$, so that $Y=0$, we have $X=\theta$, and then

$$r=2a(1+\cos\theta):$$

and, when X varies from $\frac{1}{2}\pi$ to $-\frac{1}{2}\pi$, θ varies from $\frac{1}{2}\pi$ to $-\frac{1}{2}\pi$.

Thirdly, when $Z=-\frac{1}{2}\pi+iY$, we have $X=-\frac{1}{2}\pi$, so that

$$\tan(\tfrac{1}{4}\pi+\tfrac{1}{2}\theta)=\tanh\tfrac{1}{2}Y,$$

whence
$$\tan\tfrac{1}{2}\theta=-e^{-Y},$$

so that, as Y varies from 0 to ∞, θ varies from $-\frac{1}{2}\pi$ to 0. And then

$$r=\frac{2a}{1+\cos\theta},$$

another part of the same parabola as before.

Lastly, when Y is infinite and X varies from $-\frac{\pi}{2}$ to $+\frac{\pi}{2}$, we have

$$\tan\tfrac{1}{2}(X-\theta)=\tan\tfrac{1}{2}X,$$

so that $\theta=0$; and then $r=a$, in effect the point of the z-plane corresponding to the point at infinity in the Z-plane.

We thus obtain a figure in the z-plane $ABCDA$ corresponding to the strip in the Z-plane: the boundary is partly a parabola DAB, of focus O and axis OA, and partly a cardioid with O for cusp—the inverse of the parabola with regard to a circle on the latus rectum BD as diameter: the angles at B and D are right.

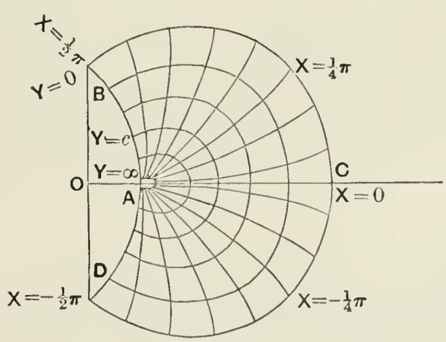

Fig. 92.

To trace the division of the space between the axes of the cardioid and of the parabola corresponding to the division of the plane strip into small squares, we can proceed as follows.

Let $e^{-Y}=c$: then we have

$$\frac{R}{a}=c\left(\frac{1}{c}+c\right)+2c\cos X,$$

or, if $R=a\rho$, then

$$\rho=1+c^2+2c\cos X;$$

and

$$\tan\tfrac{1}{2}(X-\theta)=\frac{1-c}{1+c}\tan\tfrac{1}{2}X,$$

so that

$$c=\frac{\sin\frac{1}{2}\theta}{\sin(X-\frac{1}{2}\theta)};$$

and therefore

$$\frac{\cos\frac{1}{2}\theta}{1+c\cos X}=\frac{\sin\frac{1}{2}\theta}{c\sin X}=\frac{1}{\sqrt{\rho}},$$

so that

$$c\cos X=\sqrt{\rho}\cos\tfrac{1}{2}\theta-1,\qquad c\sin X=\sqrt{\rho}\sin\tfrac{1}{2}\theta,$$

from which the curves, corresponding to c=constant and to X=constant, are at once obtained. They are exhibited in the figure, the whole of the internal space being divisible.

By combination with the transformation, which (Ex. 12, § 257) represents a strip of the foregoing kind on a circle, the relation can be obtained, leading to the representation of the figure on a circle.

Ex. 5. Shew that, if a straight line be drawn from the cusp to the point $r=a$, $\theta=0$, so as to prevent z from passing round $z=0$ or $z=a$, then the area bounded by the cardioid and this line can be represented, on a strip of the w-plane given by $Y=0$, $Y=\infty$, $X=-\pi$, $X=+\pi$, by the equation

$$iw=\log\{(z/a)^{\frac{1}{2}}-1\}. \qquad \text{(Burnside.)}$$

Ex. 6. In the same way, treating the curve (the Cissoid of Diocles) $(2r-x)\,y^2=x^3$, and taking the equations

$$x=2r\sin^2 u,\qquad y=2r\frac{\sin^3 u}{\cos u},$$

as defining the points on the curve, we may assume the consecutive curve defined by the equations

$$x=\epsilon+(2r-\epsilon)\sin^2 u,\qquad y=(2r-\epsilon)\frac{\sin^3 u_1}{\cos u_1},$$

another cissoid with the same asymptote. Proceeding as before we find the value of X to be $\tan u+\frac{1}{12}\tan^3 u$, on taking $A=-\frac{8}{3}r$.

The relation, which changes the cissoidal arc into the axis of X and a consecutive cissoidal arc into a line parallel to the axis of X at an infinitesimal distance from it, is then

$$z=2r\frac{\sin^2 w}{\cos w}e^{wi},$$

where the relation between w and Z is

$$Z=\tan w+\tfrac{1}{12}\tan^3 w.$$

Note. The method is applicable to any curve, whose equation can be expressed in the form $r=f(\theta)$: a first transformation is

$$z=f(w)\,e^{wi}.$$

The determination of w in terms of Z depends upon the character of the consecutive curve chosen; this curve also determines the details of the conformation.

266. It has been pointed out (§ 265, *Note*) that, though a curve and its consecutive in the z-plane correspond with a curve and its consecutive in the w-plane, the conformation is only effective for parts of the included areas, in which the magnification, if it is not uniform, becomes zero or infinite only at isolated points, and in which no branch-points of the transforming relation occur. The immediate vicinity of a curve C is conformable with the immediate vicinity of a corresponding curve S, arbitrarily chosen limits being assigned for the vicinity.

But, as remarked by Cayley*, when a curve is given, then the consecutive curve can be so chosen that the whole included area is conformable with the whole corresponding area in the Z-plane. For a circle can be thus represented, the ultimate limit of the squares when consecutive curves are constructed being then a point: this can be expressed by saying that the area can be *contracted* into a point. For instance, the relation

$$z(w+1)+i(w-1)=0$$

transforms the z-half-plane into the area included by a w-circle of radius unity. The lines parallel to the axis of x are internal circles all touching one another at the point $(-1, 0)$: and the lines parallel to the axis of y are circles orthogonal to these, having their centres on a line parallel to the axis of Y and all touching at the point $(-1, 0)$. Similarly for the contraction of any circle, by making it one of two systems of orthogonal circles: the form of the necessary equation is obtained as above by taking the next circle of the same system as the consecutive curve: and a circle can thus be contracted to its centre (the infinitesimal squares being bounded by concentric circles and by radii) when the w-circle is derived from a strip of the z-half-plane by the relation $w=e^{iz}$. Such a contraction of a circle is unique.

But, by Riemann's theorem, it is known that the area of a given analytical curve can be conformally represented on the area of a given circle, so that a given internal point is the homologue of the centre and a given point on the curve is the homologue of a given point on the circumference of the circle: and that the representation is unique. Hence it follows that, when an analytical curve C is given, a consecutive curve C' can be chosen in such a manner as to secure that the construction of the whole series of consecutive curves by infinitesimal squares will make the curve C contract into an assigned point†.

267. The areas, already considered in special examples, have been bounded by one or by two analytical curves: we shall now consider two special forms of areas bounded by a number of portions of analytical curves. These areas are (i) the area included within a convex rectilinear polygon, (ii) the area bounded by any number of circular arcs, and especially the area

* l.c. (p. 530, *note*), pp. 213, 214.

† For further developments, see Cayley's memoir cited p. 530, *note*.

bounded by three circular arcs. For the sake of analytical simplicity, the former will be conformally represented on the half-plane, the transformation to the circle being immediate by means of the results of § 257.

In regard to the representation* of the rectilinear polygon, convex in the sense that its sides do not cross, we shall take the case corresponding to the first of the two forms of § 264; it will be assumed that the origin in the w-plane is left unspecified and that the magnification is subject to an unspecified increase, constant over the plane. Our purpose, therefore, is to represent the w-area included by a polygon on the half of the z-plane; the boundary of the polygonal area in the w-plane is to be transformed into the axis of real quantities in the z-plane.

It follows from Schwarz's continuation-theorem (§ 36), that a function defined for a region in the positive half of a plane and acquiring continuous real values for continuous real values of the argument can be continued across the axis of real quantities: and the continuation is such that conjugate values of the function correspond to conjugate values of the variable. Moreover, the function, for real values of the variable, can be expanded in a converging series of powers, so that

$$w - w_0 = (x - c) P(x - c),$$

where P is a series of positive, integral powers with real coefficients that does not vanish when c is the value of the real variable x.

Suppose a convex polygon given in the w-plane, the area included by which is to be represented on the z-plane, and the contour of which is to be represented along the axis of x by means of a relation between w and z.

First, consider a point say β on the side $A_{r-1}A_r$ which is not an angular point. Then, if θ denote the inclination of $A_{r-1}A_r$ to the axis of u, the function

$$(w - \beta) e^{-i(\pi+\theta)}$$

is real when w lies on the side $A_{r-1}A_r$: it changes sign when w passes through β: and for all other points w, lying either in the interior or on the other sides of the polygon, it has the same properties as w. Hence, if b be a (purely real) value of z corresponding to $w = \beta$, we have

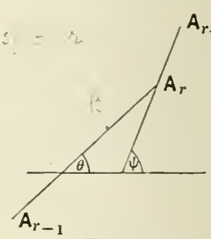

Fig. 93.

$$(w - \beta) e^{-i(\pi+\theta)} = (z - b) P(z - b),$$

* In connection with the succeeding investigations the following authorities may be consulted:

Schwarz, *Ges. Werke*, t. ii, pp. 65—83; Christoffel, *Ann. di Mat.*, 2da Ser., t. i, (1867), pp. 95—103, ib., t. iv, (1871), pp. 1—9; Schläfli, *Crelle*, t. lxxviii, (1873), pp. 63—80; Darboux, *Théorie générale des surfaces*, t. i, pp. 176—180; Phragmén, *Acta Math.*, t. xiv, (1890), pp. 229—231.

for points in the vicinity of β: the series $P(z-b)$ does not vanish for $z=b$; and, when w lies on the side $A_r A_{r-1}$, then $z=x$.

Next, consider the vicinity of an angular point of the polygon. Let γ be the coordinate of A_r, let $\mu\pi$ be the internal angle of the polygon, and let ψ be the inclination of $A_r A_{r+1}$ to the axis of u: and consider the function

$$(w-\gamma)\, e^{-i(\pi+\theta)}.$$

When w lies on the side $A_r A_{r-1}$ at a distance d from A_r, then

$$w-\gamma = de^{i(\pi+\theta)},$$

so that the function is then real and positive.

When w lies in the interior of the polygon, the function has the same properties as w, and its argument is negative.

When w lies on the side $A_r A_{r+1}$ at a distance d' from A_r, then $w-\gamma = d'e^{i\psi}$, so that the function is $d'e^{-i(\pi+\theta-\psi)}$, that is, $d'e^{-i\mu\pi}$. Hence

$$\{(w-\gamma)\, e^{-i(\pi+\theta)}\}^{\frac{1}{\mu}}$$

is real and positive along the side $A_{r-1}A_r$, and is real and negative along the side $A_r A_{r+1}$. If then $z=c$ be the value corresponding to $w=\gamma$, we can expand this function in the form $(z-c)\,Q'(z-c)$: and therefore

$$(w-\gamma)\, e^{-i(\pi+\theta)} = (z-c)^{\mu}\, R(z-c),$$

where $R\,(=Q'^{\mu})$ does not vanish for $z=c$.

These forms assume that neither b nor c is infinite. The point on the boundary of the polygon (if there be one), corresponding to $x=\infty$, can be obtained as follows. We form a new representation of the z-plane given by

$$z\zeta = -1,$$

which conformally represents the upper half of the z-plane on itself: and then, on the assumption that such point at infinity does not correspond to an angular point of the polygon, we have $\zeta=0$ corresponding to an ordinary point of the boundary, so that

$$(w-\beta')\, e^{-i(\pi+\theta')} = \zeta Q(\zeta) = \frac{1}{z}\, Q\left(\frac{1}{z}\right),$$

where Q does not vanish when $z=\infty$.

All kinds of points on the boundary of the w-polygon have been considered, corresponding to points on the axis of x.

We now consider points in the interior. If w' be such an interior point and z' be the corresponding z-point, then

$$w-w' = (z-z')\, S(z-z'),$$

where S does not vanish for $z=z'$ because at every point $\frac{dw}{dz}$ must be different from zero: for otherwise the magnification from a part of the z-plane to a part in the interior of the polygon would be zero and the representation would be ineffective.

Now in the present case, just as in the first case suggested in § 264, it is manifest that, if a particular function u give a required representation, then $Au+B$, where $|A|=1$, will give the same w-polygon displaced to a new origin and turned through an angle $=\arg. A$, that is, no change will be made in the size or in the shape of the polygon, its position and orientation in the w-plane not being essential. Hence the function to be obtained may be expected to occur in the form $w=Au+B$, so that, *in representing a figure bounded by straight lines, the function to be obtained is*

$$Z=\frac{d}{dz}\left\{\log\left(\frac{dw}{dz}\right)\right\}.$$

Now in the vicinity of a boundary-point β, not being an angular point and corresponding to a finite value of z, we have

$$w-\beta=e^{i(\pi+\theta)}(z-b)P(z-b),$$

and therefore

$$Z=P_1(z-b),$$

having $z=b$ for an ordinary (non-zero) point.

For a boundary-point β', not being an angular point and corresponding to an infinite value of z on the real axis, we have

$$w-\beta'=e^{i(\pi+\theta')}\frac{1}{z}Q\left(\frac{1}{z}\right),$$

and therefore

$$Z=-\frac{2}{z}+\frac{1}{z^2}Q_1\left(\frac{1}{z}\right),$$

where Q_1 is finite for $z=\infty$. Thus Z vanishes for such a point.

In the vicinity of an angular point γ, we have

$$w-\gamma=e^{i(\pi+\theta)}(z-c)^{\mu}R(z-c),$$

and therefore

$$Z=\frac{\mu-1}{z-c}+R_1(z-c),$$

where R_1 has $z=c$ for an ordinary point.

Lastly, for a point w' in the interior of the polygon, we have

$$w-w'=(z-z')S(z-z'),$$

and therefore

$$Z=S_1(z-z'),$$

having $z=z'$ for an ordinary point.

Hence Z, considered as a function of z, has the following properties:—

It is an analytical function of z, real for all real values of its argument, and zero when x is infinite:

It has a finite number of accidental singularities each of the first order and all of them isolated points on the axis of x: and at all other points on one side of the plane it is uniform, finite and continuous, having (except at the singularities) real continuous values for real continuous values of its argument.

The function Z can therefore be continued across the axis of x, conjugate values of the function corresponding to conjugate values of the variable: and its properties make it, by § 48, a rational, algebraical, meromorphic function of z.

Let $a, b, c, \ldots, l$ be the points (all in the finite part of the plane) on the axis of x corresponding to the angular points of the polygon, and let

$$\alpha\pi,\ \beta\pi,\ \gamma\pi,\ \ldots,\ \lambda\pi$$

be the internal angles of the polygon at the respective points: then (by § 48)

$$Z = \frac{\alpha - 1}{z - a} + \frac{\beta - 1}{z - b} + \ldots + \frac{\lambda - 1}{z - l},$$

no additive constant being required because Z has been proved to vanish for infinite values of z.

Moreover, because $\alpha\pi, \beta\pi, \ldots, \lambda\pi$ are the internal angles of the polygon, we have

$$\Sigma\,(\pi - \alpha\pi) = 2\pi,$$

so that

$$\Sigma\,(\alpha - 1) = -2,$$

a relation among the constants $\alpha, \beta, \ldots, \lambda$ in the equation

$$\frac{d}{dz}\left\{\log\left(\frac{dw}{dz}\right)\right\} = \frac{\alpha - 1}{z - a} + \ldots + \frac{\lambda - 1}{z - l};$$

and each of the quantities $\alpha, \beta, \ldots, \lambda$ is less than 2. This equation*, when integrated, gives

$$w = C\int (z-a)^{\alpha-1}(z-b)^{\beta-1}\ldots(z-l)^{\lambda-1}\,dz + C',$$

where C and C' are arbitrary constants, determinable from the position of the polygon†.

268. It may be remarked, first, that any three of the real quantities $a, b, c, \ldots, l$ can be chosen arbitrarily, subject to the restrictions that the points $a, b, c, \ldots, l$ follow in the same order along the axis of x as the angular points of the polygon and that no one of the remaining points passes to infinity. For if three definite points, say a, b, c, have been chosen, they can, by a real substitution

$$z = \frac{p\zeta + q}{r\zeta + s},$$

* This relation, as is possible with many relations in conformal representation of areas, is made the basis of some interesting applications in hydrodynamics, by Michell, *Phil. Trans.*, (1890), pp. 389—431; and in conduction of heat, by Christoffel, l.c., p. 538, *note*.

† This result was obtained independently by Christoffel and by Schwarz: l.c., p. 538, *note*.

where p, q, r, s are real quantities satisfying $ps - qr = 1$, be changed into other three, say a', b', c': and then, substituting

$$z - k = \frac{p - kr}{r\zeta + s}(\zeta - k'),$$

and using the relation $\Sigma(\alpha - 1) = -2$,

we have $w = \Gamma \int (\zeta - a')^{\alpha-1}(\zeta - b')^{\beta-1} \ldots (\zeta - l')^{\lambda-1} d\zeta + C'$,

where Γ is a new constant. By the real substitution, the axis of real quantities is preserved: and thus the new form equally effects the conformal representation of the polygon.

But, secondly, it is to be remarked that when three of the points on the axis of x are thus chosen, the remainder are then determinate in terms of them and of the constants of the polygon.

Note. The z-point at infinity has been excluded from being the homologue of one of the angular points of the w-polygon: but the exclusion is not necessary.

If $z = \infty$ be the homologue of an angular point σ, at which the internal angle is $\mu\pi$, then proceeding as before, we have

$$(w - \gamma)\, e^{-i(\pi+\theta)} = \frac{1}{z^\mu} T\left(\frac{1}{z}\right)$$

for points in the vicinity of σ; and therefore

$$\frac{d}{dz}\left\{\log\left(\frac{dw}{dz}\right)\right\} = -\frac{\mu+1}{z} + \text{terms in } \frac{1}{z^2}, \frac{1}{z^3}, \ldots.$$

Let $a, b, c, \ldots, k$ be the homologues of the other vertices where the angles are $\alpha\pi, \beta\pi, \ldots, \kappa\pi$: then the function

$$\frac{d}{dz}\left\{\log\left(\frac{dw}{dz}\right)\right\} - \frac{\alpha-1}{z-a} - \frac{\beta-1}{z-b} - \ldots - \frac{\kappa-1}{z-k}$$

is finite at $a, b, \ldots, k$. The term in $\frac{1}{z}$ in the fractional part is

$$-\frac{1}{z}\Sigma(\alpha - 1).$$

But $\mu - 1 + \Sigma(\alpha - 1) = -2$, so that the term is $\frac{\mu+1}{z}$. Hence the function for infinite values of z begins with $\frac{1}{z^2}$, and therefore it vanishes at that point. It has thus no infinities for any value of z: being a uniform function, it is therefore a constant, which (owing to the value of the function for $z = \infty$) is evidently zero: so that

$$\frac{d}{dz}\left\{\log\left(\frac{dw}{dz}\right)\right\} = \frac{\alpha-1}{z-a} + \frac{\beta-1}{z-b} + \ldots + \frac{\kappa-1}{z-k}.$$

Hence, if one of the angular points of the polygon be made to correspond to an infinite value of z, the equation which determines the conformal representation is

$$w = A\int (z-a)^{\alpha-1}(z-b)^{\beta-1}\dots(z-k)^{\kappa-1}\,dz + B,$$

where
$$\alpha - 1 + \beta - 1 + \dots + \kappa - 1 = -1 + \mu,$$

$\mu\pi$ (usually equal to zero) being the internal angle at the vertex which has its homologue at infinity.

269. The simplest example is that of a triangle of angles $\alpha\pi$, $\beta\pi$, $\gamma\pi$, so that

$$\alpha + \beta + \gamma = 1.$$

Then a particular function determining the conformal representation of this w-triangle on the half z-plane is

$$w = \int \frac{dz}{(z-a)^{1-\alpha}(z-b)^{1-\beta}(z-c)^{1-\gamma}},$$

so that
$$\frac{dz}{dw} = (z-a)^{1-\alpha}(z-b)^{1-\beta}(z-c)^{1-\gamma},$$

a differential equation of the class partially discussed in §§ 246—252.

For general values of α, β, γ the integral-function w is an Abelian transcendent of some class which is greater than 1: and then, after §§ 110, 239, z is no longer a definite function of w, and the path of integration must be specified for complete definition of the function.

If $\alpha = 0$, the only instance when the integral is a uniform function of w is when $\beta = \frac{1}{2}$, $\gamma = \frac{1}{2}$: and then the function is singly-periodic (§ 252, III.). In such a case the w-figure is a strip of the plane of finite breadth, extending in one direction to infinity and terminated in the finite part of the plane by a straight line perpendicular to the direction of infinite extension.

If no one of the quantities α, β, γ be zero, then on account of the condition $\alpha + \beta + \gamma = 1$, the only cases when the integral gives z as a uniform function of w are as follows. In each case the function is doubly-periodic.

(§ 252, III., 10)...(A): $\alpha = \frac{1}{3}, \beta = \frac{1}{3}, \gamma = \frac{1}{3}$: an equilateral triangle.

(ib., 9)...(B): $\alpha = \frac{1}{2}, \beta = \frac{1}{4}, \gamma = \frac{1}{4}$: an isosceles right-angled triangle.

(ib., 8)...(C): $\alpha = \frac{1}{2}, \beta = \frac{1}{3}, \gamma = \frac{1}{6}$: a right-angled triangle with one angle equal to $\frac{1}{3}\pi$.

The integral expressions for these cases have been given by Love*, who has also discussed a further case, (due to Schwarz, Ex. 3, § 252), in which z occurs as a two-valued doubly-periodic function of w; the triangle is then isosceles with an angle of $\frac{2}{3}\pi$, the values of α, β, γ being $\alpha = \frac{2}{3}$, $\beta = \frac{1}{6}$, $\gamma = \frac{1}{6}$.

* *Amer. Journ. of Math.*, vol. xi, (1889), pp. 158—171.

The example next in point of simplicity is furnished by a quadrilateral, in particular by a rectangle: then

$$\alpha = \beta = \gamma = \delta = \tfrac{1}{2}:$$

and the general form is

$$w = \int \{(z-a)(z-b)(z-c)(z-d)\}^{-\frac{1}{2}}\, dz,$$

so that z is a doubly-periodic function of w.

First, let it be a square: and choose ∞, 1, 0 as points on the axis of x corresponding to three of the angular points in order. The symmetry of the w-figure then enables us to choose -1 as the remaining angular point.

In the vicinity of $z = \kappa$, we have

$$\frac{d}{dz}\left\{\log\left(\frac{dw}{dz}\right)\right\} + \frac{\frac{1}{2}}{z-\kappa}$$

a finite quantity, where $\kappa = 0, 1, -1$ in turn.

For infinite values of z, we have

$$(w-\gamma)e^{-i(\pi+\theta)} = \left(\frac{1}{z}\right)^{\frac{1}{2}} T\left(\frac{1}{z}\right),$$

where T is finite for $z = \infty$: hence

$$\frac{d}{dz}\left\{\log\left(\frac{dw}{dz}\right)\right\} = -\tfrac{3}{2}\frac{1}{z} + \text{terms in } \frac{1}{z^2}, \frac{1}{z^3}, \ldots.$$

Hence the function

$$\frac{d}{dz}\left\{\log\left(\frac{dw}{dz}\right)\right\} + \tfrac{1}{2}\left(\frac{1}{z+1} + \frac{1}{z} + \frac{1}{z-1}\right)$$

is finite for $z = 0$, $z = 1$, $z = -1$: it is zero for $z = \infty$: it is not infinite for any other point in the plane. It is a uniform function of z: it is therefore a constant, equal to its value at any point, say, at $z = \infty$ where it is zero: and so

$$\frac{d}{dz}\left\{\log\left(\frac{dw}{dz}\right)\right\} = -\tfrac{1}{2}\left(\frac{1}{z+1} + \frac{1}{z} + \frac{1}{z-1}\right),$$

whence

$$w = C\int^z \frac{dz}{\{z(z^2-1)\}^{\frac{1}{2}}} + C',$$

C and C' being dependent upon the position and the magnitude of the w-square.

Again, the half z-plane is transformed into the interior of a Z-circle, of radius 1 and centre the origin, by the relation

$$Z = \frac{i-z}{i+z}.$$

Then except as to a constant factor, which can be absorbed in C, the integral in w changes to

$$\int \frac{dZ}{(1-Z^4)^{\frac{1}{2}}},$$

so that, by the relation

$$W = \int_0^Z \frac{dZ}{(1-Z^4)^{\frac{1}{2}}},$$

the interior of a Z-circle, centre the origin and radius 1, is the conformal representation of the interior of some square in the W-plane. Denoting by L the integral $\int_0^1 \frac{dx}{(1-x^4)^{\frac{1}{2}}}$, so that $2L$ is the length of a diagonal, the angular points of the square are D, A, B, C on the axes of reference: and these become d, a, b, c on the circumference of the circle. They correspond to $-1, 0, 1, \infty$ on the axis of x in the representation on the half-plane.

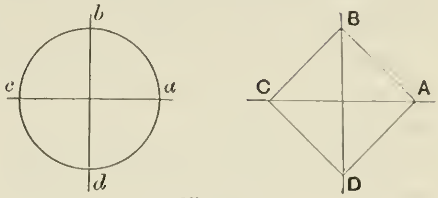

Fig. 94.

Ex. Shew that the area outside a square in the w-plane can be conformally represented on the interior of a circle in the z-plane, centre the origin and radius unity, by the equation

$$w = \int_1^z \frac{1}{z^2}(1+z^4)^{\frac{1}{2}} dz,$$

the z-origin corresponding to the infinitely distant part of the w-plane. (Schwarz.)

Secondly, let the rectangle have unequal sides. Then the symmetry of the figure justifies the choice of $\frac{1}{k}, 1, -1, -\frac{1}{k}$ as four points on the axis of x corresponding to the angular points of the rectangle when it is represented on the half-plane. We thus have

$$w = C\int_0^z \{(1-z^2)(1-k^2z^2)\}^{-\frac{1}{2}} dz + C'.$$

If the rectangle be taken so that its angular points are a, $a+2bi$, $-a+2bi$, $-a$ in order, these corresponding to $1, \frac{1}{k}, -\frac{1}{k}, -1$ respectively, then we have

$$0 = C',$$
$$a = CK,$$
$$a + 2bi = C(K + iK');$$

so that the relation is

$$\frac{w}{a}K = \int_0^z \{(1-z^2)(1-k^2z^2)\}^{-\frac{1}{2}} dz,$$

and then

$$\frac{K'}{K} = \frac{2b}{a},$$

whence

$$q = e^{-\frac{2\pi b}{a}},$$

where q is the usual Jacobian constant: this equation determines the relation between the shape of the rectangle and the magnitude of k.

In the particular case when the rectangle is a square, we have $b = a$ and so $q = e^{-2\pi}$, or $\frac{K'}{K} = 2$: and therefore* $k = 3 - \sqrt{8}$ or $\frac{1}{k} = 3 + \sqrt{8}$. The difference from the preceding representation of the square is that, there, the point $z = i$ was the homologue of the centre of the square, whereas now, as may easily be proved, the point $z = i(\sqrt{2} + 1)$ is the homologue of the centre.

But in the case of a quadrilateral in which such symmetrical forms are obviously not possible and, in the case of any convex polygon, only three points can be taken arbitrarily on the axis of x: the most natural three points to take are 0, 1, ∞ for three successive points. The values for the remaining points must be determined before the representation can be considered definite.

Thus in the case of a quadrilateral, taking ∞, 0, 1 as the homologues of D, A, B respectively and $\frac{1}{\mu}$ as the homologue of C, (where $\mu < 1$), the equation for conformal representation is

$$w = Cu + C',$$

where

$$u = \int_0^z z^{\alpha-1}(1-z)^{\beta-1}(1-\mu z)^{\gamma-1}\,dz = \int_0^z Z dz, \text{ say.}$$

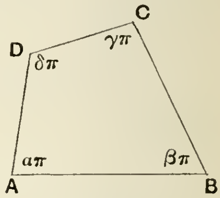

Fig. 95.

If the w-origin be taken at A, and the real axis along AB, we have

$$0 = C',$$

$$a = C\int_0^1 X dx + C',$$

$$de^{i\pi\alpha} = C\int_0^\infty X dx + C',$$

$$be^{i\pi(1-\beta)} = C\int_1^{\frac{1}{\mu}} X dx + C',$$

being the equations for the four angular points. They determine only three quantities C, C', μ, so that they coexist in virtue of a relation, which is in effect the relation between the sides and the angles of a quadrilateral.

An equation to determine μ is

$$a\int_0^\infty X dx = de^{i\pi\alpha}\int_0^1 X dx;$$

the second equation serves to determine C, because $C' = 0$.

The equation determining μ can be modified as follows†, so as to be expressed in terms of the hypergeometric series.

* This is derived at once by means of the quadric transformation in elliptic functions.

† For the analytical relations in reference to the definite integrals, see Goursat, "Sur l'équation différentielle linéaire qui admet pour intégrale la série hypergéométrique," *Ann. de l'Éc. Norm. Sup.*, 2^{me} Sér., t. x, (1881), Suppl., pp. 3—142; and for the relations between the hypergeometric series, see my *Treatise on Differential Equations*, pp. 192—201, 232, 233, the notation of which is here adopted.

Let $\frac{d}{a} e^{i\pi a} = \lambda$, so that the equation is

$$\int_0^\infty X dx = \lambda \int_0^1 X dx.$$

Now to compare these integrals with the definite integrals which are the solution the differential equation of the hypergeometric series, we take

$$a' = 1 - \gamma, \qquad \beta' = a, \qquad \gamma' = a + \beta,$$

so that $$X = x^{\beta'-1}(1-x)^{\gamma'-\beta'-1}(1-\mu x)^{-a'}.$$

And $$a' > 0 < 1, \qquad \gamma' - \beta' > 0, \qquad a' + 1 - \gamma' = 2 - \gamma - a - \beta = \delta > 0,$$

so that, as $\mu < 1$, the definite integral is finite at all the critical points.

We have

$$\int_0^1 X dx = \frac{\Gamma(\beta')\Gamma(\gamma'-\beta')}{\Gamma(\gamma')} F(a', \beta', \gamma', \mu)$$

$$= \frac{\Gamma(a)\Gamma(\beta)}{\Gamma(a+\beta)} Y_1;$$

$$\int_1^{\frac{1}{\mu}} X dx = e^{\pi i(\gamma'-\beta'-1)} \frac{\Gamma(\gamma'-\beta')\Gamma(1-a')}{\Gamma(\gamma'-a'-\beta'+1)} \mu^{\beta'-\gamma'}(1-\mu)^{\gamma'-\beta'-a'}$$

$$\times F\left(\gamma'-\beta',\ 1-\beta',\ \gamma'-a'-\beta'+1,\ \frac{\mu-1}{\mu}\right)$$

$$= e^{\pi i(\beta-1)} \frac{\Gamma(\beta)\Gamma(\gamma)}{\Gamma(\beta+\gamma)} Y_4;$$

$$\int_{\frac{1}{\mu}}^\infty X dx = e^{-\pi i(\gamma'-\beta'-1)+\pi i(1-a')} \frac{\Gamma(a'+1-\gamma')\Gamma(1-a')}{\Gamma(2-\gamma')} \mu^{1-\gamma'}(1-\mu)^{\gamma'-\beta'-1}$$

$$\times F\left(\beta'-\gamma'+1,\ 1-a',\ 2-\gamma',\ \frac{\mu}{\mu-1}\right)$$

$$= e^{-\pi i(\beta+\gamma)} \frac{\Gamma(\gamma)\Gamma(\delta)}{\Gamma(\gamma+\delta)} Y_2.$$

Hence $$(\lambda-1)\frac{\Gamma(a)\Gamma(\beta)}{\Gamma(a+\beta)} Y_1 = e^{\pi i(\beta-1)} \frac{\Gamma(\beta)\Gamma(\gamma)}{\Gamma(\gamma+\delta)} Y_4 + e^{-\pi i(\beta+\gamma)} \frac{\Gamma(\gamma)\Gamma(\delta)}{\Gamma(\gamma+\delta)} Y_2.$$

Now, if $$M_1 = \frac{\Pi(\gamma'-1)\Pi(-a')\Pi(-\beta')}{\Pi(1-\gamma')\Pi(\gamma'-a'-1)\Pi(\gamma'-\beta'-1)} = \frac{\Gamma(a+\beta)\Gamma(\gamma)\Gamma(1-a)}{\Gamma(\gamma+\delta)\Gamma(1-\delta)\Gamma(\beta)},$$

$$N_1 = \frac{\Pi(-a')\Pi(-\beta')}{\Pi(\gamma'-a'-\beta')\Pi(-\gamma')} = \frac{\Gamma(\gamma)\Gamma(1-a)}{\Gamma(\beta+\gamma)\Gamma(\gamma+\delta-1)},$$

then $$Y_1 = M_1 Y_2 + N_1 Y_4.$$

Substituting, we have

$$Y_4\left[(\lambda-1)\frac{\Gamma(a)\Gamma(\beta)}{\Gamma(a+\beta)} N_1 - e^{\pi i(\beta-1)} \frac{\Gamma(\beta)\Gamma(\gamma)}{\Gamma(\beta+\gamma)}\right]$$

$$= Y_2\left[e^{-\pi i(\beta+\gamma)} \frac{\Gamma(\gamma)\Gamma(\delta)}{\Gamma(\gamma+\delta)} - (\lambda-1)\frac{\Gamma(a)\Gamma(\beta)}{\Gamma(a+\beta)} M_1\right].$$

By using the properties of the Γ functions, the coefficient of Y_4 can be proved equal to

$$\frac{e^{\pi i a}}{a \sin a\pi} \frac{\Gamma(\beta)\Gamma(\gamma)}{\Gamma(\beta+\gamma)} \{d \sin(a+\beta)\pi - a \sin\beta\pi\} = -\frac{e^{\pi i a}\sin\gamma\pi}{a \sin a\pi} c \frac{\Gamma(\beta)\Gamma(\gamma)}{\Gamma(\beta+\gamma)},$$

and the coefficient of Y_2 can be proved* equal to

$$\frac{e^{\pi i a}}{a \sin a\pi} \frac{\Gamma(\gamma)\Gamma(\delta)}{\Gamma(\gamma+\delta)} \{d \sin(a+\beta+\gamma)\pi - a \sin(\beta+\gamma)\pi\} = -\frac{e^{\pi i a} \sin \gamma\pi}{a \sin a\pi} b \frac{\Gamma(\gamma)\Gamma(\delta)}{\Gamma(\gamma+\delta)}.$$

Moreover

$$Y_2 = y_{20} = y_4 = \mu^{1-\gamma'} (1-\mu)^{\gamma'-a'-\beta'} F\{1-a', 1-\beta', 2-\gamma', \mu\},$$

$$Y_4 = y_{24} = y_8 = \mu^{1-\gamma'} (1-\mu)^{\gamma'-a'-\beta'} F\{1-a', 1-\beta', \gamma'-a'-\beta'+1, 1-\mu\}:$$

so that

$$\frac{Y_2}{Y_4} = \frac{F\{\gamma, 1-a, \gamma+\delta, \mu\}}{F\{\gamma, 1-a, \gamma+\beta, 1-\mu\}};$$

and therefore an equation to determine μ is

$$\frac{F\{\gamma, 1-a, \gamma+\delta, \mu\}}{F\{\gamma, 1-a, \gamma+\beta, 1-\mu\}} = \frac{c}{b} \frac{\Gamma(\beta)\Gamma(\gamma+\delta)}{\Gamma(\delta)\Gamma(\gamma+\beta)}.$$

Ex. A regular polygon of n sides, in the w-plane, has its centre at the origin and one angular point on the axis of real quantities at a distance unity from the origin. Shew that its interior is conformally represented on the interior of a circle, of radius unity and centre the origin, in the z-plane by means of the relation

$$w \int_0^1 (1-x^n)^{-\frac{2}{n}} dx = \int_0^z (1-z^n)^{-\frac{2}{n}} dz. \qquad \text{(Schwarz.)}$$

270. It is natural to consider the form which the relation assumes when we pass from the convex polygon to a convex curve, by making the number of sides of the polygon increase without limit. The external angle between two consecutive tangents being denoted by $d\psi$, and the internal angle of the polygon at the point of intersection of the tangents being $\xi\pi$, we have

$$\pi - \xi\pi = d\psi,$$

so that

$$\xi - 1 = -\frac{d\psi}{\pi}.$$

Let x be the point on the axis of real quantities, which corresponds to this angular point of the polygon; then the limiting form of the relation

$$\frac{d}{dz}\left(\log \frac{dw}{dz}\right) = \Sigma \frac{\alpha - 1}{z - a}$$

is

$$\frac{d}{dz}\left(\log \frac{dw}{dz}\right) = -\frac{1}{\pi}\int \frac{d\psi}{z - x},$$

where x is the point on the real axis in the z-plane corresponding to the point on the w-curve at which the tangent makes an angle ψ with some fixed line, and the integral extends round the curve, which is supposed to be simple (that is, without singular points) and everywhere convex.

The disadvantage of the form is that x is not known as a function of ψ, and its chief use is to construct curves such that the contour is conformally represented, according to any assigned law, along the axis of real quantities

* In reducing the coefficients to these forms, limiting cases (such as $\beta+\gamma=1$) of the quadrilateral are excluded.

in the z-plane. The utility of the form is thus limited: the relation is not available for the construction of a function by which a given convex area in the w-plane can be conformally represented on the half of the z-plane*.

Ex. Let $x = \tan\frac{1}{2}\psi$: then taking the integral from $-\pi$ to $+\pi$, we have

$$\frac{d}{dz}\left(\log\frac{dw}{dz}\right) = -\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{d\psi}{z-\tan\frac{1}{2}\psi}$$
$$= -\frac{2}{\pi}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\frac{d\phi}{z-\tan\phi}.$$

The integral on the right-hand side is

$$\int_0^{\frac{1}{2}\pi}\frac{d\phi}{z-\tan\phi} - \int_{\frac{1}{2}\pi}^{0}\frac{d\phi}{z+\tan\phi}$$
$$= 2z\int_0^{\frac{1}{2}\pi}\frac{d\phi}{z^2-\tan^2\phi}$$
$$= 2z\int_0^{\infty}\frac{dy}{(1+y^2)(z^2-y^2)}$$
$$= \frac{2z}{1+z^2}\int_0^{\infty}\left\{\frac{1}{1+y^2} - \frac{1}{y^2+(zi)^2}\right\}dy$$
$$= \frac{2z}{1+z^2}\left\{\tfrac{1}{2}\pi - \tfrac{1}{2}\pi\frac{1}{zi}\right\}$$
$$= \frac{\pi}{z-i},$$

and therefore
$$\frac{d}{dz}\left(\log\frac{dw}{dz}\right) = -\frac{2}{z-i},$$

which, on further integration, leads to the ordinary expression for a circle on a half-plane.

271. In regard to the conformal representation on the half of the z-plane of figures in the w-plane bounded by circular arcs, we proceed† in a manner similar to that adopted for the conformal representation of rectilinear polygons.

It is manifest that, if $u = f(z)$ determine a conformal representation on the z-plane of a w-polygon bounded by circular arcs and having assigned angles, then

$$w = \frac{Au+B}{Cu+D},$$

where A, B, C, D may be taken subject to the condition $AD - BC = 1$, will represent on the half z-plane another such polygon with the same assigned

* See Christoffel, *Gött. Nachr.*, (1870), pp. 283—298.

† For the succeeding investigations the following authorities may be consulted:—
Schwarz, *Ges. Werke*, t. ii, pp. 78—80, 221—259.
Cayley, *Camb. Phil. Trans.*, vol. xiii, (1879), pp. 5—35.
Klein, *Vorlesungen über das Ikosaeder*, Section I., and particularly pp. 77, 78.
Darboux, *Théorie générale des surfaces*, t. i, pp. 180—192.
Klein-Fricke, *Theorie der elliptischen Modulfunctionen*, t. i, pp. 93—114.
Goursat, l.c., p. 516, note.

angles: for the homographic transformation, preserving angles unchanged, changes circles into circles or occasionally into straight lines. Hence, as in § 264, when the transforming function is being obtained, it is to be expected that it will be such as to admit of this apparent generality: and therefore, since

$$\{w, z\} = \{u, z\},$$

where $\{w, z\}$ is the Schwarzian derivative, *it follows that, in obtaining the conformal representation of a figure bounded by circular arcs, the function to be constructed is*

$$S = \{w, z\} = \frac{w'''}{w'} - \frac{3}{2}\left(\frac{w''}{w'}\right)^2.$$

We proceed as in the case of the rectilinear polygon and find the form of the appropriate function in the vicinity of points of various kinds. But one immediate simplification is possible, which enables us to use some of the earlier results.

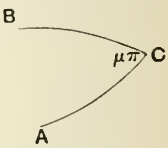

Fig. 96.

Let C be an angular point, CA and CB two circular arcs, one of which may be a straight line: if both were straight lines, the modification would be unnecessary. Invert the figure with regard to the other point of intersection of CA and CB: the two circles invert into straight lines cutting at the same angle $\mu\pi$. Take the reflexion of the inverted figure in the axis of imaginary quantities: and make any displacement parallel to the axis of real quantities: if W be the new variable, the relation between w and W is of the form

$$\frac{aW + b}{cW + d} = w,$$

where $ad - bc = 1$; and therefore

$$\{W, z\} = \{w, z\}.$$

Consider the function for the W-plane. Let Γ be the point corresponding to C, an angular point of the polygon, having $z = c$ as its homologue on the axis of x, account being taken of the possibility of having $c = \infty$; let β be any point on either of the straight lines corresponding to a point on the contour of the polygon not an angular point, having $z = b$ as its homologue on the axis of x. If a contour point not an angular point have $z = \infty$ as its homologue on the axis, denote it by β'.

Then for the vicinity of β, we have (as in § 267) a relation of the form

$$W - \beta = e^{i(\pi+\theta)}(z - b)\,P(z - b):$$

then
$$\log \frac{dW}{dz} = \text{const.} + \log P_1(z - b),$$

so that
$$\{W, z\} = P_2(z - b),$$

where P_2 is an integral function of $z - b$, converging for sufficiently small values of $|z - b|$.

For the vicinity of β', we have similarly

$$W - \beta' = e^{i(\pi+\theta)} \frac{1}{z} Q\left(\frac{1}{z}\right):$$

then
$$\frac{dW}{dz} = e^{i\theta} \frac{1}{z^2} Q_1\left(\frac{1}{z}\right),$$

and therefore
$$\{W, z\} = \frac{1}{z^4}\left\{\frac{Q_1''\left(\frac{1}{z}\right)}{Q_1\left(\frac{1}{z}\right)} - \frac{\frac{3}{2}Q_1'^2\left(\frac{1}{z}\right)}{Q_1^2\left(\frac{1}{z}\right)}\right\}$$
$$= \frac{1}{z^4} Q_2\left(\frac{1}{z}\right),$$

where Q_2 does not vanish for $z = \infty$.

In the vicinity of the angular point Γ, having a finite point on the axis of x for its homologue, we have

$$W - \Gamma = e^{i(\pi+\theta)} (z - c)^{\mu} R(z - c),$$

and, proceeding as before, we find that

$$\{W, z\} = \frac{\frac{1}{2}(1 - \mu^2)}{(z - c)^2} + \frac{C_0}{z - c} + R_2(z - c),$$

where C_0 depends on the coefficients in the series $R(z - c)$.

But if the angular point Γ have the point at infinity on the axis of x for its homologue, we have

$$W - \Gamma = e^{i(\pi+\theta)} \frac{1}{z^{\mu}} T\left(\frac{1}{z}\right):$$

then, proceeding as before, we find that

$$\{W, z\} = \frac{\frac{1}{2}(1 - \mu^2)}{z^2} + \frac{1}{z^3} T_2\left(\frac{1}{z}\right),$$

where $T_2\left(\frac{1}{z}\right)$ does not vanish when $z = \infty$.

Lastly, for a point W' in the interior having its homologue at $z = z'$, we have

$$W - W' = (z - z') S(z - z'),$$

and then
$$\{W, z\} = S_2(z - z').$$

Hence $\{W, z\}$, considered as a function of z, has the following properties :—

(i) It is an analytical function of z, real for all real values of the argument z; and if $x = \infty$ do not correspond to an angular point of the polygon, then for very large values of z

$$\{W, z\} = \frac{1}{z^4} Q_2\left(\frac{1}{z}\right),$$

where Q_2 is finite when $z = \infty$.

(ii) It has a finite number of accidental singularities, all of them isolated points on the axis of x: and at all other points on one side of the plane it is uniform finite and continuous, having (except at the accidental singularities) real continuous values for real continuous values of its argument. Its form near the singularities, and its form for infinitely large values of z, if $z=\infty$ be the homologue of an angular point, are given above.

Hence $\{W, z\}$ can be continued across the axis of x, conjugate values of $\{W, z\}$ corresponding to conjugate values of z: and thus its properties make it an algebraical rational meromorphic function of z.

Two cases have to be considered.

First, let the angular points of the polygon have their homologues at finite distances from the z-origin, say, at $a, b, \ldots, l$: and let $\alpha\pi, \beta\pi, \ldots, \lambda\pi$ be the internal angles of the polygon at the vertices. Then

$$\{W, z\} - \Sigma \frac{A}{z-a} - \tfrac{1}{2}\Sigma \frac{1-\alpha^2}{(z-a)^2}$$

has no infinity in the plane; it is a uniform analytical function of z, and must therefore be a constant, which, by the value at $z=\infty$, is seen to be zero. Hence

$$\{W, z\} = \Sigma \frac{A_0}{z-a} + \tfrac{1}{2}\Sigma \frac{1-\alpha^2}{(z-a)^2} = 2J(z),$$

the summation being for the homologues of all the angular points of the polygon. But when z is very large, we have, in this case

$$\{W, z\} = \frac{1}{z^4} Q_2\left(\frac{1}{z}\right),$$

so that, expanding $2J(z)$ in powers of $\dfrac{1}{z}$ and comparing with the latter form, we have, on equating coefficients of z^{-1}, z^{-2}, z^{-3},

$$0 = \Sigma A_0,$$

$$0 = \Sigma A_0 a + \tfrac{1}{2}\Sigma(1-\alpha^2),$$

$$0 = \Sigma A_0 a^2 + \Sigma a(1-\alpha^2),$$

relations among the constants of the problem.

Secondly, let one angular point, say a, of the polygon have its homologue on the axis of x at infinity, and let $\alpha\pi$ be the internal angle at a: and let the homologues of the others be $b, \ldots, k, l$, the internal angles of the polygon being $\beta\pi, \ldots, \kappa\pi, \lambda\pi$. Then the function

$$\{W, z\} - \Sigma \frac{B_0}{z-b} - \tfrac{1}{2}\Sigma \frac{1-\beta^2}{(z-b)^2}$$

has no infinity in the plane: it is a uniform analytical function of z, and must therefore be a constant, say M; thus

$$\{W, z\} = M + \Sigma \frac{B_0}{z-b} + \tfrac{1}{2}\Sigma \frac{1-\beta^2}{(z-b)^2}.$$

But, when z is very large, we have

$$\{W, z\} = \frac{\tfrac{1}{2}(1-\alpha^2)}{z^2}\left[1 + \frac{1}{z}T\left(\frac{1}{z}\right)\right],$$

because $x = \infty$ is the homologue of the vertex a of the polygon, the angle there being $\alpha\pi$: and $T\left(\frac{1}{z}\right)$ does not vanish when $z = \infty$. Hence, expanding in powers of $\frac{1}{z}$ and comparing coefficients, we have

$$M = 0,$$

$$\Sigma B_0 = 0,$$

$$\tfrac{1}{2}\Sigma(1-\beta^2) + \Sigma B_0 b = \tfrac{1}{2}(1-\alpha^2),$$

so that

$$\{W, z\} = \Sigma \frac{B_0}{z-b} + \tfrac{1}{2}\Sigma \frac{1-\beta^2}{(z-b)^2} = 2I(z),$$

where the summation is for the homologues of all the angular points other than a, and the constants are subject to the two conditions

$$\Sigma B_0 = 0,$$

$$\Sigma B_0 b = \tfrac{1}{2}(1-\alpha^2) - \tfrac{1}{2}\Sigma(1-\beta^2).$$

The form of the function $\{W, z\}$ is thus obtained for the two cases, the latter being somewhat more simple than the former: and the exact expansion of W in the vicinity of a singular point can be obtained with coefficients expressed in terms of the constants.

272. In either case the equation which determines W is of the third order: but the determination can be simplified by using a well-known property of linear differential equations*. If y_1 and y_2 be two solutions of the equation

$$\frac{d^2y}{dx^2} + 2P\frac{dy}{dx} + Qy = 0,$$

the quotient of which is equal to the quotient of two solutions of

$$\frac{d^2Y}{dx^2} + IY = 0,$$

where $I = Q - \frac{dP}{dx} - P^2$, being the invariant of the equation for linear transformation of the dependent variable, and where $Y/y = e^{\int P dx}$, then the equation satisfied by s, $= y_1/y_2$, is

$$\{s, x\} = 2I.$$

* See my *Treatise on Differential Equations*, pp. 89—93.

Hence for the present case, if we can determine two independent solutions Z_1 and Z_2 of the equation

$$\frac{d^2Z}{dz^2} + Z\,J(z) = 0$$

for the first case, or two independent solutions of the equation

$$\frac{d^2Z}{dz^2} + Z\,I(z) = 0$$

for the second case, then

$$W = \frac{AZ_1 + BZ_2}{CZ_1 + DZ_2}$$

is the general solution of the equation

$$\{W, z\} = 2J(z) \text{ or } 2I(z),$$

and therefore is the function by which the curvilinear w-polygon is conformally represented on the z-half-plane.

273. As a first example, consider the w-area between two circular arcs which cut at an angle $\lambda\pi$. The z-origin can be conveniently taken as the homologue of one of the angular points: and the z-point at infinity along the axis of x as the homologue of the other. Then we have

$$\{W, z\} = \frac{A}{z} + \frac{\frac{1}{2}(1-\lambda^2)}{z^2},$$

provided $\qquad A = 0,\; A\,.\,0 = \frac{1}{2}(1-\lambda^2) - \frac{1}{2}(1-\lambda^2),$

both of which conditions are satisfied by $A = 0$; and so

$$\{W, z\} = \frac{\frac{1}{2}(1-\lambda^2)}{z^2}.$$

The linear differential equation is

$$\frac{d^2Z}{dz^2} + \frac{1}{4}\,\frac{1-\lambda^2}{z^2}\,Z = 0,$$

so that $\qquad Z_1 = z^{\frac{1}{2}(1+\lambda)},\; Z_2 = z^{\frac{1}{2}(1-\lambda)};$

and therefore the general solution for W is

$$W = \frac{az^\lambda + b}{cz^\lambda + d}.$$

The (three) arbitrary constants can be determined by making $z = 0$ and $z = \infty$ correspond to the angular points of the crescent, and the direction of the line $z = z_0$ (which is the axis of x) correspond to one of the circles, the other of the circles being then determinate.

If the w-circles intersect in $-i$ (the homologue of the z-origin) and $+i$

(the homologue of $x=\infty$), and if the centre of one of the circles be at the point $(\cot\alpha, 0)$, then the relation is

$$w = i\,\frac{z^\lambda - ce^{-\alpha i}}{z^\lambda + ce^{-\alpha i}},$$

where c is an arbitrary constant, equivalent to the possible constant magnification of the z-plane without affecting the conformal representation: it can be determined by fixing homologous points on the contour of the crescent.

More generally, if the w-circles intersect in w_1 and w_2, respectively homologous to $z=0$ and $z=\infty$, then

$$z^\lambda = K\,\frac{w-w_1}{w-w_2}$$

is the form of the relation.

Evidently a segment of a circle is a special case.

274. Next, consider a triangle in the w-plane formed by three circular arcs and let the internal angles be $\lambda\pi$, $\mu\pi$, $\nu\pi$. The homologue of one of the angular points, say of that at $\mu\pi$, can be taken at $z=\infty$; of one, say of that at $\lambda\pi$, at the z-origin; and of the other, say of that at $\nu\pi$, at a point $z=1$: all on the axis of x. Then we have

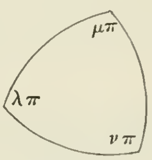

Fig. 97.

$$2I(z) = \frac{B}{z} + \frac{C}{z-1} + \tfrac{1}{2}\,\frac{1-\lambda^2}{z^2} + \tfrac{1}{2}\,\frac{1-\nu^2}{(z-1)^2},$$

where the constants B and C are subject to the relations

$$B + C = 0,$$

$$B\,.\,0 + C\,.\,1 = \tfrac{1}{2}(1-\mu^2) - \tfrac{1}{2}(1-\lambda^2) - \tfrac{1}{2}(1-\nu^2),$$

so that

$$-B = C = \tfrac{1}{2}(\lambda^2 - \mu^2 + \nu^2 - 1),$$

and therefore

$$I(z) = \tfrac{1}{4}\,\frac{1-\lambda^2}{z^2} + \tfrac{1}{4}\,\frac{1-\nu^2}{(z-1)^2} + \tfrac{1}{4}\,\frac{\lambda^2-\mu^2+\nu^2-1}{z(z-1)}.$$

But $I(z)$ is the invariant of the differential equation of the hypergeometric series*

$$\frac{d^2Z}{dz^2} + \frac{\gamma-(\alpha+\beta+1)z}{z(1-z)}\frac{dZ}{dz} - \frac{\alpha\beta}{z(1-z)}Z = 0,$$

provided $\lambda^2 = (1-\gamma)^2$, $\mu^2 = (\alpha-\beta)^2$, $\nu^2 = (\gamma-\alpha-\beta)^2$;

so that, if Z_1 and Z_2 be two particular solutions of this equation, the function which gives the conformal representation of the w-triangle on the z-half-plane is

$$w = \frac{AZ_1 + BZ_2}{CZ_1 + DZ_2}.$$

* *Differential Equations*, p. 188.

The transforming function thus depends upon the solution of the differential equation of the hypergeometric series, and for general values of λ, μ, ν which are $> 0 < 1$ we shall obtain merely general values of α, β, γ; hence the transforming function will be obtained as a quotient of two particular solutions of the equation of the series. Now according to the magnitude of $|z|$, these solutions, which are in the form of infinite series, change: and thus we have w equal to an analytical function of z, which has different branches in different parts of the plane.

The distribution of the values $z = 0, 1, \infty$ as the homologues of the three angular points was an arbitrary selection of one among six possible arrangements, which change into one another by the following scheme:—

z	$1-z$	$\frac{1}{z}$	$\frac{1}{1-z}$	$\frac{z}{z-1}$	$\frac{z-1}{z}$
0	1	∞	1	0	∞
1	0	1	∞	∞	0
∞	∞	0	0	1	1

The quantities in the first row are the homographic substitutions, conserving the positive half-plane and interchanging the arrangements.

These substitutions are the functions of z subsidiary to the derivation of Kummer's set of 24 particular solutions of the equation of the hypergeometric series.

Ex. Take the case when two of the angles of the triangle are right, say $\nu = \frac{1}{2}$, $\mu = \frac{1}{2}$, $\lambda = \frac{1}{n}$. Then, when n is finite*, a transforming relation is

$$w^n = \frac{1-(1-z)^{\frac{1}{2}}}{1+(1-z)^{\frac{1}{2}}};$$

and, when n is infinite, a transforming relation is

$$w = \log\frac{1-(1-z)^{\frac{1}{2}}}{1+(1-z)^{\frac{1}{2}}},$$

obtainable either as a limiting form of the above, or by means of the solutions $F(a, \beta, \gamma, z)$ and $F(a, \beta, a+\beta-\gamma+1, 1-z)$ of the differential equation of the hypergeometric series. In the respective cases the general relations, establishing the conformal representation, are

$$\left(\frac{aw+b}{cw+d}\right)^n = \frac{1-(1-z)^{\frac{1}{2}}}{1+(1-z)^{\frac{1}{2}}},$$

and

$$\frac{aw+b}{cw+d} = \log\frac{1-(1-z)^{\frac{1}{2}}}{1+(1-z)^{\frac{1}{2}}}.$$

* *Differential Equations*, p. 208.

The three circles, arcs of which form the triangle, divide the whole of the w-plane into eight triangles which can be arranged in four pairs, each pair having angles of the same magnitude. Thus

D, D' have angles $\lambda\pi$, $\mu\pi$, $\nu\pi$,

A, A' .... $\lambda\pi$, $(1-\mu)\pi$, $(1-\nu)\pi$,

B, B' .... $(1-\lambda)\pi$, $\mu\pi$, $(1-\nu)\pi$,

and C, C' .... $(1-\lambda)\pi$, $(1-\mu)\pi$, $\nu\pi$;

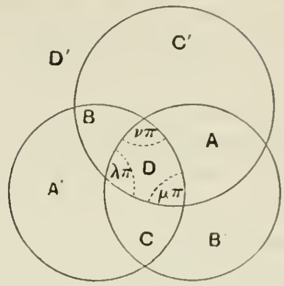

Fig. 98.

and when any one of the triangles is given, it determines the remaining seven. It is convenient then to choose that one which has the sum of its angles the least, say the triangle of reference: let it be D. Unless λ, μ, ν, each of which is $> 0 < 1$, be each $= \frac{1}{2}$, then $\lambda + \mu + \nu < \frac{3}{2}$.

We have already, in part, considered the case in which $\lambda + \mu + \nu = 1$. For, when this equation holds, inversion with the other point having $\lambda\pi$ for its angle as centre of inversion, changes* D into a triangle bounded by straight lines and having $\lambda\pi$, $\mu\pi$, $\nu\pi$ as its angles; and therefore, in that case, the problem is merely a special instance of the representation of a w-rectilinear polygon on the z-half-plane.

But there is a very important difference between the cases for which $\lambda + \mu + \nu < 1$ and those for which $\lambda + \mu + \nu > 1$: in the former, the orthogonal circle (having its centre at the radical centre of the three circles) is real, and in the latter it is imaginary. The cases must be treated separately.

275. First, we take $\lambda + \mu + \nu < 1$. Then of the two triangles, which have the same angles, one lies entirely within the orthogonal circle and the other entirely without it; and each is the inverse of the other with regard to the orthogonal circle†. Let inversion with regard to the angular point $\lambda\pi$ in A take place: then the new triangle is bounded by two straight lines cutting at an angle $\lambda\pi$ and by a circular arc cutting them at angles $\mu\pi$ and $\nu\pi$ respectively, the convex side of the arc being turned towards the straight angle. The new orthogonal circle is the inverse of the old and its centre is A, the angular point at $\lambda\pi$; its radius is the tangent from A to the arc CB, and therefore it completely includes the triangle ABC.

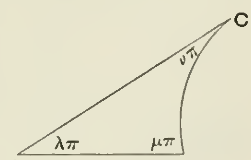

Fig. 99.

The homologue of A is, as before, taken to be the z-origin O, that of C to be the point $z = 1$, say c, and that of B to be $z = \infty$ on the axis of x, say b for $+\infty$ and b' for $-\infty$.

* The figure in the text does not apply to this case, because, as may easily be proved, the three circles must meet in a point.

† For the general properties of such systems of circles, see Lachlan, *Quart. Journ. Math.*, vol. xxi, (1886), pp. 1—59.

Suppose that we have a representation of the triangle on the positive half-plane of z. The function $\{w, z\}$ can be continued across the axis of x into a negative half-plane, if the passage be over a part of that axis, where the function is real and continuous, that is, if the passage be over Oc, or over cb, or over $b'O$; and therefore w is defined for the whole plane by $\{w, z\} = 2I(z)$, its branch-points being O, c, b. Any branch on the other side, say w_1, will give, on the negative half-plane, a representation of a triangle having the same angles, bounded by circular arcs orthogonal to the same circle, and having O, c, b for the homologues of its angular points. Thus if the continuation be over cb, the new w-triangle has CB common with the old, and the angular point A' lies beyond CB from A.

To obtain the new triangle $A'CB$ geometrically, it is sufficient to invert the triangle ACB, with regard to the centre of the circular arc CB. This inversion leaves CB unaltered; it gives a circular arc CA' instead of CA and a circular arc BA' instead of BA: the angles of $A'CB$ are the same as those of ACB. Since the orthogonal circle of ACB cuts CB at right angles and CB is inverted into itself, the orthogonal circle is inverted into itself; therefore the triangle $A'CB$ has the same orthogonal circle as the triangle ACB.

The branch w_1, by passing back across the axis round a branch-point into the positive half-plane, leads to a new branch w_2, which gives in that half-plane a representation of a triangle, again having the angles $\lambda\pi, \mu\pi, \nu\pi$ and having O, c, b for the homologues of its angular points. Thus if the passage be over Oc, the new w-triangle has $A'C$ common with $A'CB$ and the angular point B'' lies on the side of CA' remote from B: but if the passage be over cb, then we merely revert to the original triangle CAB. The new triangle has, as before, the same orthogonal circle as $A'CB$.

Proceeding in this way by alternate passages from one side of the axis of x to the other, we obtain each time a new w-triangle, having one side common with the preceding triangle and obtained by inversion with respect to the centre of that common side: and for each triangle we obtain a new branch of the function w, the branch-points being $0, 1, \infty$. If, by means of sections such as Hermite's (§ 103), we exclude all the axis of x except the part between two branch-points, the function is uniform over the whole plane thus bounded.

All these triangles lie within the orthogonal circle, and they gradually approach its circumference: but as the centres of inversion always turn that circle into itself, while the sides of the triangle are orthogonal to it, they do not actually reach the circumference. The orthogonal circle forms a *natural limit* (§ 81) to the part of the w-plane thus obtained.

Ex. Shew that all the inversions, necessary to obtain the complete system of triangles, can be obtained by combinations of inversions in the three circles of the original triangle. (Burnside.)

Each of the triangles, thus formed in successive alternation, gives a w-region conformally represented on one half or on the other of the z-plane. If, then, the original triangle be combined with the first triangle that is conformally represented on the negative half-plane, every other similar combination may be regarded as a symmetrical repetition of that initial combination: each of them can be conformally represented upon the whole of the z-plane, with appropriate barriers along the axis of x.

The number of the triangles is infinite, and with each of them a branch of the function w is associated: hence the integral relation between w and z which is equivalent to the differential relation $\{w, z\} = 2I(z)$, when $\lambda + \mu + \nu < 1$, is transcendental in w.

In the construction of the successive triangles, the successive sides passing through any point, such as C, make the same angle each with its predecessor: and therefore the repetition of the operation will give rise to a number of triangles at C each having the same angle $\lambda\pi$.

If λ be incommensurable, then no finite number of operations will lead to the initial triangle: each operation gives a new position for the homologous side and ultimately the w-plane in this vicinity is covered an infinite number of times, that is, we can regard the w-surface as made up of an infinite number of connected sheets.

If λ be commensurable, let it be equal to l/l', where l and l' are integers, prime to each other. When l is odd, $2l'$ triangles will fill up the w-space immediately round C, and the $(2l' + 1)$th triangle is the same as the first: but the space has been covered l times since $2l'\lambda\pi = 2l\pi$, that is, in the vicinity of C we can regard the w-surface as made up of l connected sheets. When l is even (and therefore l' odd), l' triangles will fill up the space round C completely, but the $(l'+1)$th triangle is not the same as the first: it is necessary to fill up the space round C again, and the $(2l' + 1)$th triangle is the same as at first; the space has then been covered l times, so that again the w-surface can be regarded as made up of l connected sheets. The simplest case is evidently that, in which λ is the reciprocal of an integer, so that $l = 1$; and the w-surface must then be regarded as single-sheeted.

Similar considerations arise according to the values of μ and of ν.

If then either λ, μ, or ν be incommensurable, the number of w-sheets is unlimited, that is, z as a function of w has an infinite number of values, or the equation between z and w is transcendental in z. Hence, *when* $\lambda + \mu + \nu < 1$ *and either* λ *or* μ *or* ν *is incommensurable, the integral relation between* w *and* z, *which is equivalent to the differential relation* $\{w, z\} = 2I(z)$, *is transcendental both in* w *and in* z.

If all the quantities λ, μ, ν be commensurable and have the forms l/l', m/m', n/n', fractions in their lowest terms, and if N be the least common multiple of l, m, n, then the number of w-sheets is N, that is, z as a function of w has N values and therefore the equation between z and w is algebraical in z, of degree N. Hence, *when* $\lambda+\mu+\nu<1$ *and* λ, μ, ν *have the forms of fractions in their lowest terms, and if* N *be the least common multiple of their numerators, the integral relation between* w *and* z *equivalent to the differential relation*

$$\{w, z\} = 2I(z)$$

is an algebraical equation of degree N *in* z, *the coefficients of which are transcendental functions of* w.

The simplest case of all arises when λ, μ, ν are the reciprocals of integers: for then $N=1$ and z is a uniform transcendental function of w, satisfying the equation

$$\{w, z\} = 2I(z);$$

or, making z the dependent and w the independent variable, we have the result:—

A function z *that satisfies the equation*

$$-\left[\frac{d^3z}{dw^3}\frac{dz}{dw}-\frac{3}{2}\left(\frac{d^2z}{dw^2}\right)^2\right]=\left[\frac{1}{2}\frac{1-\dfrac{1}{l^2}}{z^2}+\frac{1}{2}\frac{1-\dfrac{1}{n^2}}{(z-1)^2}+\frac{1}{2}\frac{\dfrac{1}{l^2}-\dfrac{1}{m^2}+\dfrac{1}{n^2}-1}{z(z-1)}\right]\left(\frac{dz}{dw}\right)^4,$$

where l, m, n *are integers, such that* $\dfrac{1}{l}+\dfrac{1}{m}+\dfrac{1}{n}<1$, *is a uniform transcendental function of* w.

Restricting ourselves to the last case, merely for simplicity of explanation, it is easy to see that the whole of the space within the orthogonal circle is divided up into triangles, with angles $\lambda\pi$, $\mu\pi$, $\nu\pi$ bounded by circular arcs which cut that circle orthogonally: and, by the inversion which connects the space external to the circle with the internal space, the whole of the outside space is similarly divided. Moreover, it has been seen that every triangle can be obtained from any one by some substitution of the form $w_r=\dfrac{a_rw+b_r}{c_rw+d_r}$: therefore the division of the interior of the circle into triangles is that which is considered, in the next chapter, for the more general case of division into polygons, the orthogonal circle of the present case being then the 'fundamental' circle. The uniform transcendental function of w is therefore *automorphic*: the infinite group of substitutions is that which serves to transform a single triangle into the infinite number of triangles within the circle*.

One or two special cases need merely be mentioned.

If any one of the three quantities λ, μ, ν be zero and if $\lambda+\mu+\nu$ is not equal to unity, the triangle can be included under the general case just treated. For let $\lambda=0$, and suppose that $\mu+\nu$ is not greater than unity:

* The figure for the example $\nu=\frac{1}{2}$, $\mu=\frac{1}{4}$, $\lambda=\frac{1}{5}$ is given by Schwarz, *Ges. Werke*, t. ii, p. 240; and the figure for the example $\nu=\frac{1}{2}$, $\mu=\frac{1}{3}$, $\lambda=\frac{1}{7}$ is given in Klein-Fricke (p. 370); both of course satisfying the conditions $\lambda+\mu+\nu<1$.

if $\mu+\nu$ were greater than unity, the triangle would be a particular instance of the class about to be discussed. The division of the area within the (real) orthogonal circle is of the same general character as before: a particular illustration is provided by the division appropriate to the elliptic modular-functions, for which $\mu=\frac{1}{2}$, $\nu=\frac{1}{3}$ (§ 284). When two triangles, one of which is obtained from the other by continuation in the z-plane across the axis of real variables, are combined, they give a w-space (corresponding to the whole of the z-plane) for which $\lambda=0$, $\mu'=\frac{1}{3}$, $\nu=\frac{1}{3}$. Since the orthogonal circle is real, it forms a natural limit to these spaces; when it is transformed into the axis of real variables in the w-plane by a homographic substitution, the positive half of the w-plane is divided as in figure 108 (p. 590).

The extreme case of the present class, for which $\lambda+\mu+\nu$ is less than unity, is given by $\lambda=0$, $\mu=0$, $\nu=0$: the triangle is then the area between three circles which touch one another. Reverting to the differential equation of the hypergeometric series, we have $\gamma=1$, $\alpha=\beta=\frac{1}{2}$; the equation is

$$\frac{d^2Z}{dz^2}+\frac{1-2z}{z(1-z)}\frac{dZ}{dz}-\frac{\frac{1}{4}}{z(1-z)}Z=0,$$

which is the differential equation of the Jacobian quarter-periods in elliptic functions with modulus equal to $z^{\frac{1}{2}}$. If

$$K=\int_0^{\frac{1}{2}\pi}(1-z\sin^2\phi)^{-\frac{1}{2}}\,d\phi,\quad K'=\int_0^{\frac{1}{2}\pi}\{1-(1-z)\sin^2\phi\}^{-\frac{1}{2}}\,d\phi,$$

then
$$w=\frac{K'}{K},$$

or, more generally,
$$w=\frac{aK+bK'}{cK+dK'},$$

a relation between w and z which gives the conformal representation of the w-triangle upon the z-half-plane.

276. We now pass to the consideration of the case in which the triangle with angles $\lambda\pi$, $\mu\pi$, $\nu\pi$ has no real orthogonal circle: the other associated triangles have therefore not a real orthogonal circle. In this case, the sum of the angles of the triangle is greater than π, so that we have

$$\lambda+\mu+\nu>1 \text{ from the pair } D \text{ and } D',$$
$$-\lambda+\mu+\nu<1 \text{ from the pair } A \text{ and } A',$$
$$\lambda-\mu+\nu<1 \text{ from the pair } B \text{ and } B',$$
$$\lambda+\mu-\nu<1 \text{ from the pair } C \text{ and } C',$$

as the conditions which attach to the quantities λ, μ, ν. As before, we invert with respect to the angular point $\lambda\pi$ in A: then the new triangle D is bounded by two straight lines and a circle, the intersection of the lines being in the interior of the circle, because the orthogonal circle is imaginary. Let d be distance of L from the centre of the circle, θ the angle OLN, r the radius of the circle: then

$$d \sin \theta = - r \cos \nu\pi, \quad d \sin (\lambda\pi - \theta) = - r \cos \mu\pi,$$

which determine d and θ. Let $R^2 = r^2 - d^2$, so that iR is the radius of the (imaginary) orthogonal circle.

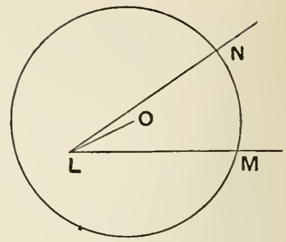

Fig. 100.

With L as centre and radius equal to R describe a sphere: let P be the extremity of the radius through L perpendicular to the plane. Then P can be taken as the centre for projecting the plane on the sphere stereographically*; so that, if Q be a point on the plane, Q' its projection on the sphere, $PQ \,.\, PQ' = 2R^2$. The projection of LN is a great circle through P, the projection of LM is another great circle through P inclined at $\lambda\pi$ to the former: and since PO is equal to the radius of the plane circle, so that its diameter subtends a right angle at P, the stereographic projection of that plane circle is a great circle on the sphere, making angles $\nu\pi$ and $\mu\pi$ with the former great circles. There is thus, on the sphere, a triangle bounded by arcs of great circles, that is, a spherical triangle in the ordinary sense, whose angles are $\lambda\pi$, $\mu\pi$, $\nu\pi$: and this spherical triangle is conformally represented on the z-half-plane, its angular points L, N, M finding their homologues in $z = 0, 1, \infty$ respectively.

Just as in the former case, the successive passages, backwards and forwards across the z-axis, give in the w-plane new triangles with angles $\lambda\pi$, $\mu\pi$, $\nu\pi$, all with the same imaginary orthogonal circle of radius iR and centre L: each of these, when stereographically projected on the sphere with P as the centre, becomes a spherical triangle of angles $\lambda\pi$, $\mu\pi$, $\nu\pi$ bounded by arcs of great circles, every triangle having one side common with its predecessor: and the triangles are equal in area.

Moreover, the triangles thus obtained correspond alternately to the positive half and the negative half of the z-plane: and it is convenient to consider two such contiguous triangles, connected with the variable w, as a single combination for the purposes of division of the spherical surface, each combination corresponding to the whole of the z-plane.

The repetition of the analytical process leads to the distribution of the surface of the sphere into such triangles: and the nature of the analytical relation between w and z depends on the nature of this distribution.

If λ, μ, or ν be incommensurable, then the number of triangles is

* Lachlan, (l.c., p. 557, *note*), p. 43.

infinite, so that the relation is transcendental in w: and the surface of the sphere is covered an infinite number of times; that is, corresponding to z there is an infinite number of sheets, so that the relation is transcendental in z. Thus, when $\lambda+\mu+\nu$ is greater than 1 and any one of the three quantities λ, μ, ν is incommensurable, the integral relation between w and z, which is equivalent to

$$\{w, z\} = 2I(z),$$

is transcendental both in w and in z.

If the quantities λ, μ, ν be commensurable, the simplest possible cases arise in connection with the division of the surface by the central planes associated with the inscribed regular solids. These planes give the divisions into triangles, which are equiangular with one another.

First, suppose that the spherical surface is divided completely and covered only once by the two sets of triangles, corresponding to the upper half and the lower half of the z-plane respectively. One of the sets, say N in number, will occupy one half of the surface in the aggregate: and similarly for the other set, also N in number. Hence

$$R^2(\lambda+\mu+\nu-1)\pi = \text{the area of a triangle}$$

$$= \frac{1}{N}\,(\text{area of a hemisphere}),$$

so that

$$\lambda+\mu+\nu-1=\frac{2}{N}.$$

Then, in passing round an angular point, say $\lambda\pi$, the triangles will alternately correspond to the upper and the lower halves: hence, of the whole angle 2π, one half will belong to one set of triangles and the other half to the other set. Hence $\pi \div \lambda\pi$ is an integer, that is, λ is the reciprocal of an integer, say $\frac{1}{l}$. Similarly for μ, which must be of the form $\frac{1}{m}$; and for ν, which must be of the form $\frac{1}{n}$; where m and n are integers. Thus

$$\frac{1}{l}+\frac{1}{m}+\frac{1}{n}-1=\frac{2}{N}.$$

The only possible solutions of this equation are

(I.)* $\lambda=\frac{1}{2}$, $\mu=\frac{1}{2}$, $n=$ any integer, $N=2n$;

(II.) $\lambda=\frac{1}{2}$, $\mu=\frac{1}{3}$, $\nu=\frac{1}{3}$, $N=12$;

(IV.) $\lambda=\frac{1}{2}$, $\mu=\frac{1}{3}$, $\nu=\frac{1}{4}$, $N=24$;

(VI.) $\lambda=\frac{1}{2}$, $\mu=\frac{1}{3}$, $\nu=\frac{1}{5}$, $N=60$.

277. In each of these cases there is a finite number of triangles: with each triangle a branch of w is associated, so that there is only a finite number

* The reason for the adoption of these numbers to distinguish the cases will appear later, in § 279.

of branches of w: the sphere is covered only once, and therefore there is only a single z-sheet. Hence the integral relation between w and z is of the first degree in z: and it is algebraical in w, of degrees $2n$, 12, 24, 60 respectively.

The regular solids, with which these sets of triangles are respectively associated, are easily discerned.

I. We have $\lambda, \mu, \nu = \frac{1}{2}, \frac{1}{2}, \frac{1}{n}$. The solid is a double pyramid, having its summits at the two poles of the sphere: the common base is an equatorial polygon of $2n$ sides: the sides of the various triangles, in the division of the sphere, are made by the half-meridians of longitude, through the angular points of the polygon from the respective poles to the equator, and by arcs of the equator subtended by the sides of the polygon.

Fig. 101.

II. We have $\lambda, \mu, \nu = \frac{1}{2}, \frac{1}{3}, \frac{1}{3}$. The solid is the tetrahedron; and the division of the surface of the sphere, by the planes of symmetry of the solid, into 24 triangles, 12 of each set, is indicated, in fig. 102, on the (visible) half of the sphere, the other (invisible) half of the sphere being the reflexion, through the plane of the paper, of the visible half.

The angular summits of the tetrahedron are T, the middle points of its edges are S, the centres of its faces are F: all projected on the surface of the sphere from the centre. If desired, the summits of the tetrahedron may be taken at F: the centres of the faces are then T.

Each of the angles at T is $\frac{1}{3}\pi$: each of the angles at F is $\frac{1}{3}\pi$: each of the angles at S is $\frac{1}{2}\pi$.

The shaded triangles (only six of which are visible, being half of the aggregate) correspond to one half of the z-plane; and the unshaded triangles correspond to the other half of the z-plane.

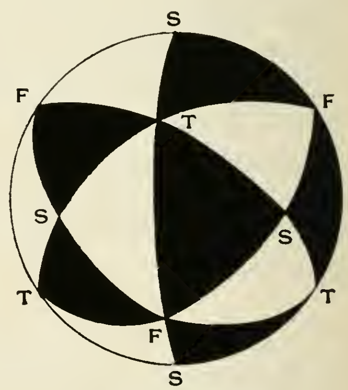

Fig. 102.

IV. We have $\lambda, \mu, \nu = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}$. The solid is the cube or the octahedron. These two solids can be placed so as to have the same planes of symmetry, by making the centres of the eight faces of the octahedron to be the summits of the cube. In the figure (fig. 103), the points O are the summits of the octahedron: the points C are the summits of the cube and the centres of the faces of the octahedron: and the points S are the middle points of the edges: all projected from the centre of the sphere.

The shaded triangles (the visible twelve being one half of the aggregate) correspond to one half of the z-plane; the unshaded triangles correspond to the other half of the z-plane.

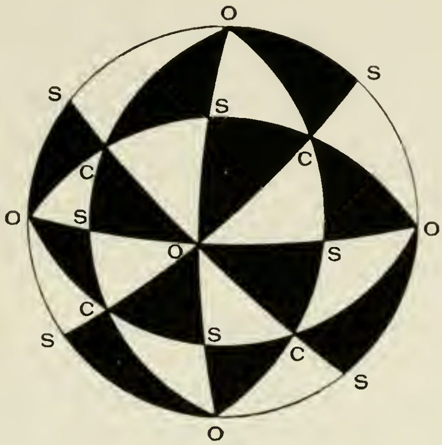

Fig. 103.

Each of the angles at O is $\frac{1}{4}\pi$: each of the angles at C is $\frac{1}{3}\pi$: each of the angles at S is $\frac{1}{2}\pi$; and it may be noted that the triangles COC are the triangles in the tetrahedral division of the spherical surface, the point O in the present triangle COC being the point S in a triangle STF and the two points C being the points F and T in the former figure (fig. 102).

VI. We have $\lambda, \mu, \nu = \frac{1}{2}, \frac{1}{3}, \frac{1}{5}$.

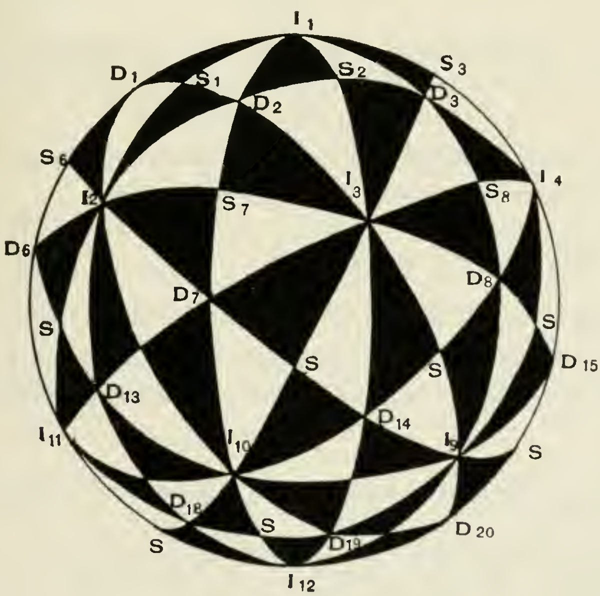

Fig. 104.

The solid is the icosahedron or the dodecahedron. These two solids can be placed so as to have the same planes of symmetry, by making the centres of the twenty faces of the icosahedron the vertices of the dodecahedron. In the figure (fig. 104) the vertices of the icosahedron are the points I: those of the dodecahedron are the points D: and the middle points of the edges are the points S. The shaded triangles (the visible thirty, six in each lune through a vertex of the icosahedron, being one half of their aggregate) correspond to one half of the z-plane: the unshaded triangles, equal in number and similarly distributed, correspond to the other half of the z-plane. The angles at the vertices I of the icosahedron are $\frac{1}{5}\pi$; those at the vertices D of the dodecahedron are $\frac{1}{3}\pi$; and those at the middle points S of the edges (the same for both solids) are $\frac{1}{2}\pi$.

278. Having obtained the division of the surface, we now proceed to determine the functions, which establish the conformal representation.

In all these cases, z is a uniform algebraical function of w: therefore when we know the zeros and the infinities of z as a function of w, each in its proper degree, we have the function determined save as to a constant factor. This factor can be determined from the value of w when $z=1$.

The variable w belongs to the stereographic projection of the point of the spherical surface on the equatorial plane, the south pole being the pole of projection. If X, Y, Z be the coordinates of the point on the spherical surface, the radius being unity, then

$$w=\frac{X+iY}{1+Z}.$$

For a point in longitude l and latitude $\frac{1}{2}\pi-\delta$, we have $X=\cos l\sin\delta$, $Y=\sin l\sin\delta$, $Z=\cos\delta$: so that, if preferable, another form for w is

$$w=e^{il}\tan\tfrac{1}{2}\delta.$$

In our preceding investigation, the angle at $\lambda\pi$ was made to correspond with $z=0$, that at $\nu\pi$ with $z=1$, that at $\mu\pi$ with $z=\infty$.

Case I. We take $\lambda=\dfrac{1}{n}$, $\mu=\frac{1}{2}$, $\nu=\frac{1}{2}$.

For the angular points $\mu\pi$ we have $\delta=\frac{1}{2}\pi$; $l=0, \dfrac{2\pi}{n}, \dfrac{4\pi}{n}, \ldots$, each point belonging to two triangles of the same set, that is, triangles represented on the same half of the plane: thus the various w-points in the plane are

$$e^{\frac{2\pi i}{n}r},$$

for $r=0, 1, \ldots, n-1$, each occurring twice. Hence $z=\infty$, when the function

$$\prod_{r=0}^{n-1}\left(w-e^{\frac{2\pi i}{n}r}\right)^2$$

vanishes, that is, $z=\infty$, when $(w^n-1)^2$ vanishes.

For the angular points $\nu\pi$, we have $\delta = \frac{1}{2}\pi$; $l = \frac{\pi}{n}, \frac{3\pi}{n}, \frac{5\pi}{n}, \ldots$, each point belonging to two triangles of the same set: thus the various w-points in the plane are

$$e^{\frac{\pi i}{n}(2r+1)},$$

for $r = 0, 1, \ldots, n-1$, each occurring twice. Hence $z = 1$, when the function

$$\prod_{r=0}^{n-1} \{w - e^{\frac{\pi i}{n}(2r+1)}\}^2$$

vanishes, that is, $z = 1$, when $(w^n + 1)^2$ vanishes.

Now z is a uniform function of w: hence we can take

$$1 - z = K\frac{(w^n + 1)^2}{(w^n - 1)^2},$$

where K is a constant, easily seen to be unity: because, when $w = 0$ (corresponding to the common vertex $\lambda\pi$ at the North pole) and when $w = \infty$ (corresponding to the common vertex $\lambda\pi$ at the South pole), z vanishes, as required. The relation is often expressed in the equivalent form

$$z : z - 1 : 1 = -4w^n : -(w^n + 1)^2 : (w^n - 1)^2,$$

which gives the conformation on the half z-plane of a w-triangle bounded by circular arcs, the angles being $\frac{\pi}{n}$, $\frac{1}{2}\pi$, $\frac{1}{2}\pi$. The simplest case is that in which the triangle is a sector of a circle with an angle $\frac{\pi}{n}$ at the centre.

The preceding relation is a solution of the equation

$$\{w, z\} = \frac{1}{2}\left[\frac{1 - \frac{1}{n^2}}{z^2} + \frac{1 - \frac{1}{4}}{(z-1)^2} + \frac{\frac{1}{n^2} - 1}{z(z-1)}\right].$$

If we choose $\lambda = \frac{1}{2}$, $\mu = \frac{1}{2}$, $\nu = \frac{1}{n}$; so that $z = 0$, when $(w^n + 1)^2$ vanishes, $z = \infty$, when $(w^n - 1)^2$ vanishes, and $z = 1$, when w^n vanishes, the relation establishing the conformal representation will be

$$z : z - 1 : 1 = (w^n + 1)^2 : 4w^n : (w^n - 1)^2:$$

this relation is a solution of the equation

$$\{w, z\} = \frac{1}{2}\left[\frac{1 - \frac{1}{4}}{z^2} + \frac{1 - \frac{1}{n^2}}{(z-1)^2} + \frac{\frac{1}{n^2} - 1}{z(z-1)}\right].$$

Case II. We take $\lambda = \frac{1}{2}$; so that $z = 0$ must give the points S, each of them twice, since there are two triangles of the same set at S: $\mu = \frac{1}{3}$ (and these are taken at T), so that $z = \infty$ must give the points T, each of them thrice: and $\nu = \frac{1}{3}$ (and these are taken at F), so that $z = 1$ must give the points F, each of them thrice.

Taking the plane of the paper as the meridian from which longitudes are measured, the coordinates of the four w-points in the plane, corresponding to T by stereographic projection, are

$$\frac{\frac{\sqrt{2}}{\sqrt{3}}}{1-\frac{1}{\sqrt{3}}}, \quad \frac{-\frac{\sqrt{2}}{\sqrt{3}}}{1-\frac{1}{\sqrt{3}}}, \quad \frac{i\frac{\sqrt{2}}{\sqrt{3}}}{1+\frac{1}{\sqrt{3}}}, \quad \frac{-i\frac{\sqrt{2}}{\sqrt{3}}}{1+\frac{1}{\sqrt{3}}},$$

say w_1, w_2, w_3, w_4. Then $z = \infty$ gives each of these points thrice: that is, $z = \infty$, when $\{(w - w_1)\ldots(w - w_4)\}^3$ vanishes, or $z = \infty$, when

$$(w^4 - 2w^2\sqrt{3} - 1)^3$$

vanishes.

The coordinates of the four points corresponding to F, are

$$\frac{\frac{\sqrt{2}}{\sqrt{3}}}{1+\frac{1}{\sqrt{3}}}, \quad \frac{-\frac{\sqrt{2}}{\sqrt{3}}}{1+\frac{1}{\sqrt{3}}}, \quad \frac{i\frac{\sqrt{2}}{\sqrt{3}}}{1-\frac{1}{\sqrt{3}}}, \quad \frac{-i\frac{\sqrt{2}}{\sqrt{3}}}{1-\frac{1}{\sqrt{3}}}.$$

Hence $z = 1$, when

$$(w^4 + 2w^2\sqrt{3} - 1)^3$$

vanishes.

The coordinates of the six points corresponding to S are 0, $e^{(2r+1)\frac{\pi i}{4}}$ (for $r = 0, 1, 2, 3$) and ∞: hence $z = 0$, when

$$w^2(w^4 + 1)^2$$

vanishes.

Moreover, z is a uniform function of w: and therefore

$$1 - z = \frac{(w^4 + 2w^2\sqrt{3} - 1)^3}{(w^4 - 2w^2\sqrt{3} - 1)^3},$$

the constant multiplier on the right-hand side being determined as unity by the relation between the points S and the value $z = 0$.

The relation is often expressed in the equivalent form

$$z : z - 1 : 1 = 12\sqrt{3}\, w^2(w^4 + 1)^2 : (w^4 + 2w^2\sqrt{3} - 1)^3 : -(w^4 - 2w^2\sqrt{3} - 1)^3;$$

it gives the conformation on the z-half-plane of a triangle in the w-plane, bounded by circular arcs, the angles of the triangle being $\frac{1}{3}\pi$, $\frac{1}{3}\pi$, $\frac{1}{2}\pi$.

The simplest case is that of a portion cut out of a sector of a circle of central angle 30°, by the arc and two lines at right angles to one another symmetrical with respect to the arc.

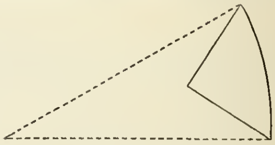

Fig. 105.

It has been assumed that the plane of the paper is the meridian. Another convenient meridian to take is one which passes through a point S on the equator: in that case, the preceding analysis applies if a rotation through an angle $\frac{1}{4}\pi$ be made. The effect of this rotation is to give the new variable W for any point in the form

$$W = we^{\frac{i\pi}{4}},$$

so that $w^2 = -iW^2$. The relation then takes the form

$$\begin{aligned} z : z-1 : 1 \\ &= 12\sqrt{-3}\, W^2(W^4-1)^2 : (W^4 + 2W^2\sqrt{-3}+1)^3 : -(W^4 - 2W^2\sqrt{-3}+1)^3; \end{aligned}$$

but there is no essential difference between the two relations.

The lines by which the w-plane is divided into triangles, each conformally represented on one or other half of the z-plane, are determined by $z = z_0$, that is, by

$$\frac{(w^4 - 2w^2\sqrt{3} - 1)^3}{w^2(w^4+1)^2} = \frac{(w_0^4 - 2w_0^2\sqrt{3} - 1)^3}{w_0^2(w_0^4+1)^2}.$$

The figure is the stereographic projection of the division of the sphere, and it can be obtained as in § 257 (Ex. 13, Ex. 16).

Case IV. We take $\lambda = \frac{1}{3}$, so that $z = 0$ must give the eight points C; each is given three times, because at C there are three triangles of the same set: we take $\nu = \frac{1}{4}$, so that $z = 1$ must give the six points O, each four times: and $\mu = \frac{1}{2}$, so that $z = \infty$ must give the twelve points S, each of them twice.

We take the plane of the paper as the meridian. The points O are $0, 1, i, -1, -i, \infty$; each four times. Hence $z = 1$, when the function

$$\{w(w^4-1)\}^4$$

vanishes.

The points C are the eight points $\dfrac{\pm 1 \pm i}{\pm\sqrt{3} - 1}$: the product of the eight corresponding factors is

$$w^8 + 14w^4 + 1:$$

and each occurs thrice, so that $z = 0$, when the function

$$(w^8 + 14w^4 + 1)^3$$

vanishes.

The points S are (i) the four points $\dfrac{\pm 1}{\pm\sqrt{2} - 1}$ in the plane of the paper, giving a corresponding product

$$w^4 - 6w^2 + 1:$$

(ii) the four points $\dfrac{\pm i}{\pm\sqrt{2} - 1}$ in the meridian plane, perpendicular to the plane of the paper, giving a corresponding product

$$w^4 + 6w^2 + 1:$$

and (iii) the four points $e^{\frac{\pi i}{4}(2r+1)}$, (for $r=0, 1, 2, 3$), in the equator, giving a corresponding product

$$w^4+1.$$

Each of these points occurs twice: and therefore $z=\infty$, when the function

$$\{(w^4+1)(w^4-6w^2+1)(w^4+6w^2+1)\}^2,$$

that is, when the function

$$(w^{12}-33w^8-33w^4+1)^2$$

vanishes.

Hence
$$z=\frac{(w^8+14w^4+1)^3}{(w^{12}-33w^8-33w^4+1)^2},$$

the constant multiplier being determined as unity, by taking account of the value unity for z: and

$$1-z=-\frac{108w^4(w^4-1)^4}{(w^{12}-33w^8-33w^4+1)^2}.$$

The relation can be expressed in the equivalent form

$$z : z-1 : 1=(w^8+14w^4+1)^3 : 108w^4(w^4-1)^4 : (w^{12}-33w^8-33w^4+1)^2;$$

it gives the conformation on half of the z-plane of a w-triangle bounded by circular arcs and having its angles equal to $\frac{1}{2}\pi$, $\frac{1}{3}\pi$, $\frac{1}{4}\pi$ respectively.

The lines, by which the w-plane is divided into the triangles, are given by $z=z_0$, that is, by

$$\frac{(w^8+14w^4+1)^3}{w^4(w^4-1)^4}=\frac{(w_0{}^8+14w_0{}^4+1)^3}{w_0{}^4(w_0{}^4-1)^4}.$$

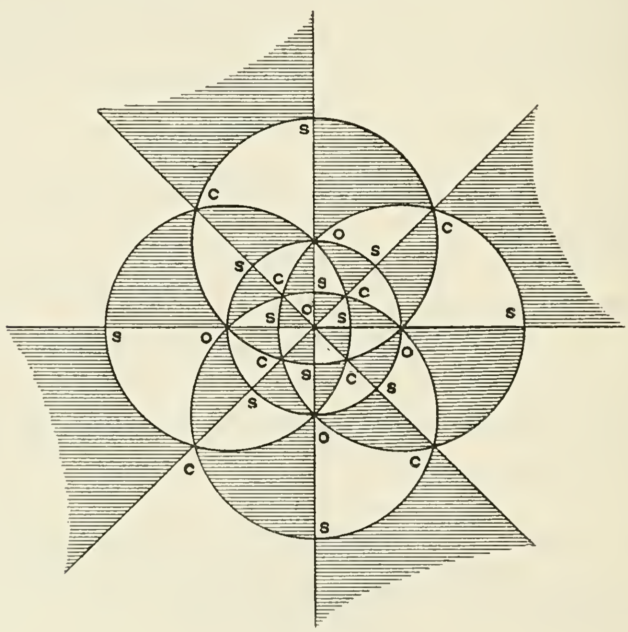

Fig. 106.

The division is indicated in Fig. 106, being the stereographic projection of the divided spherical surface of Fig. 103, with respect to the south pole, taken to be diametrically opposite to the central point O.

Case VI. We take $\lambda = \frac{1}{3}$, so that $z = 0$ must give the twenty points D, each of them thrice; $\nu = \frac{1}{5}$, so that $z = 1$ must give the twelve points I, each of them five times; and $\mu = \frac{1}{2}$, so that $z = \infty$ must give the thirty points S, each of them twice.

Let an edge of the icosahedron subtend an angle θ at the centre of the sphere: then its length is $2r \sin \frac{1}{2}\theta$. Also, five edges are the sides of a pentagon inscribed in a small circle, distant θ from a summit: hence the radius of this circle is $r \sin \theta$ and the length of the edge is $2r \sin \theta \sin \frac{1}{5}\pi$, so that

$$2 \sin \tfrac{1}{2}\theta = 2 \sin \theta \sin \tfrac{1}{5}\pi,$$

whence $$\tan \tfrac{1}{2}\theta = \tfrac{1}{2}(\sqrt{5} - 1), \quad \cot \tfrac{1}{2}\theta = \tfrac{1}{2}(\sqrt{5} + 1).$$

Let α denote $e^{\frac{2\pi i}{10}}$. Then the value of w corresponding to the north pole I is 0; the values of w for the projections on the equatorial plane of the five points I nearest the north pole are

$$\tan \tfrac{1}{2}\theta, \quad \alpha^2 \tan \tfrac{1}{2}\theta, \quad \alpha^4 \tan \tfrac{1}{2}\theta, \quad \alpha^6 \tan \tfrac{1}{2}\theta, \quad \alpha^8 \tan \tfrac{1}{2}\theta:$$

the values of w for the projections on the equatorial plane of the five points I nearest the south pole are

$$\alpha \cot \tfrac{1}{2}\theta, \quad \alpha^3 \cot \tfrac{1}{2}\theta, \quad \alpha^5 \cot \tfrac{1}{2}\theta, \quad \alpha^7 \cot \tfrac{1}{2}\theta, \quad \alpha^9 \cot \tfrac{1}{2}\theta:$$

and for projection of the south pole the value of w is infinity. The product of the corresponding factors is

$$\begin{aligned} & w \,.\, \prod_{r=0}^{4} (w - \alpha^{2r} \tan \tfrac{1}{2}\theta) \,.\, \prod_{r=0}^{4} (w - \alpha^{2r+1} \cot \tfrac{1}{2}\theta) \,.\, 1 \\ &= w (w^5 - \tan^5 \tfrac{1}{2}\theta)(w^5 + \cot^5 \tfrac{1}{2}\theta) \\ &= w (w^{10} + 11w^5 - 1) \end{aligned}$$

after substitution. Each point I occurs five times; and therefore $z = 1$, when the function

$$w^5 (w^{10} + 11w^5 - 1)^5$$

vanishes.

The points D lie by fives on four small circles with the diameter through the north pole and the south pole for axis. The polar distance of the small circle nearest the north pole is $\tan \delta = 3 - \sqrt{5}$, and of the circle next to it is $\tan \delta' = 3 + \sqrt{5}$, so that

$$\tan \tfrac{1}{2}\delta = \frac{\sqrt{15 - 6\sqrt{5}} - 1}{3 - \sqrt{5}}, \qquad \tan \tfrac{1}{2}\delta' = \frac{\sqrt{15 + 6\sqrt{5}} - 1}{3 + \sqrt{5}}.$$

The function corresponding to the projections of the five points nearest the north pole is

$$w^5 + \tan^5 \tfrac{1}{2}\delta,$$

and to the projections of the five nearest the south pole is

$$w^5 - \cot^5 \tfrac{1}{2}\delta\,;$$

while, for the projections of the other two sets of five, the products are

$$w^5 + \tan^5 \tfrac{1}{2}\delta'$$

and $$w^5 - \cot^5 \tfrac{1}{2}\delta'$$

respectively. Each occurs thrice. Hence $z = 0$, when the function

$$\{(w^5 + \tan^5 \tfrac{1}{2}\delta)(w^5 - \cot^5 \tfrac{1}{2}\delta)(w^5 + \tan^5 \tfrac{1}{2}\delta')(w^5 - \cot^5 \tfrac{1}{2}\delta')\}^3,$$

that is, when $$(w^{20} - 228w^{15} + 494w^{10} + 228w^5 + 1)^3,$$

which is the reduced form of the preceding product, vanishes.

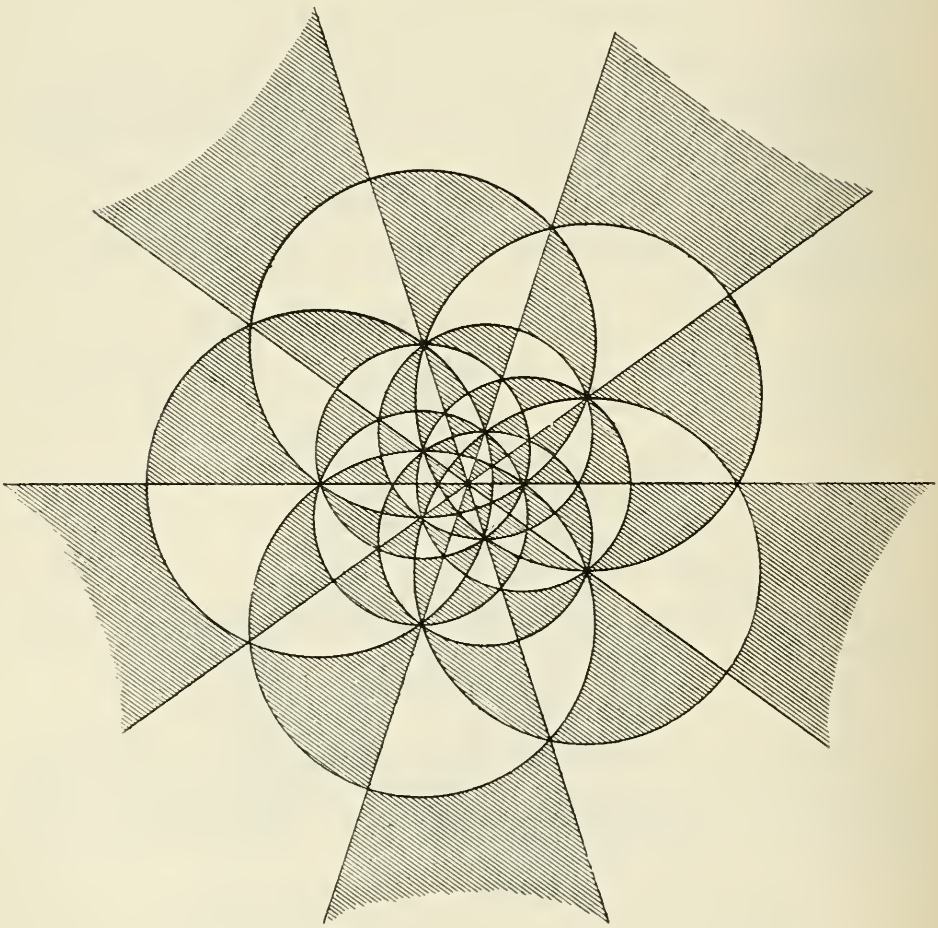

Fig. 107.

The points S lie by tens on the equator, by fives on four small circles having the polar axis for their axis. Proceeding in the same way with the products for their projections, it is found that $z = \infty$, when the function

$$\{w^{30} + 1 + 522w^5(w^{20} - 1) - 10005w^{10}(w^{10} + 1)\}^2$$

vanishes.

Hence $$z = \frac{(w^{20} - 228w^{15} + 494w^{10} + 228w^5 + 1)^3}{\{w^{30} + 1 + 522w^5(w^{20} - 1) - 10005w^{10}(w^{10} + 1)\}^2},$$

the constant factor being found to be unity, through the value of $1 - z$ which is $$1 - z = \frac{1728w^5(w^{10} + 11w^5 - 1)^5}{\{w^{30} + 1 + 522w^5(w^{20} - 1) - 10005w^{10}(w^{10} + 1)\}^2}.$$

These relations give the conformal representation on half of the z-plane of a w-triangle, bounded by circular arcs and having angles $\frac{1}{2}\pi$, $\frac{1}{3}\pi$, $\frac{1}{5}\pi$.

The lines, by which the w-plane is divided into the triangles, are given by $z = z_0$, that is, by

$$\frac{(w^{20} - 228w^{15} + 494w^{10} + 228w^5 + 1)^3}{w^5(w^{10} + 11w^5 - 1)^5} = \frac{(w_0^{20} - 228w_0^{15} + 494w_0^{10} + 228w_0^5 + 1)^3}{w_0^5(w_0^{10} + 11w_0^5 - 1)^5}.$$

The division is indicated in figure 107, which is the stereographic projection* of the divided spherical surface of figure 104, with I_{12} as the pole of projection.

279. The preceding are all the cases, in which simultaneously z is a uniform function of w, and w is an algebraical function of z: they arise when the surface of the sphere has been completely covered once with the two sets of triangles corresponding to the upper half and the lower half of the z-plane.

But an inspection of the figures at once shews that they are not the only cases to be considered, if the surface of the sphere may be covered more than once.

In the configuration arising through the double-pyramid, the surface of the sphere will be covered completely and exactly m times, if the angles at the poles be $2m\pi/n$, where m is prime to n. The corresponding relation between w and z is obtained from the simpler form by changing n into n/m.

In the tetrahedral configuration (fig. 102) the surface of the sphere will be exactly and completely covered twice by triangles FFT (or by triangles TTF, it being evident that these give substantially the same division of the surface). The relation between w and z will then be of the same degree, 12, as before in w, for the number of different triangles in the two w-sheets is still twelve of each kind: because there are two w-sheets corresponding to the single z-plane, that relation will be of the second degree in z. The values of the angles are determined by

$$\text{(III.)} \quad \lambda,\ \mu,\ \nu = \tfrac{2}{3},\ \tfrac{1}{3},\ \tfrac{1}{3}.$$

* In regard to all the configurations thus obtained as stereographic projections of a spherical surface, divided by the planes of symmetry of a regular solid, Möbius's "Theorie der Symmetrischen Figuren," (*Ges. Werke*, t. ii, especially pp. 642—699) may be consulted with advantage; and Klein-Fricke, *Elliptische Modulfunctionen*, vol. i. pp. 102—106.

Again, in the octahedral configuration, the surface of the sphere will be exactly and completely covered twice by triangles *OCO*. The relation between w and z will be of degree 24 in w and degree 2 in z: and the values of the angles are determined by

$$\text{(V.)} \quad \lambda,\ \mu,\ \nu = \tfrac{2}{3},\ \tfrac{1}{4},\ \tfrac{1}{4}.$$

Similarly, a number of cases are obtainable from the icosahedral configuration, in the following forms:

(VII.) $\lambda, \mu, \nu = \frac{2}{5}, \frac{1}{3}, \frac{1}{3}$ with triangles such as $I_1D_1D_2$;
(VIII.) $\lambda, \mu, \nu = \frac{2}{3}, \frac{1}{5}, \frac{1}{5}$ .... $D_1I_1I_2$;
(IX.) $\lambda, \mu, \nu = \frac{1}{2}, \frac{2}{5}, \frac{1}{5}$ .... $S_1I_1I_3$;
(X.) $\lambda, \mu, \nu = \frac{3}{5}, \frac{1}{3}, \frac{1}{5}$ .... $D_1I_1I_3$;
(XI.) $\lambda, \mu, \nu = \frac{2}{5}, \frac{2}{5}, \frac{2}{5}$ .... $I_1I_2I_3$;
(XII.) $\lambda, \mu, \nu = \frac{2}{3}, \frac{1}{3}, \frac{1}{5}$ .... $I_1D_6D_{13}$;
(XIII.) $\lambda, \mu, \nu = \frac{4}{5}, \frac{1}{5}, \frac{1}{5}$ .... $I_1I_2I_{10}$;
(XIV.) $\lambda, \mu, \nu = \frac{1}{2}, \frac{2}{5}, \frac{1}{3}$ .... $I_1S_6D_7$;
(XV.) $\lambda, \mu, \nu = \frac{3}{5}, \frac{2}{5}, \frac{1}{3}$ .... $I_1I_3D_6$.

Other cases appear to arise: but they can be included in the foregoing, by taking that supplemental triangle which has the smallest area. Thus, apparently, $I_1D_1I_{10}$ would be a suitable triangle, with $\lambda, \mu, \nu = \frac{2}{3}, \frac{2}{5}, \frac{1}{5}$: it is replaced by $I_{12}D_{20}I_{10}$, an example of case (X.) above.

These, with the preceding cases numbered* (I.), (II.), (IV.), (VI.), form the complete set of distinct ways of appropriate division of the surface of the sphere.

It is not proposed to consider these cases here: full discussion will be found in the references already given. The nature, however, of the relation, which is always of the form

$$f(z) = F(w),$$

where f and F are rational functions, may be obtained for any particular case without difficulty. Thus, for (III.), we have

$$\{w, z\} = \tfrac{1}{2}\left[\frac{1-\frac{1}{4}}{z^2} + \frac{1-\frac{1}{9}}{(1-z)^2} + \frac{\frac{1}{4}-\frac{1}{9}+\frac{1}{9}-1}{z(z-1)}\right],$$

when

$$z : 1-z : 1 = -12\sqrt{3}\,w^2(w^4+1)^2 : (w^4+2w^2\sqrt{3}-1)^3 : (w^4-2w^2\sqrt{3}-1)^3.$$

Again, if

$$z : 1-z : 1 = (Z+1)^2 : -4Z : (Z-1)^2,$$

* These numbers are the numbers originally assigned by Schwarz, *Ges. Werke*, t. ii, p. 246, and used by Cayley, *Camb. Phil. Trans.*, vol. xiii, pp. 14, 15.

a special case of § 278, I., by taking $n = 1$, then

$$\{Z, z\} = \tfrac{1}{2}\frac{1-\frac{1}{4}}{z^2}.$$

Hence
$$\{w, Z\} = \left(\frac{dz}{dZ}\right)^2 [\{w, z\} - \{Z, z\}]$$
$$= \frac{16(Z+1)^2}{(Z-1)^6} \cdot \left[\frac{\frac{1}{2}(1-\frac{1}{9})}{(1-z)^2} + \frac{\frac{1}{2}(\frac{1}{4}-1)}{z(z-1)}\right]$$
$$= \tfrac{1}{2}\left[\frac{1-\frac{1}{9}}{Z^2} + \frac{1-\frac{4}{9}}{(Z-1)^2} + \frac{\frac{1}{9}-\frac{1}{9}+\frac{4}{9}-1}{Z(Z-1)}\right],$$

so that $\lambda = \frac{1}{3}$, $\nu = \frac{2}{3}$, $\mu = \frac{1}{3}$. Hence the relation

$$(Z+1)^2 : -4Z : (Z-1)^2$$
$$= -12\sqrt{3}\, w^2(w^4+1)^2 : (w^4 + 2w^2\sqrt{3} - 1)^3 : (w^4 - 2w^2\sqrt{3} - 1)^3$$

gives the conformation of triangles bounded by circular arcs and having angles $\frac{1}{3}\pi$, $\frac{1}{3}\pi$, $\frac{2}{3}\pi$.

The foregoing are the only cases, for $\lambda + \mu + \nu > 1$, in which the integral relation between w and z is algebraical both in w and in z.

In all other cases in which λ, μ, ν are commensurable, this integral relation is algebraical in z and transcendental in w.

It is to be noticed, in anticipation of Chapter XXII., that, since every triangle in any of the divisions of the spherical surface, or of the plane, can be transformed into another triangle, the functions which occur in these integral relations are functions characterised by a group of substitutions. When the functions are algebraical, the groups are finite, and the functions are then the *polyhedral functions*: when the functions are transcendental, the groups are infinite and the functions are then of the general automorphic type.

The case in which $\lambda + \mu + \nu = 1$ has already been considered: the spherical representation is no longer effective, for the radius of the sphere becomes infinite and the triangle is a plane rectilinear triangle. The equation may still be used in the form

$$\{w, z\} = 2I(z)$$

with the condition $\lambda + \mu + \nu = 1$. A special solution of the equation is then given by

$$\frac{dw}{dz} = z^{\lambda-1}(1-z)^{\nu-1},$$

leading to the result of § 268, the homologue of the angular point $\mu\pi$ being at $z = \infty$.

280. It is often possible by the preceding methods to obtain a relation between complex variables that will represent a given curve in one plane on an assigned curve in the other: there is no indication of the character of the relation for an arbitrary curve or a family of curves. But in one case, at any rate, it is possible to give an indication of the limitations on the functional form of the relation.

Let there be a family of plane algebraical curves, determined as potential curves by a variable parameter*: and let their equation be

$$F(x, y, u) = 0,$$

where u is the variable parameter, which, when it is expressed in terms of x and y by means of the equation, satisfies the potential-equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Since u is a potential, it is the real part of a function w of $x + iy$: and the lines $u =$ constant are parallel straight lines in the w-plane. It therefore appears that the functional relation between w and z must represent the w-plane conformally on the z-plane, so that the series of parallel lines in the one plane is represented by a family of algebraical curves in the other: let the relation, which effects this transformation, be

$$\chi(z, w) = 0.$$

Let the algebraical curve, which corresponds to some particular value of u, say $u = 0$, be

$$F(x, y, 0) = f(x, y) = 0,$$

which in general is not a straight line. Let a new complex ζ be determined by the equation

$$f\left(\zeta, \frac{z - \zeta}{i}\right) = 0:$$

this equation is algebraical, and therefore ζ can be regarded as a function of w, say $\psi(w)$, between which and z, regarded as a function of w, say $\phi(w)$, there is an algebraical equation.

Now when $u = 0$, z describes the curve

$$f(x, y) = 0:$$

hence at least one branch of the function ζ, defined by

$$f\left(\zeta, \frac{z - \zeta}{i}\right) = 0,$$

* Such curves are often called *isothermal*, after Lamé. The discussion of the possible functional relations, that lead to algebraical isothermal curves, is due to Schwarz, *Ges. Werke*, t. ii, pp. 260—268: see also Hans Meyer, "Ueber die von geraden Linien und von Kegelschnitten gebildeten Schaaren von Isothermen; so wie über einige von speciellen Curven dritter Ordnung gebildete Schaaren von Isothermen," (a Göttingen dissertation, Zürich, Zürcher and Furrer, 1879); Cayley, *Quart. Journ. Math.*, vol. xxv, pp. 208—214; and the memoir by Von der Mühll, cited p. 500.

can be taken as equal to x when $u=0$, that is, *there is one branch of the function ζ which is purely real when w is purely imaginary.*

The curves in the z-plane are algebraical: when this plane is conformally represented on the ζ-plane by the foregoing branch, which is an algebraical function of z, the new curves in the ζ-plane are algebraical curves, also determined as potential curves by the variable parameter u. And the ζ-curve corresponding to $u=0$ is (the whole or a part of) the axis of real quantities. In order that the conformal representation may be effected by the functions, they must allow of continuous variation: hence lines on opposite sides of $u=0$ correspond to lines on opposite sides of the axis of real quantities. The functional relation between $\zeta=\xi+i\eta$ and $w=u+iv$ is therefore such that

$$\xi+i\eta=\psi(u+iv),$$
$$\xi-i\eta=\psi(-u+iv).$$

The equation of the ζ-curves, which are obtained from varying values of u, is algebraical: and therefore, when we substitute in it for ξ and η their values in terms of $\psi(u+iv)$ and $\psi(-u+iv)$, we obtain an algebraical equation between $\psi(u+iv)$ and $\psi(-u+iv)$, the coefficients of which are functions of u though not necessarily algebraical functions of u. Let $\theta=-2u$; and let ψ_2, ψ_3 denote $\psi(w)$, $\psi(w+\theta)$ respectively; then the equation can be represented in the form

$$g(\psi_2, \psi_3, \theta)=0,$$

algebraical and rational in ψ_2 and ψ_3, but not necessarily algebraical in θ.

Because the functions allow continuous variation, we can expand ψ_3 in powers of θ: hence

$$g\left(\psi_2, \psi_2+\theta\frac{d\psi_2}{dw}+\tfrac{1}{2}\theta^2\frac{d^2\psi_2}{dw^2}+\ldots\ldots, \theta\right)=0.$$

When this equation, which is satisfied for all values of w and of θ, where w and θ are independent of one another, is arranged in powers of θ, the coefficients of the various powers of θ must vanish separately. The coefficient independent of θ, when equated to zero, can only lead to an identity, for it will obviously involve only ψ_2: any non-evanescent equation would determine ψ_2 as a constant. Similarly, the coefficient of every power of θ, which involves none of the derivatives of ψ_2, must vanish identically. The coefficient of the lowest power of θ, which does not vanish identically, involves ψ_2, $\dfrac{d\psi_2}{dw}$ and constants: but, because the equation $g(\psi_2, \psi_3, \theta)=0$ is algebraical in ψ_3, the second and higher derivatives of ψ_2, associated with the second and higher powers of θ in the expansion of ψ_3, cannot enter into the coefficient of this power of θ. Hence we have

$$h\left(\psi_2, \frac{d\psi_2}{dw}\right)=0,$$

an algebraical equation between ψ_2 and $\frac{d\psi_2}{dw}$, the coefficients of which are constants.

The coefficient of the next power of θ will involve $\frac{d^2\psi_2}{dw^2}$, and so on for the powers in succession. Instead of using the equations, obtained by making these coefficients vanish, to deduce an algebraical equation between ψ_2 and any one of its derivatives, we use $h=0$. Thus for $\frac{d^2\psi_2}{dw^2}$, the equation would be obtained by eliminating ψ_2' between the (algebraical) equations

$$h(\psi_2, \psi_2') = 0, \quad \frac{\partial h}{\partial \psi_2}\psi_2' + \frac{\partial h}{\partial \psi_2'}\psi_2'' = 0;$$

and so for others.

Returning now to the equation

$$g(\psi_2, \psi_3, \theta) = 0,$$

in which, as it is algebraical in ψ_2 and ψ_3, only a limited number of coefficients, say k, are functions of θ, we can remove these coefficients as follows. Let $k-1$ differentiations with regard to w be effected: the resulting equations, with $g=0$, are sufficient to determine these k coefficients algebraically in terms of ψ_2, ψ_3 and their derivatives. But the coefficients are functions of θ only and do not depend upon w: hence the values obtained for them must be the same whatever value be assigned to w. Let, then, a zero value be assigned: ψ_2 and its derivatives become constants; ψ_3 becomes $\psi(\theta)$, say ψ_1, and all its derivatives become derivatives of ψ_1; so that the coefficients can be algebraically expressed in terms of ψ_1 and its derivatives. When these values are substituted in $g=0$, it takes the form

$$g_1(\psi_2, \psi_3, \psi_1, \psi_1', \psi_1'', \ldots) = 0,$$

algebraical in each of the quantities involved. But between ψ_1 and each of its derivatives there subsists an algebraical equation with constant coefficients: by means of these equations, all the derivatives of ψ_1 can be eliminated from $g_1=0$, and the final form is then an algebraical equation

$$G(\psi_2, \psi_3, \psi_1) = 0,$$

involving only constant coefficients. But

$$\psi_1 = \psi(\theta), \quad \psi_2 = \psi(w), \quad \psi_3 = \psi(w+\theta);$$

and therefore the function $\psi(w)$ possesses an algebraical addition-theorem.

Now $\psi(w)$ and $\phi(w)$ are connected by the algebraical equation

$$f\left(\psi, \frac{\phi - \psi}{i}\right) = 0;$$

therefore $\phi(w)$ possesses an algebraical addition-theorem. But, by § 151, when a function $\phi(w)$ possesses an algebraical addition-theorem, it is an algebraical function either of w, or of $e^{\mu w}$, or of an elliptic function of w, the various constants that arise being properly chosen: and hence *the only equations*

$$\chi(z, w) = 0,$$

which can give families of algebraical curves in the z-plane as the conformal equivalent of the parallel lines, u = constant, in the w-plane, are such that z is connected by an algebraical equation either with w, or with a simply-periodic function of w, or with a doubly-periodic function of w.

There are three sets of *fundamental* systems, as Schwarz calls them, of algebraical curves determined as potential curves by a variable parameter: they are curves such that all the others can be derived from them solely by algebraical functions.

The first set is fundamental for the case when z is an algebraical function of w: it is given by

$$u = \text{constant},$$

being a series of parallel straight lines.

The second set is fundamental for the case when z is an algebraical function of $e^{\mu w}$; if W denote $e^{\mu w}$, then z is an algebraical function of W, and all the associated curves in the z-plane are conformal representations of the algebraical curves in the W-plane. If $\mu = \alpha + \beta i$, where α and β are real, then

$$(\alpha^2 + \beta^2) u = \tfrac{1}{2} \alpha \log (X^2 + Y^2) + \beta \tan^{-1} \frac{Y}{X},$$

a relation which can lead to algebraical curves in the W-plane only if α or β be zero. If α be zero, then μ is a pure imaginary, and the W-curves are straight lines, concurrent in the origin: if β be zero, then μ is real, and the W-curves are circles with the origin for a common centre. Hence the set of fundamental systems for the case, when z is an algebraical function of $e^{\mu w}$, consists of an infinite series of concurrent straight lines and an infinite series of concentric circles, having for their common centre the point of concurrence of the straight lines.

The third set is fundamental for the case when z is an algebraical function of a doubly-periodic function, say, of $\operatorname{sn}(\mu w)$.

Ex. Prove that either the modulus k is real or that an algebraical transformation of argument to another elliptic function having a real modulus is possible: and shew that the set of fundamental curves are quartics, which are the stereographic projection of confocal sphero-conics. (Schwarz, Siebeck, Cayley.)

We thus infer that *all families of algebraical curves, determined as potential curves by a variable parameter, are conformal representations of* *one or other of these sets of fundamental systems, by equations which are algebraical.*

But though it is thus proved that the relation between z and w must express z as an algebraical function either of w, or of $e^{\mu w}$, or of sn μw, in order that a group of algebraical curves may be the conformal representation in the z-plane of the lines $u =$ constant in the w-plane, the same limitation does not apply, if we take a single algebraical curve in the z-plane as the conformal representation of a single line in the w-plane.

Let $w = \dfrac{1-W}{1+W}$: then the lines in the W-plane, which correspond to the parallel lines, $u =$ constant, in the w-plane, are the system of circles

$$\left(W + \frac{u}{u+1}\right)\left(W_0 + \frac{u}{u+1}\right) = \frac{1}{(u+1)^2}.$$

Now consider a relation

$$\frac{2K}{\pi} Z = \text{sn}^{-1}(Wk^{-\frac{1}{2}}),$$

where Z is as yet some unspecified function of z: then

$$k^{-\frac{1}{2}} W = \text{sn}\left(\frac{2K}{\pi} Z\right).$$

Hence
$$\frac{1}{k} W W_0 = \text{sn}\left(\frac{2K}{\pi} Z\right) \text{sn}\left(\frac{2K}{\pi} Z_0\right),$$

so that, if W describe the circle corresponding to $u = 0$, we have

$$\frac{1}{k} = \text{sn}\left(\frac{2K}{\pi} Z\right) \text{sn}\left(\frac{2K}{\pi} Z_0\right),$$

whence
$$Z - Z_0 = \frac{i\pi K'}{2K}.$$

If $Z = \sin^{-1} z$, and therefore $Z_0 = \sin^{-1} z_0$, then

$$2x = z + z_0 = 2 \sin \tfrac{1}{2}(Z + Z_0) \cos \frac{i\pi K'}{4K} = (q^{-\frac{1}{4}} + q^{\frac{1}{4}}) \sin \tfrac{1}{2}(Z + Z_0),$$

$$2iy = z - z_0 = 2 \cos \tfrac{1}{2}(Z + Z_0) \sin \frac{i\pi K'}{4K} = i(q^{-\frac{1}{4}} - q^{\frac{1}{4}}) \cos \tfrac{1}{2}(Z + Z_0),$$

so that
$$\frac{y^2}{(q^{-\frac{1}{4}} + q^{\frac{1}{4}})^2} + \frac{x^2}{(q^{-\frac{1}{4}} - q^{\frac{1}{4}})^2} = \tfrac{1}{4},$$

an ellipse, agreeing with the result in § 257, Ex. 6. This is obtained from the relation

$$k^{-\frac{1}{2}} \frac{1-w}{1+w} = \text{sn}\left(\frac{2K}{\pi} \sin^{-1} z\right),$$

which is not included in the general forms of relation obtained in the preceding investigation.

But the equation

$$\left\{k^{\frac{1}{2}} \operatorname{sn}\left(\frac{2K}{\pi} Z\right) + \frac{u}{u+1}\right\}\left\{k^{\frac{1}{2}} \operatorname{sn}\left(\frac{2K}{\pi} Z_0\right) + \frac{u}{u+1}\right\} = \frac{1}{(u+1)^2}$$

does not lead to an algebraical relation between x and y for a general (non-zero) value of u. Neither the conditions of the proposition nor its limitations apply to this case.

The problem of determining the kinds of functional relation which will represent a single algebraical curve in the z-plane upon a single line of the w-plane is wider than that which has just been discussed: it is, as yet, unsolved.

CHAPTER XXI.

Groups of Linear Substitutions.

281. The properties of the linear substitution

$$w = \frac{az+b}{cz+d},$$

considered in Chap. XIX. as bearing upon the conformal representation of two planes, were discussed solely in connection with the geometrical relations of the conformation: but the applications of these properties have a significance, which is wider than their geometrical aspect.

The essential characteristic of singly-periodic functions and of doubly-periodic functions, each with additive periodicity, is the reproduction of the function when its argument is modified by the addition of a constant quantity. This modification of argument, uniform and uniquely reversible, is only a special case of a more general modification which is uniform and uniquely reversible, viz., of the foregoing linear substitution. This substitution may therefore be regarded as the most general expression of linear periodicity, in a wider sense: and all functions, characterised by the property in the general form or in special forms, may be called *automorphic*.

Our immediate purpose is the consideration of all the points in the plane, which can be derived from a given point z and from one another by making z subject to a set of linear substitutions. The set may be either finite or infinite in number; it is supposed to contain every substitution which can be formed by combining two or more substitutions. Such a set is called a *group*.

The substitution is often denoted by $S(z)$, or by

$$\left(z, \frac{az+b}{cz+d}\right);$$

it is said to be in its *normal* form, when the real part of a (if a be a complex constant) is positive and $ad - bc = 1$.

The ideas of the theory of groups of substitutions are necessary for a proper consideration of the properties of automorphic functions. What is contained in the present chapter is merely sufficient for this requirement, being strictly limited to such details as arise in connection with these special functions. Information on the fuller development of the theory of groups, which owes its origin as a distinct branch of mathematics to Galois, will be found in appropriate treatises such as those of Serret*, Jordan†, Netto‡, and Klein§: and in memoirs by Klein‖, Poincaré**, Dyck††, and Bolza‡‡. The account of the properties of groups contained in the present chapter is based upon the works of Klein and Poincaré just quoted.

A substitution can be repeated; a convenient symbol for representing the substitution, that arises from n repetitions of S, is S^n. Hence the various integral powers of S, considered in § 258, are substitutions, indicated by the symbols S^2, S^3, S^4,....

But we have negative powers of S also. The definition of $S^0(z)$ is given by

$$SS^0(z) = S(z),$$

so that $S^0(z) = z$ and it is often called the *identical* substitution: the definition of $S^{-1}(z)$ is given by

$$SS^{-1}(z) = S^0(z) = z,$$

so that $S^{-1}(z)$ is a substitution the inverse of S; in fact, if $w = S(z) = \dfrac{az+b}{cz+d}$, then $z = S^{-1}w = \dfrac{-dw+b}{cw-a}$. And then, from $S^{-1}z$, by repetition we obtain S^{-2}, S^{-3}, S^{-4},....

If some of all the substitutions to which a variable z is subject be not included in S and its integral powers, then we have a new substitution T and its integral powers, positive and negative. The variable is then subject to combinations of these substitutions: and, as two general linear substitutions are not interchangeable, that is, we do not have $T(Sz) = S(Tz)$ in general, therefore among the substitutions to which z is subject there must occur all those of the form

$$\ldots S^{\alpha}T^{\beta}S^{\gamma}T^{\delta}\ldots,$$

where α, β, γ, δ, ... are positive or negative integers.

If, again, there be other substitutions affecting z, that are not included among the foregoing set, let such an one be U: then there are also powers of U and combinations of S, T, U (with integral indices) operating in any order: and so on. The substitutions S, T, U, ... are called *fundamental:* the sum of the moduli of α, β, γ, δ, ... of any substitution, compounded from the fundamental substitutions, is called the *index* of that substitution; and the aggregate of all the substitutions, fundamental and composite, is the group.

* *Cours d'Algèbre Supérieure*, t. ii, Sect. iv, (Paris, Gauthier-Villars).

† *Traité des substitutions*, (ib., 1870).

‡ *Substitutionentheorie und ihre Anwendung auf die Algebra*, (Leipzig, Teubner, 1882).

§ *Vorlesungen über das Ikosaeder*, (ib., 1884).

‖ *Math. Ann.*, t. xxi, (1883), pp. 141—218, where references to earlier memoirs by Klein are given.

** *Acta Math.*, t. i, (1882), pp. 1—62, pp. 193—294; ib., t. iii, (1883), pp. 49—92.

†† *Math. Ann.*, t. xx, (1882), pp. 1—44, ib., t. xxii, (1883), pp. 70—108.

‡‡ *Amer. Journ. of Math.*, vol. xiii, (1890), pp. 59—144.

There may however be relations among the substitutions of the group, depending on the fundamental substitutions; they are, ultimately, relations among the fundamental substitutions, though they are not necessarily the simplest forms of those relations. Hence, as we may have a relation of the form

$$\ldots S^a \ldots T^b \ldots U^c \ldots (z) = z,$$

the index of a composite substitution is not a determinate quantity, being subject to additions or subtractions of integral multiples of quantities of the form $(a) + (b) + (c) + \ldots$, there being one such quantity for every relation: we shall assume the index to be the smallest positive integer thus obtainable.

282. There are certain classifications which may initially be associated with such groups, in view of the fact that the arguments are the arguments of uniform automorphic functions satisfying the equation

$$f(Sz) = f(z):$$

in this connection, the existence of such functions will be assumed until their explicit expressions have been obtained.

Thus a group may contain only a finite number of substitutions, that is, the fundamental substitutions may lead, by repetitions and combinations, only to a finite number of substitutions. Hence the fundamental substitutions, and all their combinations, are periodic in the sense of § 260, that is, they reproduce the variables after a finite number of repetitions.

Or a group may contain an infinite number of substitutions: these may arise either from a finite number of fundamental substitutions, or from an infinite number. The latter class of infinite groups will not be considered in the present connection, for a reason that will be apparent (p. 598, note) when we come to the graphical representations. It will therefore be assumed that the infinite groups, which occur, arise through a finite number of fundamental substitutions.

A group may be such as to have an *infinitesimal* substitution, that is, there may be a substitution $\dfrac{az+b}{cz+d}$, which gives a point infinitesimally near to z for every value of z. It is evident there will then be other infinitesimal substitutions in the group; such a group is said to be *continuous*. If there be no infinitesimal substitution, then the group is said to be *discontinuous*, or *discrete*.

But among discontinuous groups a division must be made. The definition of group-discontinuity implies that there is no substitution, which gives an infinitesimal displacement for every value of z: but there may be a number of special points in the plane for regions in the immediate vicinity of which there are infinitesimal displacements. Such groups are called *improperly* *discontinuous* in the vicinity of such points: all other groups are called *properly discontinuous.* For instance, with the group of real substitutions

$$\frac{\alpha z+\beta}{\gamma z+\delta},$$

where $\alpha, \beta, \gamma, \delta$, are integers such that $\alpha\delta-\beta\gamma=1$, it is easy to see that, when z_1 and z_2 are real, we can make the numerical magnitude of

$$\frac{\alpha z_1+\beta}{\gamma z_1+\delta}-\frac{\alpha z_2+\beta}{\gamma z_2+\delta}$$

as small a non-evanescent quantity as we please by proper choice of $\alpha, \beta, \gamma, \delta$: thus the group is improperly discontinuous, because for real values of the variable it admits infinitesimal transformations. But such infinitesimal transformations are not possible, when z does not lie on the axis of real quantities, that is, when z is complex: so that, for all complex values of z, the group is properly discontinuous.

The various points, derived from a single point by linear substitutions, will, in subsequent investigations, be found to be arguments of a uniform function. Continuous groups would give a succession of points infinitely close together; that is, for these points, we should have $f(z)$ unaltered in value for a line or a small area of points and therefore constant everywhere. We shall therefore consider only discontinuous groups.

A group containing only a finite number of substitutions is easily seen to be discontinuous: hence the groups which are to be considered in the present connection are the discontinuous groups which arise from a finite number of fundamental substitutions*.

The constants of all linear substitutions of the form $\dfrac{az+b}{cz+d}$ are supposed subject to the relation $ad-bc=1$. This condition holds for all combinations, if it hold for the components of the combination. For let

$$S=\frac{\alpha z+\beta}{\gamma z+\delta},\quad T=\frac{az+b}{cz+d};$$

then

$$ST=\frac{(\alpha a+\beta c)z+\alpha b+\beta d}{(\gamma a+\delta c)z+\gamma b+\delta d}=\frac{Az+B}{Cz+D},$$

whence

$$AD-BC=(\alpha\delta-\beta\gamma)(ad-bc)=1.$$

It is easy to see that $ST\,(=U)$ and $TS\,(=V)$ are of the same class, that is, they are elliptic, parabolic, hyperbolic or loxodromic together: but there is no limitation on the class arising from the character of the component substitutions.

* These discontinuous, or discrete, groups will be considered from the point of view of automorphic functions. But the theory of such groups, which has many and wide applications quite outside the range of the subject of this treatise, can be applied to other parts of our subject. Thus it has been connected with the discussion of Riemann's surfaces by Dyck, *Math. Ann.*, t. xvii, (1880), pp. 473—509, and by Hurwitz (l.c., p. 406, note).

Moreover, if $U=V$, so that S and T are interchangeable, then

$$\frac{a-d}{\alpha-\delta}=\frac{c}{\gamma}=\frac{b}{\beta};$$

that is, S and T have the same fixed points. They can be applied in any order; and, for any given number of occurrences of S and a given number of occurrences of T, the composite substitution will give the same point. Thus if $S=z+\omega$, then $T=z+\omega'$; if $S=kz$, then $T=k'z$. The class of functions, which have their argument subject to interchangeable substitutions of the former category, have already been considered: they are the periodic functions with additive periodicity. The group is $S^mT^{m'}$, $(=z+m\omega+m'\omega')$, for all integral values of m and of m'.

The latter class of functions have what may be called a factorial periodicity, that is, they resume their value when the argument is multiplied by a constant*.

283. Some examples have already been given of groups containing a finite number of substitutions†, in the case of certain periodic elliptic substitutions. The effect of such substitutions is (p. 514) to change a crescent-shaped part of the plane having its angles at the (conjugate) fixed points of the substitution into consecutive crescent-shaped parts: and so to cover the whole plane in the passage of a substitution through the elements constituting its period. They form the simplest discontinuous group—in that they have only one fundamental substitution and only a finite number of derived substitutions.

The groups which are next in point of simplicity are those with only two substitutions that are fundamental and only a finite number that are composite. Both of the fundamental substitutions must be periodic, and therefore elliptic, by § 260. Taking one of these groups as an example, one of its fundamental substitutions has ± 1 as its fixed points and it is periodic of the second order: it is evidently

$$w=Sz=\frac{1}{z}.$$

The other has $\frac{1}{2}$ and ∞ as its fixed points, and it is periodic of the second order: it is evidently

$$w=Tz=1-z.$$

* Functions having this property are discussed in Rausenberger's *Theorie der periodischen Functionen*, (Leipzig, Teubner, 1884): in particular, in Section VI.

† The complete theory of finite groups of linear substitutions is discussed, partly in its geometrical relation with polyhedral functions, by Klein, *Math. Ann.*, t. ix, (1876), pp. 183—188, and, in its algebraical aspect, by Gordan, *Math. Ann.*, t. xii, (1877), pp. 23—46. A reference to these memoirs will shew that the previous chapter contains all the essentially distinct finite groups of linear substitutions.

Evidently $S^2z=z$, $T^2z=z$, $(S=S^{-1},\ T=T^{-1})$, so that we have already all the powers of the fundamental substitutions taken separately.

But it is necessary to combine them. We have $Uz=STz=\dfrac{1}{1-z}$, a new substitution: and then

$$U^2z=\frac{z-1}{z},\quad U^3z=z,$$

so that U is periodic of the third order. Again

$$Vz=TSz=\frac{z-1}{z},$$

which is not a new substitution, for $Vz=U^2z$: and it is easy to see that there is only one other substitution, which may be taken to be either TUz or SVz: it gives

$$TUz=SVz=\frac{z}{z-1},$$

again periodic of the second order.

Hence the group consists of the six substitutions for z given by

$$z,\ \frac{1}{z},\ 1-z,\ \frac{1}{1-z},\ \frac{z-1}{z},\ \frac{z}{z-1},$$

taking account of the identical substitution.

These finite discontinuous groups are of importance in the theory of polyhedral functions: to some of their properties we shall return later.

Next, and as the last special illustration for the present, we form a discontinuous group with two fundamental substitutions but containing an infinite number of composite* substitutions. As one of the two that are fundamental, we take

$$w=Tz=-\frac{1}{z},$$

which is elliptic and periodic of the second order. As the other, we take

$$w=Sz=z+1,$$

which is parabolic and not periodic. All the substitutions are real.

Evidently $T^2z=z$, so that $T=T^{-1}$: and $S^mz=z+m$, where m is any integer. Then all the composite substitutions, are either of the form $\dots S^pTS^nTS^mz$ or of the form $\dots S^pTS^nTS^mTz$, both of these being included in $\dfrac{az+b}{cz+d}$, *where a, b, c, d are integers*, such that $ad-bc=1$.

Ex. Prove the converse—that the substitution $\dfrac{az+b}{cz+d}$, where a, b, c, d are integers such that $ad-bc=1$, is compounded of the substitutions S and T.

* One such group has already occurred: its fundamental (parabolic) substitutions were

$$w=Sz=z+\omega,\qquad w=Tz=z+\omega'.$$

This group, again, is of the utmost importance: it arises in the theory of the elliptic modular-functions. As with the polyhedral groups, the general discussion of the properties will be deferred: but it is advantageous to discuss one of its properties now, because it forms a convenient introduction to, and illustration of, the corresponding part of the theory of groups of general substitutions.

284. In the discussion of the functions with additive periodicity, it was found convenient to divide the plane into an infinite number of regions such that a region was changed into some other region when to every point of the former was applied a transformation of the form $z + m\omega + m'\omega'$, that is, a substitution: and the regions were so chosen that no two homologous points, that is, points connected by a substitution, were within one region, and each region contained one point homologous with an assigned point in any region of reference.

Similarly, in the case when the variable is subject to the substitutions of an infinite group, it is convenient to divide the plane into an infinite number of regions; each region is to be associated with a substitution which, applied to the points of a region of reference, gives all the points of the region, and each region is to contain one and only one point derived from a given point by the substitutions of the group. It is a condition that the complete plane is to be covered once and only once by the aggregate of the regions.

When the discontinuous group has only the two fundamental substitutions, $Sz = z + 1$ and $Tz = -\dfrac{1}{z}$, the division of the plane is easy: the difficulty of determining an initial region of reference is slight, relatively to that which has to be overcome in more general groups*.

The ordinates of z and $w\,(=Sz)$ are positive together or negative together; and similarly for the ordinates of z and $w\,(=Tz)$: so that it will suffice to divide the half-plane on the positive side of the axis of real quantities.

For the repetitions of the substitution S, it is evidently sufficient to divide the plane into a series of strips, bounded by straight lines parallel to the axis of y at unit distance apart.

For the application of the substitution T, we have to invert with regard to a circle of radius 1 and centre the origin and to take the reflexion of the inversion in the axis of y.

In these circumstances, we can choose as an initial region of reference, the space bounded by the conditions

$$\frac{1}{2} \geqslant x \geqslant -\frac{1}{2}, \quad x^2 + y^2 \geqslant 1.$$

* In addition to the references already given, a memoir by Hurwitz, *Math. Ann.*, t. xviii, (1881), pp. 531—544, may be consulted for this group.

It is sufficient to prove that any point in this region when subjected to a substitution of the group, necessarily of the form $\frac{az+b}{cz+d}$, where a, b, c, d are integers such that $ad-bc=1$, is transformed to some point without the region, and that the aggregate of the regions covers the half-plane.

If c be 0, then $a=1=d$ and the transformation is only some power of S, which transforms the point out of the region.

If c be $+1$, then, since $ad-bc=1$, we have

$$w-a=-\frac{1}{z+d},$$

a and d being integers. For any point z within the region, $|z+d|$, which is the distance of the point from some point $0, \pm 1, \pm 2, \ldots$ on the axis of x, is >1: hence

$$|w-a|<1,$$

that is, the distance of w from some point $0, \pm 1, \pm 2, \ldots$ on the axis is <1, and therefore the transformed point is without the region.

Similarly, if c be -1.

If $|c|$ be >1, then $$w-\frac{a}{c}=-\frac{1}{c^2}\frac{1}{z+\frac{d}{c}}.$$

As z is within the region, $\left|z+\frac{d}{c}\right| \geqslant \frac{\sqrt{3}}{2}$: and therefore

$$\frac{\sqrt{3}}{2}\left|w-\frac{a}{c}\right| \leqslant \frac{1}{c^2} \leqslant \frac{1}{4},$$

so that $$\left|w-\frac{a}{c}\right| \leqslant \frac{1}{2\sqrt{3}} < \tfrac{1}{2}\sqrt{3}.$$

Hence the distance of w from some point of the axis is $<\frac{1}{2}\sqrt{3}$, that is, the transformed point is without the region.

The exceptions are points on the boundary of the region. The boundary $x=-\frac{1}{2}$ is transformed by S to $x=+\frac{1}{2}$: the boundary $x^2+y^2=1$ is transformed by T into itself: but all other points are transformed into others without the region.

We now apply the substitutions S and T to this region and to the resulting regions. Each substitution is uniform and is reversible: so that to a given point in the initial region there is one, and only one, point in each other region.

The accompanying diagram (Fig. 108) gives part of the division of the plane into regions, the substitutions associated with each region being placed in the region in the figure; it is easy to see that the aggregate of regions completely covers the half-plane. All the linear boundaries of S^n, for different integral values of n, are changed by the substitution T into circles having their centres on the axis of x and touching at A: thus the boundary between S and S^2 is transformed into the boundary between

TS and TS^2. All the lines which bound the regions are circles having their centres on the axis of x or are straight lines perpendicular to that axis; and the configuration of each strip is the same throughout the diagram.

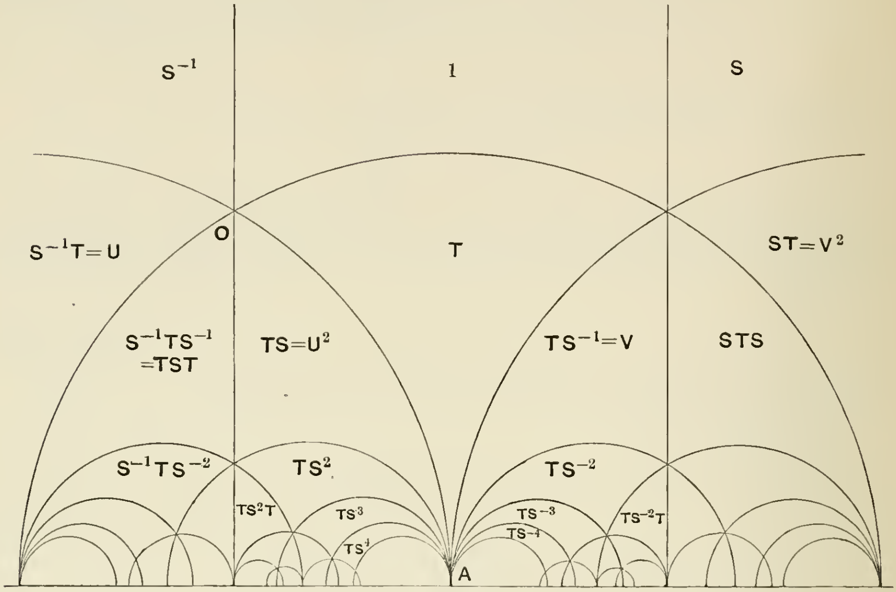

Fig. 108.

It will be noticed that in one region there are two symbols, viz., $S^{-1}TS^{-1}$ and TST: the region can be constructed either by S^{-1} applied to TS^{-1} or by T applied to ST. It therefore follows that

$$TST = S^{-1}TS^{-1}.$$

Hence $$S \,.\, TST \,.\, S = S \,.\, S^{-1}TS^{-1} \,.\, S = T,$$

or, since $T^2 = 1$, we have $$STSTST = 1 = TSTSTS,$$

a relation among the fundamental substitutions. Thus the symbol of any region is not unique: and, as a matter of fact, if we pass clockwise in a small circuit round O from the initial region, we find the regions to be 1, T, TS, TST, $TSTS$, $TSTST$, $TSTSTS$, the seventh being the same as the first and giving the above relation.

By means of this relation it will be found possible to identify the non-unique significations of the various regions. At each point there are six regions thus circulating always, either in the form ΘS, ΘST, ΘSTS, ... or in the form ΘT, ΘTS, ΘTST, ... And by successive transformations, the space towards the axis of x is distributed into regions.

The decision of the region to which a boundary should be assigned will be made later in the general investigation; it will prove a convenient step towards the grouping of edges of a region in conjugate pairs.

Note. It may be proved in the same way that, for any discontinuous group of substitutions, the plane of the variable can be divided into regions of a similar character. As will subsequently appear, there is considerable freedom of choice of an initial region of reference, which may be called a *fundamental* region.

285. We pass now to the consideration of the more general discontinuous groups, based on the composition of a finite number of fundamental substitutions. By means of these groups and in connection with them, the plane of the variable can be divided into regions, one corresponding to each substitution of the group. The regions are said to be *congruent* to one another: the infinite series of points, one in each of the congruent regions, which arise from z when all the substitutions of the group are applied to z, are said to be *corresponding* or *homologous* points: and the point in R_0 of the series is the *irreducible* point of the series. As remarked before, the correspondence between two regions is uniform: interiors transform to interiors, boundaries to boundaries.

Two regions are said* to be *contiguous*, when a part of their boundaries is common to both. Each region, lying entirely in the finite part of the plane, is closed: the boundary is made up of a succession of lines which may for convenience be called *edges*, and the meeting-point of two edges may for convenience be called a *corner*.

Such a group, when all the substitutions are real, is called† *Fuchsian*, by Poincaré; the preceding example will furnish a simple illustration, useful for occasional reference. All the substitutions are of the form

$$\frac{a_s z + b_s}{c_s z + d_s},$$

which form will be denoted by $f_s(z)$. We shall suppose that an infinite group of real substitutions is given, and that it is known independently to be a discontinuous group: we proceed to consider the characteristic properties of the associated division of the plane, which is to be covered once and only once by the aggregate of the regions. The fundamental region is denoted by R_0: the region, which results when the substitution $f_m(z)$ is applied to the points of R_0, will be denoted by R_m.

So long as we deal with real substitutions, it is sufficient to divide the half-plane above the axis of x into regions: and this axis may be looked upon as a boundary of the plane. Since the group is infinite, the division into regions must extend in all directions in the plane to its finite or infinite boundaries: for we should otherwise have infinitesimal transformations. Thus

* Poincaré uses the term *limitrophes*.

† *Math. Ann.*, t. xix, p. 554, t. xx, pp. 52, 53: *Acta Math.*, t. i, p. 62. The same term is applied to a less limited class of groups; see p. 606, note.

the edge of a region is either the edge of a contiguous region, and then it is said to be of the *first kind;* or it is a part of the boundary of the plane, that is, in the present case it is a part of the axis of x: and then it is said to be of the *second kind.* Since all real substitutions transform a point above the axis of x into another point above the axis of x, it follows that all edges congruent with an edge of the first kind (an edge lying off the axis of x) themselves lie off the axis of x, that is, are of the first kind: and similarly all edges congruent with an edge of the second kind are themselves of the second kind.

The corners, being the extremities of the edges, are of three categories. If a corner be an extremity of two edges of the first kind and not on the axis of x, then it is of the *first category:* and the infinite series of corners homologous with it are of the first category. If it be common to two edges of the first kind and lie on the axis of x, then it is of the *second category:* and the infinite series of corners homologous with it are of the second category. If it be common to two edges, one of the first and one of the second kind, it is of the *third category;* of course it lies on the axis of x and the infinite series of corners homologous with it are of the third category. We do not consider two edges of the second kind as meeting: they would, in such a case, be regarded as a single edge.

Each edge of the first kind belongs to two regions. We do not assign such an edge to either of the regions, but we use this community of region to range edges as follows. Let the edge be E_p, common to R_0 and R_p; then, making the substitution inverse to $f_p(z)$, say $f_p^{-1}(z)$, R_p becomes R_0, R_0 becomes R_{-p}, and E_p becomes $f_p^{-1}(E_p)$, which is necessarily an edge of the first kind and is common to the new regions R_{-p} and R_0, that is, it is an edge of R_0. Let it be E_p': then E_p and E_p' may be the same or they may be different.

If E_p and E_p' be different, then we have a pair of edges congruent to one another: two such congruent edges of the same region are said to be *conjugate.* Since the substitutions are of the linear type, the correspondence being uniform, not more than one edge of a region can be conjugate with a given edge of that region.

If E_p and E_p' be the same, then the substitution transforms E_p into itself: hence some point on E_p must be transformed into itself. As the edge is of the first kind so that the point is above the axis of X, the substitution is elliptic and has this point as the fixed point of the substitution in the positive half-plane. The two parts of E_p can be regarded as two edges: and the common point as the corner, evidently of the first category. Because the directions of the edges measured away from the point are inclined at an angle π, it follows that the multiplier of the elliptic substitution is $e^{\pi i}$, or -1. An illustration of this occurs in the special example of § 284, where the circular boundary of the initial region of reference is changed into itself by the fundamental substitution $wz=-1$, that is,

$$\frac{w-i}{w+i}=-\frac{z-i}{z+i}.$$

Hence *the edges of the first kind are even in number and can be arranged in conjugate pairs.*

Further, a point on an edge of the first kind is transformed into a point on the conjugate edge—uniquely, unless the point be a corner, when it belongs to two edges. Hence *points on edges of the first kind other than corners correspond in pairs.*

An edge of the second kind is transformed into one of the second kind, but belonging to a different polygon: there is no correspondence between points on edges of the second kind belonging to the same polygon.

Each corner, as the point common to two edges, belongs to at least three regions. As a point of one edge, it will have as its homologue an extremity of the conjugate edge: as a point of another edge, it will have as its homologue an extremity of the edge conjugate to that other: and these homologues may be the same or they may be different. Hence *several corners of a given region may be homologous: the set of homologous corners of a given region is called a cycle.* Since points of a series homologous with a given point all belong to one category, it is convenient to *arrange the cycles in connection with the categories of the component points.*

The number of edges of the first kind is even, say $2n$: and they can be arranged in pairs of conjugates, say E_1, E_{n+1}; E_2, E_{n+2}; .... Then since E_{n+p} is the conjugate of E_p, and $f_{n+p}(z)$ is the substitution which changes R_0 into R_{n+p}, $f_{n+p}(z)$ is a substitution changing E_p into E_{n+p}. After the preceding explanation, $f_p^{-1}(z)$ is also a substitution changing E_p into its conjugate E_{n+p}: hence we have

$$f_{n+p}(z)=f_p^{-1}(z).$$

Hence *for a division of the plane, each region of which has $2n$ edges of the first kind, the group contains n fundamental substitutions:* the remaining n substitutions, necessary to construct the remaining contiguous regions, are obtained by taking the first inverses of the fundamental substitutions.

The edge E_p has been taken as the edge common to R_0 and R_p, the region derived from R_0 by the substitution $f_p(z)$. Every region will have an edge congruent to E_p: if R_i be one such region, then the region, on the other side of that line and having that line for an edge (the edge is, for that other region, the congruent of the conjugate of E_p), is obtainable from R_0 by the substitution $f_i\{f_p(z)\}$. We thus have an easy method of determining the substitution to be associated with the region, by considering the edges which are crossed in passing to the region: and, conversely, when the substitutions are associated with the regions, the correspondence of the edges is known.

As in the special example, there are relations among the fundamental substitutions. The simplest mode of determining them is to describe a small circuit round each corner of R_0 in succession: in the description of the circuit, the symbol of each new region can be derived by a knowledge of the edge last crossed and when the circuit is closed the last symbol is the symbol also of R_0, so that a relation is obtained.

286. The only limitations as yet assigned to the initial region (and therefore to each of the regions) of the plane are (i) that it contains only one point homologous with z, and (ii) that the even number of edges of the first kind can be arranged in congruent conjugate pairs. But now, without detracting from the generality of the division, we can modify the initial region in such a way that all the edges of the first kind are arcs of circles with their centres on the axis of x. For let $C...AFB...DGC$ be a region with CGD and AFB for conjugate edges; join CD by an arc of a circle CED with its centre on the axis of x: and apply to CED the substitution inverse to that which gives the region in which E lies: let AHB be the result, being also (§ 258) an arc of a circle with its centre on the axis of x. Then the part $AFBHA$, say S_0, is transformed to $CGDEC$, say S_0', by the substitution which causes a passage from R_0 across CGD into another region: every point in S_0 has a homologue in S_0': and there is, by the hypothesis that R_0 is the initial region, no homologue in R_0 of a point in S_0 except the point itself. If, then, we take away S_0 from R_0 and add S_0', we have a new region

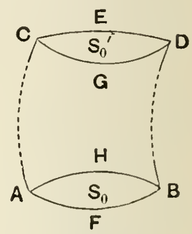

Fig. 109.

$$R_0' = R_0 + S_0' - S_0.$$

It satisfies all the conditions which apply to the regions so far obtained: there is no point in R_0' homologous with a point in it, and the conjugate edges CGD and AFB are replaced by conjugate edges CED, AHB congruent by the same substitution as the former pair. And the new conjugate edges are circles having their centres on the axis of x.

Proceeding in this way with each pair of conjugate edges that are not arcs of circles having their centres on the axis of x, and replacing it by a pair of conjugate edges congruent by the same substitution and consisting of arcs of circles having their centres on the axis of x, we ultimately obtain a region in which all the edges of the first kind are arcs of circles having their centres on the axis of x. These can, of course, be arranged in conjugate pairs, congruent by the assigned fundamental substitutions. Straight lines perpendicular to the axis of x count as circles with centres at $x = \infty$ on that axis: all other straight lines, not being parts of the axis of x, can be replaced by circles.

The edges of the second kind are left unaltered.

A region, thus bounded, is called a *normal polygon*.

Further, this normal polygon may be taken *convex*, that is, edges do not cross one another. If the preceding reduction of a region to the form of a normal polygon should lead to a cross polygon, then, as is usual in dealing with the area of such cross figures, part of the area is to be considered negative: and therefore, for every point in this negative part, there must be two points in the positive part. Hence, in the positive part, there are

(i) points, none of which has a homologue in the negative part, or in the positive part except itself: their aggregate gives a normal polygon Q:

(ii) two sets of points, each set of which consists of the homologues of points in the negative part, and makes up a positive normal polygon; let the polygons be T_1 and T_2.

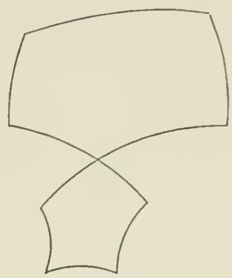

Fig. 110.

The negative part is a normal polygon T, to which T_1 and T_2 are each congruent.

We now change R by adding a normal polygon T and subtracting a normal polygon T_1: thus for the new region we have a positive (that is, a convex) polygon Q, and a positive (convex) polygon T_2. No point in Q has a homologue in T_2: hence T_2 and Q together make up a region such that homologues of all points within it lie outside: this region is a normal polygon, and it is convex. Hence we may take as the initial region of reference a *normal convex polygon, that is, a convex polygon bounded by arcs of circles having their centres on the axis of x, or by portions of the axis of x: the number of arc-edges is even, and they can be arranged in conjugate pairs.*

Simplicity is obtained by securing that the curves, which compose the boundary, are as like one another in character as possible. The substitutions are linear and they change boundaries into boundaries: the whole plane is to be covered: and there are no gaps between a bounding edge and the homologue of the conjugate bounding edge. The only curves, which satisfy this condition of leaving no gaps, and which are of the same character after any number of linear transformations, are circles and straight lines.

287. We have seen that two (or more than two) corners of a convex polygon may be homologous: it is now necessary to arrange all the corners in their cycles. Let AB and ED be two conjugate edges of a normal polygon, and let $\dfrac{az+b}{cz+d}$ be the substitution which changes AB into ED; then, as usual, we have

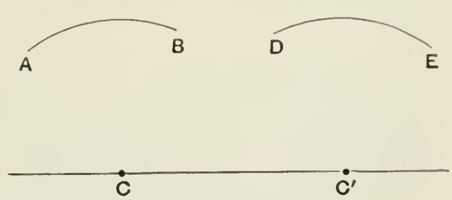

Fig. 111.

$$w-\frac{a}{c}=-\frac{ad-bc}{c^2}\frac{1}{z+\frac{d}{c}}=-\frac{1}{c^2}\frac{1}{z+\frac{d}{c}},$$

so that
$$\arg.\left(w-\frac{a}{c}\right)+\arg.\left(z+\frac{d}{c}\right)=\pi.$$

This at once shews that, whatever be the value of $\frac{a}{c}$ and of $\frac{d}{c}$, the points A, E are homologous, and likewise the points B, D. Hence to obtain a corner homologous to a given corner we start from the corner, describe the edge of the polygon beginning there, then describe in the same direction* the conjugate edge: the extremity of that edge is a homologous corner.

The process may now be reapplied, beginning with the last point; and it can be continued, each stage adding one point to the cycle, until we either return to the initial point or until we are met by an edge of the second kind. In the former case we have a completed cycle, which may be regarded as a *closed* cycle. In the latter case we can proceed no further, as edges of the second kind are not ranged in conjugate pairs; but, resuming at the initial point we apply the process with a description in the reverse direction until we again arrive at an edge of the second kind: again we have a cycle, which may be regarded as an *open* cycle.

In the case of a closed cycle, if one of the included points be of the first category, then all the points are of the first category: the cycle itself is then said to be of the first category. If one of the points be of the second category, then since no edge of the second kind is met in the description, all the edges met are of the first kind; and therefore all the points, lying on the axis of x and being the intersections of edges of the first kind, are of the second category: the cycle itself is then said to be of the second category.

Open cycles will contain points of the third category: they may also contain points of the second category for points both of the second and of the third categories lie on the axis of x, and homology of the points does not imply conjugacy of all edges of which they are extremities. Such cycles are said to be of the third category.

It thus appears that the cycles can be derived when the arrangement in conjugate pairs of edges of the first kind is given; and it is easy to see that the number of open cycles is equal to the number of edges of the second kind.

We may take one or two examples. For a quadrilateral, in which the conjugate pairs are 1, 4; 2, 3—the numbers being as in the figure—we have by the above process A, AB, DA, A: that is, A is a cycle by itself. Then B, BC, CD, D, DA, AB, B: that is, B and D form a cycle; and then C, CD, BC, C, that is, C is a cycle by itself. The cycles are therefore three, namely, A; B, D; C.

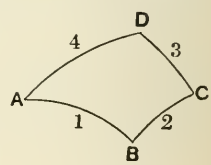

Fig. 112.

* This is necessary: the direction is easily settled for a complete polygon the sides of which are described in positive or in negative direction throughout.

For a hexagon, in which the conjugate pairs are 1, 5; 2, 4; 3, 6, the cycles are two, namely, A, F, D, C and B, E. If the conjugate pairs be

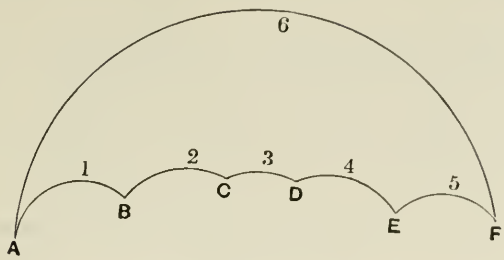

Fig. 113.

1, 6; 2, 5; 3, 4, the cycles are four, namely, A; B, F; C, E; D. If the conjugate pairs be 1, 4; 2, 5; 3, 6 the cycles are two, namely, A, C, E; B, D, F.

For a pentagon, with one edge of the second kind as in the figure and

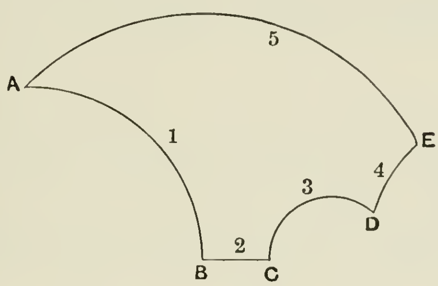

Fig. 114.

having 1, 3; 4, 5 as the conjugate pairs, the cycles are three, namely, E; A, D; B, C; the last being open and of the third category.

For a quadrilateral as in the figure, having three corners on the axis of x and 1, 2; 3, 4 as the arrangement of its conjugate pairs, the cycles are D; A, C; B; the last two being of the second category.

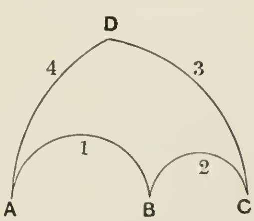

Fig. 115.

We have now to consider the angles of the polygons taken internally. It is evident that at any corner of the second category, the angle is zero, for it is the angle between two circles meeting on their line of centres; and that at any corner of the third category the angle is right. There therefore remain only the angles at corners of the first category. Let $A_1, A_2, \ldots, A_n$ be the corners in a cycle of the first category and denote the angles by the same letters.

Since A_1 and A_2 are homologous corners, they are extremities of conjugate edges. Apply to the plane, in the vicinity of A_2, the substitution which changes the edge ending in A_2 to its conjugate ending in A_1: then the point A_2 is transferred to the point A_1; one edge at A_2 coincides with its conjugate at A_1 and the other edge at A_2 makes an angle A_2 with it, because of the substitution which conserves angles. The latter edge was the edge which followed A_2 in the cycle for the derivation of A_3: we take its conjugate ending in A_3, and treat these and the points A_2 and A_3 as before for A_1 and A_2 and their conjugate edges, namely, by using the substitutions transforming conjugate edges and passing from A_3 to A_2 and then those from A_2 to A_1.

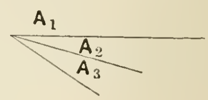

Fig. 116.

Proceeding in this way round the cycle, we shall have

(1) a series of lines at the point, each line between two angles being one of the conjugate edges on which the two corners lie:

(2) the angles corresponding to the corners taken in cyclical order.

Hence after n such operations we shall again reach an angle A_1. If the edge do not coincide with the first edge, we repeat the set of n operations: and so on.

Now all these substitutions lead to the construction of the various regions meeting in A, which are to occupy all the plane round A, and no two of which are to contain a point which does not lie on an edge. Hence after the completion of some set of operations, say the pth set, the edges of A_1 will coincide with their edges of the first angle A_1; and therefore

$$p(A_1 + A_2 + \dots + A_n) = 2\pi,$$

so that

$$A_1 + A_2 + \dots + A_n = \frac{2\pi}{p}.$$

Hence *the sum of the angles at the corners, in a cycle of the first category, is a submultiple of* 2π.

Further, if q be the number of polygons at A, we have

$$np = q.$$

Corollary 1. *For a cycle of the second category—it is a closed cycle—both p and q are infinite.*

The cycle contains only a finite number of corners, because the polygon has only a finite number* of edges: as each corner is of the second category,

* If the number be infinite, the edges must be infinitesimal in length unless the perimeter of each of the polygons is infinite: each of these alternatives is excluded.

The reason for finiteness (§ 282) in the number of fundamental substitutions in the group is now obvious: their number is one-half of the number of edges of the first kind.

the angle is zero: and therefore the repetition of the set of operations can be effected without limit. Hence p is infinite; and, as n polygons at a corner are given by each set of operations, the number q of polygons is infinite.

COROLLARY 2. *Corresponding to every cycle of the first category, there is a relation among the fundamental substitutions of the group.*

Let f_{12} be the substitution interchanging the conjugate edges through A_1 and A_2; f_{23} the substitution interchanging the conjugate edges through A_2 and A_3; and so on. Let U denote

$$f_{12}^{-1} \cdot f_{23}^{-1} \cdot f_{34}^{-1} \cdots \cdot f_{n-1,n}^{-1}(z);$$

then

$$U^p(z) = z.$$

For U is the substitution which reproduces the polygon with the angle A_1 at A_1; and this substitution is easily seen, after the preceding explanation, to be periodic of order p. Moreover, this substitution U is elliptic.

288. The following characteristics of the fundamental region have now been obtained:

(i) It is a convex polygon, the edges of which are either arcs of circles with their centres on the axis of x or are portions of the axis of x:

(ii) The edges of the former kind are even in number and can be arranged in conjugate pairs: there is a substitution for which the edges of a conjugate pair are congruent; if this substitution change one edge a of the pair into a', it changes the given region into the region on the other side of a':

(iii) The corners of the polygon can be arranged in cycles of one or other of three categories:

(iv) The angles at corners in a cycle of the second category are zero: each of the angles at corners in a cycle of the third category is right: the sum of the angles at corners in a cycle of the first category is a submultiple of 2π.

Let there be an infinite discontinuous group of substitutions, such that its fundamental substitutions are characterised by the occurrence of the foregoing properties in the edges and the angles of the geometrically associated region: and let the whole group of substitutions be applied to the region.

Then the half-plane on the positive side of the axis of x is covered: no part is covered more than once, and no part is unassigned to regions. It is easy to see in a general way how this given condition is satisfied by the various properties of the regions. Since the edges of the first kind in the initial region can be arranged in conjugate pairs, it is so with those edges in every region: and the substitution, which makes them congruent, makes one of them to coincide with the homologue of the other for the neighbouring region, so that no part is unassigned. No part is covered twice, for the initial region is a normal convex polygon and therefore every region is a normal convex polygon: the edges are homologous from region to region, and form a common boundary. The angle of intersection with a given arc is sufficient to fix the edge of the consecutive polygon: for an arc of a circle, making on one side an assigned angle with a given arc and having its centre on the axis, is unique. At every corner of any polygon, there will be a number of polygons: the corners which coincide there are, for the different polygons, the corners homologous with a cycle in the original region: and the angles belonging to those corners fill up, either alone or after an exact number of repetitions, the full angle round the point.

We have seen that the substitution, which passes from a polygon at a point to the same polygon, after n polygons, reproduces the angular point at the same time as it reproduces the polygon; the point is a fixed point of an elliptic substitution. Similarly, if the point belong to a cycle of the second category, n is infinite and the substitution does not change the point, which is therefore a fixed point of the substitution; as the fixed point is on the axis, the substitution is parabolic (§ 292).

The preceding are the essential properties of the regions, which are sufficient for the division of the half-plane when a group is given, and therefore by reflexion through the axis of x, they are sufficient for the division of the other half-plane.

The position of corners of the first category, and the orientation of edges meeting in those corners, are determinate when the group is supposed given: within certain limits, half of the corners of the third category can be arbitrarily chosen.

289. In the preceding investigation, the group has been supposed given: the problem was the appropriate division of the plane. The converse problem occurs when a fundamental region, with properties appropriate for the division of the half-plane, is given: it is the determination of the group. The fundamental substitutions of the group are those which transform an edge into its conjugate, and they are to be real—conditions which, by § 258, are sufficient for their construction. The whole group of substitutions is obtained by combining those that are fundamental. The complete division of the half-plane is effected, by applying to each polygon in succession the series of fundamental substitutions and of their first inverses.

It is evident that a given division of the plane into regions determines the group uniquely: but, as has already been seen in the general explanation, the existence of a group with the requisite properties does not imply a unique division of the plane.

As an example, let the fundamental substitutions be required when a quadrilateral as in Fig. 112, having 1, 2; 3, 4 for the conjugate pairs of edges, is given as a fundamental region. The cycles of the corners are B; D; A, C; so that

$$B=\frac{2\pi}{l},\quad D=\frac{2\pi}{m},\quad A+C=\frac{2\pi}{n},$$

where l, m, n are integers.

The simplest case has already been treated, § 284: there, $l=2$, $m=\infty$, $n=3$, $A=C$; the region is a triangle, really a quadrilateral with two edges as conterminous arcs of the same circle. We shall therefore suppose this case excluded; we take the case next in point of simplicity, viz. $l=2$, $A=C$. Then AB and BC are conterminous arcs of one circle: we shall take the centre of this circle to be the origin, its radius unity and B on the axis of y; then B is a fixed point of the substitution, which changes AB into BC. The substitution is

$$w=-\frac{1}{z};$$

it is one of the two fundamental substitutions.

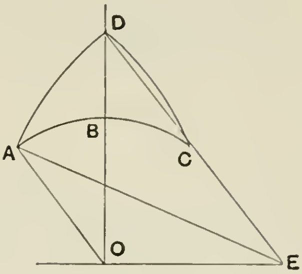

Fig. 117.

Evidently $A=\frac{\pi}{n}$, $ADB=\frac{\pi}{m}$. Let E be the centre of the circle AD, and ρ its radius: then $OAE=\frac{\pi}{n}$, $ODE=\frac{\pi}{2}-\frac{\pi}{m}$, and so

$$\rho^2+1-2\rho\cos\frac{\pi}{n}=OE^2=\rho^2\cos^2\frac{\pi}{m},$$

whence
$$\rho\sin^2\frac{\pi}{m}=\cos\frac{\pi}{n}+\left(\cos^2\frac{\pi}{n}-\sin^2\frac{\pi}{m}\right)^{\frac{1}{2}},$$

the negative sign of the radical corresponding to the case when D lies below ABC. The radius ρ must be real and therefore

$$\frac{1}{n}+\frac{1}{m}\leqslant\tfrac{1}{2}:$$

we omit the case of $m=\infty$, and therefore $n>2$.

The fundamental substitution, which changes AD into CD, has D and the complex conjugate to D for its fixed points: these points are $\pm i\rho\sin\frac{\pi}{m}$. The argument of the multiplier is $\frac{2\pi}{m}$, being the angle ADC: hence the substitution is

$$\frac{w-i\rho\sin\frac{\pi}{m}}{w+i\rho\sin\frac{\pi}{m}}=\frac{z-i\rho\sin\frac{\pi}{m}}{z+i\rho\sin\frac{\pi}{m}}e^{\frac{2\pi i}{m}},$$

which reduces to

$$w=\frac{z\cos\frac{\pi}{m}+\rho\sin^2\frac{\pi}{m}}{-\frac{z}{\rho}+\cos\frac{\pi}{m}},$$

where ρ has the value given by the above equation.

This substitution, and the substitution $w=-\frac{1}{z}$, are the fundamental substitutions of the group. The special illustration in § 284 gives

$$m=\infty,\ \rho=\infty,\ n=3,\ \rho\sin^2\frac{\pi}{m}=2\cos\frac{\pi}{n}=1\ ;$$

the special form therefore is

$$w=z+1.$$

Taking $\cos\frac{\pi}{m}=a$, $\cos\frac{\pi}{n}=b$, $\Delta=(a^2+b^2-1)^{\frac{1}{2}}$, we have $\rho(1-a^2)=b+\Delta$; the second fundamental substitution is

$$w=Sz=\frac{az+\Delta+b}{(\Delta-b)z+a}.$$

It is easy to see that

$$T^2=1,\ S^m=1,\ (TS)^n=1,$$

where $Tz=-\frac{1}{z}$; the complete figure can be constructed as in § 284.

An interesting figure occurs for $m=4$, $n=6$.

In the same way it may be proved that, if an elliptic substitution have $re^{\pm\theta i}$ for its common points and 2Θ for the argument of its multiplier, its expression is

$$w=\frac{Az+B}{Cz+D},$$

where $A=\frac{\sin(\theta-\Theta)}{\sin\theta}$, $B=r\frac{\sin\Theta}{\sin\theta}$, $C=-\frac{1}{r}\frac{\sin\Theta}{\sin\theta}$, $D=\frac{\sin(\theta+\Theta)}{\sin\theta}$.

Taking now the more general case where $B=\frac{2\pi}{l}$, $D=\frac{2\pi}{m}$, $A+C=\frac{2\pi}{n}$, let B (in figure 112) be the point $be^{\beta i}$, and A the point ae^{ai}. Then the substitution which transforms AB into BC is the above, when $\theta=\beta$, $r=b$, $\Theta=B$, so that, if C be $ce^{\gamma i}$,

$$ce^{\gamma i}=\frac{a\sin(\beta-B)\,e^{ai}+b\sin B}{-\frac{a}{b}\sin Be^{ai}+\sin(\beta+B)},$$

giving two relations among the constants.

Similarly two more relations will arise out of the substitution which transforms CD into DA. And three relations are given by the conditions that the sum of the angles at A and C is an aliquot part of 2π, and that each of the angles B and D is an aliquot part of 2π.

290. All the substitutions hitherto considered have been real: we now pass to the consideration of those which have complex coefficients. Let

$$\frac{\alpha z+\beta}{\gamma z+\delta}$$

be such an one, supposed discontinuous: then the effect on a point is obtained by displacing the origin, inverting with respect to the new position, reflecting through a line inclined to the axis of x at some angle, and again displacing the origin. The displacements of the origins do not alter the character of relations of points, lines and curves: so that the essential parts of the transformation are an inversion and a reflexion.

Let a group of real substitutions of the character considered in the preceding sections be transformed by the foregoing single complex substitution: a new group

$$\left(\frac{\alpha z+\beta}{\gamma z+\delta},\ \frac{a\dfrac{\alpha z+\beta}{\gamma z+\delta}+b}{c\dfrac{\alpha z+\beta}{\gamma z+\delta}+d}\right)$$

will thus be derived. The geometrical representation is obtained through transforming the old geometrical representation by the substitution

$$\left(\frac{\alpha z+\beta}{\gamma z+\delta},\ z\right),$$

so that the new group is discontinuous.

The original group left the axis of x unchanged, that is, the line $z=z_0$ was unchanged; hence the substitutions

$$\left(\frac{\alpha z+\beta}{\gamma z+\delta},\ \frac{a\dfrac{\alpha z+\beta}{\gamma z+\delta}+b}{c\dfrac{\alpha z+\beta}{\gamma z+\delta}+d}\right)$$

will leave unchanged the line which is congruent with $z=z_0$ by the substitution $\left(\dfrac{\alpha z+\beta}{\gamma z+\delta},\ z\right)$. This line is

$$\frac{-\delta z+\beta}{\gamma z-\alpha}=\frac{-\delta_0 z_0+\beta_0}{\gamma_0 z_0-\alpha_0},$$

or it may be taken in the form

$$\text{imaginary part of } \frac{-\delta z+\beta}{\gamma z-\alpha}=0.$$

It is a circle, being the inverse of a line; it is unaltered by the substitutions of the new group, and it is therefore called* the *fundamental circle* of this group. The group is still called Fuchsian (p. 606, note).

The half-planes on the two sides of the axis of x are transformed into the two parts of the plane which lie within and without the fundamental circle respectively: let the positive half-plane be transformed into the part within the circle.

With the group of real substitutions, points lying above the axis of x are transformed into points also lying above the axis of x, and points below into points below: hence with the new group, points within the fundamental circle are transformed into points also within the circle, and points without into points without.

* Klein uses the word *Hauptkreis*.

The division of the half-plane into curvilinear polygons is changed into a division of the part within the circle into curvilinear polygons. The sides of the polygons either are circles having their centres on the axis of x, that is, cutting the axis orthogonally, or they are parts of the axis of x: hence the sides of the polygons in the division of the circle either are arcs of circles cutting the fundamental circle orthogonally or they are arcs of the fundamental circle.

The division of the part of the plane without the circle is the transformation of the half-plane below the axis of x, which is a mere reflexion in the axis of x of the half-plane above: thus the division is characterised by the same properties as characterise the division of the part within the fundamental circle. But when the division of the part within the circle is given, the actual division of the part without it can be more easily obtained by inversion with the centre of the fundamental circle as centre and its radius as radius of inversion.

This process is justified by the proposition that conjugate complexes are transformed by the substitution $\left(\dfrac{\alpha z+\beta}{\gamma z+\delta}, z\right)$ into points which are the inverses of one another with regard to the fundamental circle. For a system of circles can be drawn through two conjugate complexes, cutting the real axis orthogonally: when the transformation is applied, we have a system of circles, orthogonal to the fundamental circle and passing through the two corresponding points. The latter are therefore inverses with regard to the fundamental circle.

This proposition can also be proved in the following elementary manner.

Let OC, the axis of x, be inverted, with A as the centre of inversion, into a circle: P and Q be two conjugate complexes, and let AP cut axis of x in C: let CQ cut the diameter of the circle in R. Since OC bisects PQ, it bisects AR; and therefore the centre of the circle is the inverse of R.

Let p and q be the inverses of P and Q: join pq, qr. Then the angle $pqQ = CPQ = CQP$, and $Aqr = CRO$: thus pqr is a straight line. Also

$$\frac{qr}{Aq} = \frac{QR}{AR} = \frac{AP}{AR} = \frac{Ar}{Ap},$$

and

$$\frac{pr}{Ap} = \frac{PR}{AR} = \frac{AQ}{AR} = \frac{Ar}{Aq},$$

so that $$rp \, . \, rq = Ar^2.$$

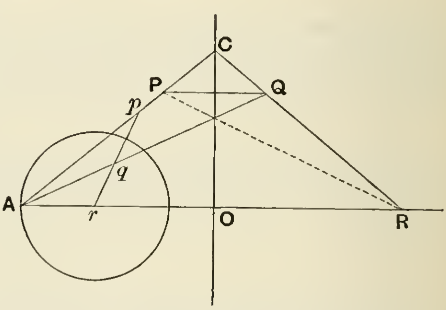

Fig. 118.

Thus p and q are inverses of each other, relative to r and with the radius of the fundamental circle as radius. Transference of origin and reflexion in a straight line do not alter these properties: and therefore p and q, the transformations of the conjugate P and Q, are inverses of one another with regard to the fundamental circle.

Hence with the present group, constructed from an infinite discontinuous group of real substitutions transformed by a single complex substitution, the fundamental circle has the same importance as the axis of real quantities in the group of real substitutions. It is of finite radius, which will be taken to be unity: its centre will be taken to be the origin. The area within it is divided into regions congruent with one another by the substitutions of the group: the whole of the area is covered by the polygons, but no part is covered more than once.

All the points, homologous with a given point z within the circle, lie within the circle: each polygon contains only one of such a set of homologous points.

The angular points of a polygon can be arranged in cycles which are of three categories. The sum of the angles at points in a cycle of the first category is unchanged by the substitution; it is equal to an aliquot part of 2π. At points in a cycle of the second category each angle is zero: at points in a cycle of the third category each angle is right.

In fact, all the properties obtained for the division of the plane into polygons now hold for the division of the circle into polygons associated with the group

$$\left(\frac{\alpha z+\beta}{\gamma z+\delta},\ \frac{a\dfrac{\alpha z+\beta}{\gamma z+\delta}+b}{c\dfrac{\alpha z+\beta}{\gamma z+\delta}+d}\right),$$

provided we make the changes that are consequent on the transformation of the axis of x into the fundamental circle.

The form of the substitution

$$w=\frac{\alpha z+\beta}{\gamma z+\delta},$$

which secures that the fundamental circle in the w-plane shall be of radius unity and centre the origin, is easily obtained.

It has been proved that inverse points with respect to the circle correspond to conjugate complexes; hence $w=0$ and $w=\infty$ correspond to two conjugate complexes, say λ and λ_0, and therefore

$$w=\kappa\,\frac{z-\lambda}{z-\lambda_0},$$

where $|\kappa|=1$ because the radius of the fundamental circle is to be unity. The presence of this factor κ is equivalent to a rotation of the w-plane about the origin. As the origin is the centre of the fundamental circle, the circle is unaltered by such a change: and therefore, without affecting the generality of the substitution, we may take $\kappa=1$, so that now

$$w=\frac{z-\lambda}{z-\lambda_0},$$

where λ is an arbitrary complex constant. The substitution is not in its canonical form, which however can at once be deduced.

291. It has been seen, in § 260, that, when any real substitution is parabolic or hyperbolic, then practically an infinite number of points coincide with the fixed point of the substitution when it is repeated indefinitely, whatever be the point z initially subjected to the transformation; this fixed point lies on the axis of x, and is called an essential singularity of the substitution. When we consider such points in reference to automorphic functions, which are such as to resume their value when their argument is subjected to the linear substitutions of the group, then at such a point the function resumes the value which it had at the point initially transformed; that is, in the immediate vicinity of such a fixed point of the substitution, the function acquires any number of different values: such a point is an essential singularity of the function. Hence the essential singularities of the group are the essential singularities of the corresponding function.

Now all the essential singularities of a discontinuous group lie on the axis of x when the group is real; the line may be or may not be a continuous line of essential singularity. If, for example, x be any such point for the group of §§ 283, 284 which is characteristic of elliptic modular-functions, then all the others for that group are given by

$$\frac{ax+b}{cx+d},$$

where a, b, c, d are integers, subject to the condition $ad - bc = 1$: and therefore all the essential singularities are given by rational linear transformations. For points on the real axis, this group is improperly discontinuous: and therefore for this group the axis of x is a continuous line of essential singularity.

Hence when we use the transformation $\left(\frac{\alpha z + \beta}{\gamma z + \delta}, z\right)$ to deduce the division of the fundamental circle into regions, the essential singularities of the new group are points on the circumference of the fundamental circle: the circumference is or is not a continuous line of essential singularity for the function or the group*, according as the group is properly or improperly discontinuous for the circle.

292. It is convenient to divide the groups into families, the discrimination adopted by Poincaré being made according to the categories of cycles of angular points in the polygons into which the group divides the plane. The group is of the

1st family, if the polygon have cycles of the	1st	category only,
2nd	2nd	...............,
3rd	3rd	...............,
4th	2nd and 3rd	...............,

* Poincaré calls the group Fuchsian, both when all the coefficients are real and when they arise from the transformation of such an infinite group by a single substitution that has imaginary coefficients. A convenient resumé of his results is given by him in a paper, *Math. Ann.*, t. xix, (1882), pp. 553—564.

5th family, if the polygon have cycles of the 1st and 3rd categories only,
6th .... 1st and 2nd ....,
7th .... all three categories.

Thus in the polygons, associated with groups of the 1st, the 2nd, and the 6th families, all the edges are of the first kind; in the polygons associated with groups of the remaining families, edges of the second kind occur.

A subdivision of some of the families is possible. It has been proved that the sum of the angles in a cycle of the first category is a submultiple of 2π. If the sum is actually 2π, the cycle is said to belong to the first sub-category: if it be less than 2π (being necessarily a submultiple), the cycle is said to belong to the second sub-category. And then, if all the cycles of the polygon belong to the first sub-category, the group is said to belong to the first order in the first family: if the polygon have any cycle belonging to the second sub-category, the group is said to belong to the second order in the first family.

It has been proved in § 288 that a corner belonging to a cycle of the second category is not changed by the substitution which gives the contiguous polygons in succession; the corner is a fixed point of the substitution, so that the substitution is either parabolic or hyperbolic. In his arrangement of families, Poincaré divided the cycles of the second category into cycles of two sub-categories, according as the substitution is parabolic or hyperbolic: but Klein has proved* that there are no cycles for hyperbolic substitutions, and therefore the division is unnecessary. The families of groups, the polygons associated with which have cycles of the second category, are the second, the fourth, the fifth and the seventh.

There is one very marked difference between the set of families, consisting of the first, the second and the sixth, and the set constituted by the remainder.

No polygon associated with a real group in the former set has an edge of the second kind: and therefore the only points on the axis taken account of in the division of the plane are the essential singularities of the group. The domain of any ordinary point on the axis in the vicinity of each of the essential singularities is infinitesimal: and therefore the axis of x is taken account of in the division of the plane only in so far as it contains essential singularities of the group and the functions. This, of course, applies equally to the transformed configuration in which the conserved line is the fundamental circle: and therefore, in the division of the area of the circle, its circumference is taken account of only in so far as it contains essential singularities of the groups and the functions.

But each polygon associated with a real group in the second set of families has an edge of the second kind: the groups still have all their

* *Math. Ann.*, t. xl, (1892), p. 132.

essential singularities on the axis of x (or on the fundamental circle) and at least some of them are isolated points; so that the domain of an ordinary point on the axis is not infinitesimal. Hence parts of the axis of x (or of the circumference of the fundamental circle) fall into the division of the bounded space.

293. There is a method of ranging groups which is of importance in connection with the automorphic functions determined by them.

The upper half of the plane of representation has been divided into curvilinear polygons; it is evident that the reflexion of the division, in the axis of real quantities, is the division of the lower half of the plane. Let the polygon of reference in the upper half be R_0 and in the lower half be R_0', obtained from R_0 by reflexion in the axis of real quantities. Then, if the group belong to the set, which includes the first, the second and the sixth families, R_0 and R_0' do not meet except at those isolated points, which are polygonal corners of the second category. But if the group belong to the set which includes the remaining families, then R_0 and R_0' are contiguous along all edges of the second kind, and they may be contiguous also at isolated points as before.

In the former case R_0 and R_0' may be regarded as distinct spaces, each fundamental for its own half-plane. Let R_0 have $2n$ edges which can be arranged in n conjugate pairs, and let q be the number of cycles all of which are closed; each point in one edge corresponds to a single point in the conjugate edge. Let the surface included by the polygon R_0 be deformed and stretched in such a manner that conjugate edges are made to coincide by the coincidence of corresponding points. A closed surface is obtained. For each pair of edges in the polygon there is a line on the surface, and for each cycle in the polygon there is a point on the surface in which lines meet; and the lines make up a single curvilinear polygon occupying the whole surface. The process is reversible; and therefore the connectivity of the surface is an integer which may properly be associated with the fundamental polygon.

When two consecutive edges are conjugate, their common corner is a cycle by itself. The line, made up of these two edges after the deformation, ends in the common corner which has become an isolated point; this line can be obliterated without changing the connectivity. The obliteration annuls two edges and one cycle of the original polygon: that is, it diminishes n by unity and q by unity. Let there be r such pairs of consecutive edges. The deformed surface is now occupied by a single polygon, with $n-r$ sides and $q-r$ angular points; so that, if its connectivity be $2N+1$, we have (§ 165)

$$\begin{aligned}2N &= 2 + (n-r) - 1 - (q-r)\\ &= n + 1 - q.\end{aligned}$$

The group may be said to be of *class* N.

In the latter case, the combination of R_0 and R_0' may be regarded as a single region, fundamental for the whole plane. Let R_0 have $2n$ edges of the first kind and m of the second kind, and let q be the number of closed cycles: the number of open cycles is m. Then R_0' has $2n$ edges of the first kind and q closed cycles; it has, in common with R_0, the m edges of the second kind and the m open cycles. The correspondence of points on the edges of the first kind is as before. Let the surface included by R_0 and R_0' taken together be deformed and stretched in such a manner that conjugate edges coincide by the coincidence of corresponding points on those edges. A closed surface is obtained. As the process is reversible, the connectivity of the surface thus obtained is an integer which may properly be associated with the fundamental polygon.

This integer is determined as before. For each pair of edges of the first kind in either polygon, a line is obtained on the surface; so that $2n$ lines are thus obtained, n from R_0 and n from R_0'. Each of the common edges of the second kind is a line on the surface, so that m lines are thus obtained. The total number of lines is therefore $2n+m$. For each of the closed cycles there is a point on the surface in which lines, obtained through the deformation of edges of the first kind, meet: their number is $2q$, each of the polygons providing q of them. For each of the open cycles there is a point on the surface in which one of the m lines divides one of the n lines arising through R_0 from the corresponding line arising through R_0': the number of these points is m. The total number of points is therefore $2q+m$.

The total number of polygons on the surface is 2. Hence, if the connectivity be $2N+1$, we have (§ 165)

$$\begin{aligned} 2N &= 2 + 2n + m - (2q+m) - 2 \\ &= 2n - 2q. \end{aligned}$$

The group may be said to be of *class* N.

Thus for the generating quadrilateral in figure 112 (p. 596), the class of the group is zero when the arrangement of the conjugate pairs is 1, 2; 3, 4: and it is unity when the arrangement of the pairs is 1, 3; 2, 4. For the generating hexagon in figure 113 (p. 597), the class of the group is zero when the arrangement of the conjugate pairs is 1, 6; 2, 5; 3, 4: and it is unity when the arrangement of the pairs is 1, 4; 2, 5; 3, 6. For the generating pentagon in figure 114 (p. 597), the class of the group is zero when the arrangement of the conjugate pairs is 1, 3; 4, 5: and it is two, when the arrangement of the pairs is 1, 4; 3, 5. For a generating polygon, bounded by $2n$ semi-circles each without all the others and by the portions of the axis of x, the number of closed cycles is zero: hence $N=n$.

294. In all the groups, which lead to a division of a half-plane or of a circle into polygons, the substitutions have real coefficients or are composed of real substitutions and a single substitution with complex coefficients: and thus the variation in the complex part of the coefficients in the group is strictly limited. We now proceed to consider groups of substitutions

$$\left(z, \frac{\alpha z + \beta}{\gamma z + \delta}\right),$$

in which the coefficients are complex in the most general manner: such groups, when properly discontinuous, are called *Kleinian,* by Poincaré.

The Fuchsian groups conserve a line, the axis of x, or a circle, the fundamental circle: the Kleinian groups do not conserve such a line or circle, common to the group. Every substitution can be resolved into two displacements of origin, an inversion and a reflexion, as in § 258. The inversion has for its centre the point $-\delta/\gamma$, being the origin after the first displacement; the reflexion is in the line through this point making with the real axis an angle $\pi - 2 \arg. \gamma$. The only line left unaltered by these processes is one which makes an angle $\frac{1}{2}\pi - \arg. \gamma$ with the real axis and passes through the point; and the final displacement to the point α/γ will in general displace this line. Moreover, arg. γ is not the same for all substitutions; there is therefore no straight line thus conserved common to the group.

Similar considerations shew that there is no fundamental circle for the group, persisting untransformed through all the substitutions.

Hence the Kleinian groups conserve no fundamental line and no fundamental circle: when they are used to divide the plane, the result cannot be similar to that secured by the Fuchsian groups. As will now be proved, they can be used to give relations between positions in space, as well as relations between positions merely in a plane.

The lineo-linear relation between two complex variables, expressed as a linear substitution, has been proved (§ 261) to be the algebraical equivalent of any even number of inversions with regard to circles in the plane of the variables: this analytical relation, when developed in its geometrical aspect, can be made subservient to the correlation of points in space.

Let spheres be constructed which have, as their equatorial circles, the circles in the system of inversions just indicated; let inversions be now carried out with regard to these spheres, instead of merely with regard to their equatorial circles. It is evident that the consequent relations between points in the plane of the variable z are the same as when inversion is carried out with regard to the circles: but now there is a unique transformation of points that do not lie in the plane. Moreover, the transformation possesses the character of conformal representation, for it conserves angles and it secures the similarity of infinitesimal figures: points lying above the plane of z

invert into points lying above the plane of z, so that the plane of z is common to all these spherical inversions and therefore common to the substitutions, the analytical expression of which is to be associated with the geometrical operation; and a sphere, having its centre in the plane of the complex z is transformed into another sphere, having its centre in that plane, so that the equatorial circles correspond to one another.

Through any point P in space, let an arbitrary sphere be drawn, having its centre in the plane of the complex variable, say, that of the coordinates ξ, η. It will be transformed, by the various inversions indicated, into another sphere, having its centre also in the plane of ξ, η and passing through the point Q obtained from P as the result of all the inversions; and the equatorial planes will correspond to one another.

Let the sphere through Q be

$$(\xi'-a)^2+(\eta'-b)^2+\zeta'^2=r'^2,$$

or

$$\xi'^2+\eta'^2+\zeta'^2-2a\xi'-2b\eta'+k=0.$$

Hence, if Q be determined by

$$z'=\xi'+i\eta',\quad z_0'=\xi'-i\eta',\quad \rho'^2=\xi'^2+\eta'^2+\zeta'^2=z'z_0'+\zeta'^2,$$

this equation is

$$\rho'^2+h_0z'+hz_0'+k=0,$$

where $-h$, $-h_0=a+ib$, $a-ib$ respectively. The equatorial circle of this sphere is evidently given by $\zeta'=0$, so that its equation is

$$z'z_0'+h_0z'+hz_0'+k=0;$$

this circle can be obtained from the equatorial circle of the sphere through P by the substitution $z'=\dfrac{\alpha z+\beta}{\gamma z+\delta}$. Hence the latter circle, by § 258, is given by

$$zz_0(\alpha\alpha_0+h_0\alpha\gamma_0+h\alpha_0\gamma+k\gamma\gamma_0)+z_0(\alpha_0\beta+h_0\beta\gamma_0+h\alpha_0\delta+k\gamma_0\delta)$$
$$+z(\alpha\beta_0+h_0\alpha\delta_0+h\beta_0\gamma+k\gamma\delta_0)+\beta\beta_0+h_0\beta\delta_0+h\beta_0\delta+k\delta\delta_0=0:$$

and therefore the equation of the sphere through P is

$$\rho^2(\alpha\alpha_0+h_0\alpha\gamma_0+h\alpha_0\gamma+k\gamma\gamma_0)+z_0(\alpha_0\beta+h_0\beta\gamma_0+h\alpha_0\delta+k\gamma_0\delta)$$
$$+z(\alpha\beta_0+h_0\alpha\delta_0+h\beta_0\gamma+k\gamma\delta_0)+\beta\beta_0+h_0\beta\delta_0+h\beta_0\delta+k\delta\delta_0=0.$$

The quantities h, h_0, k are arbitrary quantities, subject to only the single condition that the sphere passes through the point Q: there is no other relation that connects them. Hence the equation of the sphere through P must, as a condition attaching to the quantities h, h_0, k, be substantially the equivalent of the former condition given by the equation of the sphere through Q. In order that these two equations may be the same for h, h_0, k, the variables ρ'^2, z', z_0' of the point Q and those of P, being ρ^2, z, z_0, must give practically the same coefficients of h, h_0, k in the two equations, and therefore

$$\begin{aligned} \rho'^2 &: \rho^2\alpha\alpha_0 + z_0\alpha_0\beta + z\alpha\beta_0 + \beta\beta_0 \\ = z' &: \rho^2\alpha\gamma_0 + z_0\beta\gamma_0 + z\alpha\delta_0 + \beta\delta_0 \\ = z_0' &: \rho^2\alpha_0\gamma + z_0\alpha_0\delta + z\beta_0\gamma + \beta_0\delta \\ = 1 &: \rho^2\gamma\gamma_0 + z_0\gamma_0\delta + z\gamma\delta_0 + \delta\delta_0. \end{aligned}$$

These are evidently the equations which express the variables of a point Q in space in terms of the variables of the point P, when it is derived from P by the generalisation of the linear substitution

$$w' = \frac{\alpha w + \beta}{\gamma w + \delta}:$$

they may be called the equations of the substitution. It is easy to deduce that

$$\frac{\zeta'}{\zeta} = \frac{1}{\rho^2\gamma\gamma_0 + z_0\gamma_0\delta + z\gamma\delta_0 + \delta\delta_0},$$

which may be combined with the preceding equations of the substitution.

Also, the magnification for a single inversion is ds_1/ds, or r_1/r, where r_1 and r are the distances of the arcs from the centre of the sphere relative to which the inversion is effected. But $r_1/r = \zeta_1/\zeta$, where ζ_1 and ζ are the heights of the arcs above the equatorial plane; hence the magnification is ζ_1/ζ, for a single inversion. For the next inversion it is ζ_2/ζ_1, and therefore it is ζ_2/ζ for the two together; and so on. Hence the final magnification m for the whole transformation is

$$m = \frac{\zeta'}{\zeta} = \frac{1}{\zeta^2\gamma\gamma_0 + (\gamma z + \delta)(\gamma_0 z_0 + \delta_0)}$$

$$= \frac{1}{\zeta^2|\gamma|^2 + |\gamma z + \delta|^2},$$

a quantity that diminishes as the region recedes from the equatorial plane.

It is justifiable to regard the equations obtained as merely the generalisation of the substitution: they actually include the substitution in its original application to plane variables. When the variables are restricted to the plane of ξ, η, we have $\rho^2 = zz_0$, and therefore

$$z' = \frac{zz_0\alpha\gamma_0 + z_0\beta\gamma_0 + z\alpha\delta_0 + \beta\delta_0}{zz_0\gamma\gamma_0 + z_0\gamma_0\delta + z\gamma\delta_0 + \delta\delta_0} = \frac{\alpha z + \beta}{\gamma z + \delta},$$

on the removal of the factor $\gamma_0 z_0 + \delta_0$ common to the numerator and the denominator; and ζ' vanishes when $\zeta = 0$. The uniqueness of the result is an *a posteriori* justification of the initial assumption that one and the same point Q is derived from P, whatever be the inversions that are equivalent to the linear substitution.

Ex. 1. Let an elliptic substitution have u and v as its fixed points.

Draw two circles in the plane, passing through u and v and intersecting at an angle equal to half the argument of the multiplier. The transformation of the plane, caused by the substitution, is equivalent to inversions at these circles; the corresponding transformation of the space above the plane is equivalent to inversions at the spheres, having these circles as equatorial circles. It therefore follows that every point on the line of intersection of the spheres remains unchanged: hence *when a Kleinian substitution is elliptic, every point on the circle, in a plane perpendicular to the plane of x, y and having the line joining the common points of the substitution as its diameter, is unchanged by the substitution.* Poincaré calls this circle C the *double* (or *fixed*) *circle* of the elliptic substitution.

Ex. 2. Prove that, when a Kleinian substitution is hyperbolic, the only points in space, which are unchanged by it, are its double points in the plane of x, y; and shew that it changes any circle through those points into itself and also any sphere through those points into itself.

Ex. 3. Prove that, when the substitution is loxodromic, the circle C, in a plane perpendicular to the plane x, y and having as its diameter the line joining the common points of the substitution, is transformed into itself, but that the only points on the circumference left unchanged are the common points.

Ex. 4. Obtain the corresponding properties of the substitution when it is parabolic.

(All these results are due to Poincaré.)

295. The process of obtaining the division of the z-plane by means of Kleinian groups is similar to that adopted for Fuchsian groups, except that now there is no axis of real quantities or no fundamental circle conserved in that plane during the substitutions: and thus the whole plane is distributed. The polygons will be bounded by arcs of circles as before: but a polygon will not necessarily be simply connected. Multiple connectivity has already arisen in connection with real groups of the third family by taking the plane on both sides of the axis.

As there are no edges of the second kind for polygons determined by Kleinian groups, the only cycles of corners of polygons are closed cycles; let $A_0, A_1, \ldots, A_{n-1}$ in order be such a cycle in a polygon R_0. Round A_0 describe a small curve, and let the successive polygons along this curve be $R_0, R_1, \ldots, R_{n-1}, R_n, \ldots$. The corner A_0 belongs to each of these polygons: when considered as belonging to R_m, it will in that polygon be the homologue of A_m as belonging to R_0, if $m < n$; but, as belonging to R_n, it will, in that polygon, be the homologue of A_0 as belonging to R_0. Hence the substitution, which changes R_0 into R_n, has A_0 for a fixed point.

This substitution may be either elliptic or parabolic, (but not hyperbolic, § 292): that it cannot be loxodromic may be seen as follows. Let $\rho e^{i\omega}$ be the multiplier, where (§ 259) ρ is not unity and ω is not zero: and let Σ_0 denote the aggregate of polygons $R_0, R_1, \ldots, R_{n-1}$, Σ_1 the aggregate $R_n, \ldots, R_{2n-1}$, and so on. Then Σ_0 is changed to Σ_1, Σ_1 to Σ_2, and so on, by the substitution. Let p be an integer such that $p\omega \geqslant 2\pi$; then, when the substitution has been applied p times, the aggregate of the polygons is Σ_p, and it will cover the whole or part of one of the aggregates $\Sigma_0, \Sigma_1, \ldots$. But, because ρ^p is not unity, Σ_p does not coincide with that aggregate or the part of that aggregate: the substitution is not then properly discontinuous, contrary to the definition of the group. Hence there is no loxodromic substitution in the group. If the substitution be elliptic, the sum of the angles of the cycle must be a submultiple of 2π; when it is parabolic, each angle of the cycle is zero.

In the generalised equations whereby points of space are transformed into one another, the plane of x, y is conserved throughout: it is natural therefore to consider the division of space on the positive side of this plane into regions $P_0, P_1, \ldots$, such that P_0 is changed into all the other regions in turn by the application to it of the generalised equations. The following results can be obtained by considerations similar to those before adduced in the division of a plane*.

The boundaries of regions are either portions of spheres, having their centres in the plane of x, y, or they are portions of that plane: the regions are called polyhedral, and such boundaries are called *faces*. If the face is spherical, it is said to be of the *first kind:* if it is a portion of the plane of x, y, it is said to be of the *second kind.* Faces of the second kind, being in the plane of x, y and transformed into one another, are polygons bounded by arcs of circles.

The intersections of faces are *edges.* Again, an edge is of the *first kind,* when it is the intersection of two faces of the first kind: it is of the *second kind,* when it is the intersection of a face of the first kind with one of the second kind. An edge of the second kind is a circular arc in the plane of x, y: an edge of the first kind, being the intersection of two spheres with their centres in the plane of x, y, is a circular arc, which lies in a plane perpendicular to the plane of x, y and has its centre in that plane.

The extremities of the edges are *corners* of the polyhedra. They are of three categories:

(i) those which are above the plane of x, y and are the common extremities of at least three edges of the first kind:

(ii) those which lie in the plane of x, y and are the common extremities of at least three edges of the first kind:

(iii) those which lie in the plane of x, y and are the common extremities of at least one edge of the first kind and of at least two edges of the second kind.

* See, in particular, Poincaré, *Acta Math.*, t. iii, pp. 66 et seq.

Moreover, points at which two faces touch can be regarded as *isolated* corners, the edges of which they are the intersections not being in evidence.

Faces of a polyhedron, which are of the first kind, are conjugate in pairs: two conjugate faces are congruent by a fundamental substitution of the group.

Edges of the first kind, being the limits of the faces, arrange themselves in cycles, in the same way as the angles of a polygon in the division of the plane. If $E_0, E_1, \ldots, E_{n-1}$ be the n edges in a cycle, the number of regions which have an edge in E_0 is a multiple of n: and the sum of the dihedral angles at the edges in a cycle (the dihedral angle at an edge being the constant angle between the faces, which intersect along the edge) is a submultiple of 2π.

The relation between the polyhedral divisions of space and the polygonal divisions of the plane is as follows. Let the group be such as to cause the fundamental polyhedron P_0 to possess n faces of the second kind, say F_{01}, $F_{02}, \ldots, F_{0n}$. Every congruent polyhedron will then have n faces of the second kind; let those of P_s be $F_{s1}, F_{s2}, \ldots, F_{sn}$. Every point in the plane of x, y belongs to some one of the complete set of faces of the second kind: and, except for certain singular points and certain singular lines, no point belongs to more than one face, for the proper discontinuity of the group requires that no point of space belongs to more than one polyhedron.

Then the plane of x, y is divided into n regions, say $D_1, D_2, \ldots, D_n$; each of these regions is composed of an infinite number of polygons, consisting of the polygonal faces F. Thus D_r is composed of $F_{0r}, F_{1r}, F_{2r}, \ldots$; and these polygonal areas are such that the substitution S_s transforms F_{0r} into F_{sr}. Hence it appears that, by a Kleinian group, the whole plane is divided into a finite number of regions; and that each region is divided into an infinite number of polygons, which are congruent to one another by the substitutions of the group.

296. The preceding groups of substitutions, that have complex coefficients, have been assumed to be properly discontinuous.

Ex. Prove that, if any group of substitutions with complex coefficients be improperly discontinuous, it is improperly discontinuous only for points in the plane of x, y.

(Poincaré.)

One of the simplest and most important of the improperly discontinuous groups of substitutions, is that compounded from the three fundamental substitutions

$$z' = Sz = z + 1, \quad z' = Tz = -\frac{1}{z}, \quad z' = Vz = z + i,$$

where i has the ordinary meaning. All the substitutions are easily proved to be of the form

$$\frac{\alpha z + \beta}{\gamma z + \delta},$$

where $\alpha\delta - \beta\gamma = 1$, and α, β, γ, δ are complex integers, that is, are represented by $m + ni$, where m and n are integers. This is the evident generalisation of the modular-function group: consequently there is at once a suggested generalisation to a polyhedron of reference, bounded by

$$\tfrac{1}{2} \geqslant \xi \geqslant -\tfrac{1}{2}, \quad \tfrac{1}{2} \geqslant \eta \geqslant -\tfrac{1}{2}, \quad \xi^2 + \eta^2 + \zeta^2 \geqslant 1,$$

which will thus have one spherical and four (accidentally) plane faces.

The following method of consideration of the points included by the polyhedron of reference differs from that which was adopted for the polygon of reference in the plane.

If possible, let a point (ξ, η, ζ) lying within the above region be transformed by the equations generalised from some one substitution of the group, say from $\dfrac{\alpha z + \beta}{\gamma z + \delta}$, into another point of the region, say ξ', η', ζ'. Then we have

$$\tfrac{1}{2} > \xi > -\tfrac{1}{2}, \quad \tfrac{1}{2} > \eta > -\tfrac{1}{2}, \quad \xi^2 + \eta^2 + \zeta^2 > 1.$$

From the last, it follows that $\zeta > \dfrac{1}{\sqrt{2}}$: and similarly for ξ', η', ζ', by the hypothesis that the point is in the region. Now

$$\frac{\zeta'}{\zeta} = \frac{1}{\rho^2\gamma\gamma_0 + z_0\gamma_0\delta + z\gamma\delta_0 + \delta\delta_0} = \frac{1}{|\gamma\zeta|^2 + |\gamma z + \delta|^2},$$

and therefore
$$1/(\zeta\zeta') = |\gamma|^2 + \frac{1}{\zeta^2}|\gamma z + \delta|^2.$$

Hence, as ζ and ζ' are both $> \dfrac{1}{\sqrt{2}}$, we have $|\gamma|^2 < 2$: so that, because γ is a complex integer, we have

$$\gamma = 0, \ \pm 1, \ \pm i$$

as the only possible cases.

If $\gamma = 0$, then since $\alpha\delta - \beta\gamma = 1$, we have $\alpha\delta = 1$ and α, δ are complex integers: thus either

$$\left.\begin{matrix}\alpha = 1\\ \delta = 1\end{matrix}\right\}, \text{ or } \left.\begin{matrix}\alpha = -1\\ \delta = -1\end{matrix}\right\}, \text{ or } \left.\begin{matrix}\alpha = i\\ \delta = -i\end{matrix}\right\}, \text{ or } \left.\begin{matrix}\alpha = -i\\ \delta = i\end{matrix}\right\}.$$

For the first of these sub-cases we have, from the equations of the substitution,

$$z' = z + \beta,$$

where β is a complex integer: if the new point lie within the region, then $\beta = 0$, and we have

$$z' = z, \quad \zeta' = \zeta,$$

which is merely an identity.

For the second, we have $z' = z - \beta$: leading to the same result.

For the third, we have, since $\delta_0 = i$,

$$z' = -z + i\beta.$$

But as $|\xi'|$, $|\eta'|$, $|\xi|$, $|\eta|$ are all less than $\frac{1}{2}$, we have $\beta = 0$, and so

$$\xi' = -\xi, \quad \eta' = -\eta\,; \text{ and } \zeta' = \zeta.$$

For the fourth case, we have

$$z' = -z - i\beta,$$

leading to the same result as the third. Hence, if $\gamma = 0$, the only point lying within the region is given by

$$\xi' = -\xi, \quad \eta' = -\eta, \quad \zeta' = \zeta:$$

determined by the substitution $w' = \dfrac{iw}{-i}$, which is $TVT^{-1}V^{-1}TV$.

If $|\gamma| = 1$, that is, $\gamma\gamma_0 = 1$, then

$$\frac{\zeta}{\zeta'} = \rho^2 + z_0\gamma_0\delta + z\gamma\delta_0 + \delta\delta_0.$$

Of the two quantities ζ and ζ', one will be not greater than the other: we choose ζ to be that one and consider the accordingly associated substitution*: thus $\zeta/\zeta' \leqslant 1$, $\rho^2 > 1$, and so

$$z_0\gamma_0\delta + z\gamma\delta_0 + \delta\delta_0 < 0,$$

say

$$z_0\frac{\delta}{\gamma} + z\frac{\delta_0}{\gamma_0} + \frac{\delta}{\gamma}\frac{\delta_0}{\gamma_0} < 0.$$

Now $|\gamma| = 1$, so that $\dfrac{\delta}{\gamma}$ is of the form $p + iq$, where p and q are integers: thus we have

$$p^2 + q^2 + 2p\xi + 2q\eta < 0,$$

which is impossible because $2\xi < 1$, $2\eta < 1$.

Hence it follows that within the region there are only two equivalent points, derived by the generalised equations from the substitution

$$w' = \frac{iw}{-i};$$

and that all points within the region can be arranged in equivalent pairs

$$\xi, \eta, \zeta \quad \text{and} \quad -\xi, -\eta, \zeta.$$

If the region be symmetrically divided into two, so that the boundaries of a new region are

$$\tfrac{1}{2} \geqslant \xi \geqslant 0, \quad \tfrac{1}{2} \geqslant \eta \geqslant -\tfrac{1}{2}, \quad \xi^2 + \eta^2 + \zeta^2 \geqslant 1, \quad \zeta > 0,$$

then no point within the new region is equivalent to any other point in the region†. As in the division of the plane by the modular group, it is easy to see that the whole space above the plane of ξ, η is divided by the group: therefore *the region is a polyhedron of reference for the group composed of the fundamental substitutions* S, T, V.

* Were it ζ', all that would be necessary would be to take the inverse substitution.

† Bianchi, *Math. Ann.*, t. xxxviii, (1891), pp. 313—324, t. xl, (1892), pp. 332—412; Picard, ib., t. xxxix, (1891), pp. 142—144; Mathews, *Quart. Journ. Math.*, vol. xxv, (1891), pp. 289—296.

The preceding substitutions, with complex integers for coefficients, are of use in applications to the discussion of binary quadratic forms in the theory of numbers. The special division of all space corresponds, of course, to the character of the coefficients in the substitutions : other divisions for similar groups are possible, as is proved in Poincaré's memoir already quoted.

These divisions all presuppose that the group is infinite : but similar divisions for only finite groups (and therefore with only a finite number of regions) are possible. These are considered in detail in an interesting memoir by Goursat*; the transformations conserve an imaginary sphere instead of a real plane as in Poincaré's theory.

Ex. Shew that, for the infinite group composed of the fundamental substitutions

$$z' = -\frac{1}{z}, \quad z' = z+1, \quad z' = z+\epsilon,$$

where ϵ is a primitive cube root of unity, a fundamental region for the division of space above the plane of z, corresponding to the generalised equations of the group, is a symmetrical third of the polyhedron extending to infinity above the sphere

$$\xi^2 + \eta^2 + \zeta^2 = 1,$$

and bounded by the sphere and the six planes

$$2\xi = \pm 1, \quad \xi + \eta\sqrt{3} = \pm 1, \quad \xi - \eta\sqrt{3} = \pm 1. \qquad \text{(Bianchi.)}$$

* "Sur les substitutions orthogonales et les divisions regulières de l'espace," *Ann. de l'Éc. Norm. Sup.*, 3[me] Sér., t. vi, (1889), pp. 9—102. See also Schönflies, *Math. Ann.*, t. xxxiv, (1889), pp. 172—203 : other references are given in these papers.

CHAPTER XXII.

Automorphic Functions.

297. As was stated in the course of the preceding chapter, we are seeking the most general form of the arguments of functions which secures the property of periodicity. The transformation of the arguments of trigonometrical and of elliptic functions, which secures this property, is merely a special case of a linear substitution: and thus the automorphic functions to be discussed are such as identically satisfy the equation

$$F(S_i z) = f(z),$$

where S_i is any one of an assigned group of linear substitutions of which only a finite number are fundamental.

Various references to authorities will be given in the present chapter, in connection with illustrative examples of automorphic functions: but it is, of course, beyond the scope of the present treatise, dealing only with the generalities of the theory of functions, to enter into any detailed development of the properties of special classes of automorphic functions such as, for instance, those commonly called polyhedral and those commonly called elliptic-modular. Automorphic functions, of types less special than those just mentioned, are called *Fuchsian functions* by Poincaré, when they are determined in association with a Fuchsian group of substitutions, and *Kleinian functions*, when they are determined in association with a Kleinian group: as our purpose is to provide only an introduction to the theory, the more general term *automorphic* will be adopted.

The establishment of the general classes of automorphic functions is effected by Poincaré in his memoirs in the early volumes of the *Acta Mathematica*, and by Klein in his memoir in the 21st volume of the *Mathematische Annalen*: these have been already quoted (p. 583 note): and Poincaré gives various historical notes* on the earlier scattered occurrences of automorphic functions and discontinuous groups. Other memoirs that may be consulted with advantage are those of Von Mangoldt†, Weber‡, Schottky§, Stahl||,

* *Acta Math.*, t. i, pp. 61, 62, 293: ib., t. iii, p. 92. Poincaré's memoirs occur in the first, third, fourth and fifth volumes of this journal: a great part of the later memoirs is devoted to their application to linear differential equations.

† *Gött. Nachr.*, (1885), pp. 313—319; ib., (1886), pp. 1—29.

‡ *Gött. Nachr.*, (1886), pp. 359—370.

§ *Crelle*, t. ci, (1887), pp. 227—272.

|| *Math. Ann.*, t. xxxiii, (1889), pp. 291—309.

Schlesinger* and Ritter†: and there are two by Burnside‡, of special interest and importance in connection with the third of the seven families of groups (§ 292).

298. We shall first consider functions associated with finite discrete groups of linear substitutions.

There is a group of six substitutions

$$z,\ \frac{1}{z},\ 1-z,\ \frac{1}{1-z},\ \frac{z-1}{z},\ \frac{z}{z-1},$$

which (§ 283) is complete. Forming expressions $z-x$, $z-\frac{1}{x}$, $z-(1-x)$, $z-\frac{1}{1-x}$, $z-\frac{x-1}{x}$, $z-\frac{x}{x-1}$ and multiplying them together, we can express their product in the form

$$(z^2-z)^2\left\{\frac{(z^2-z+1)^3}{(z^2-z)^2}-\frac{(x^2-x+1)^3}{(x^2-x)^2}\right\},$$

so that

$$A(z)=\frac{(z^2-z+1)^3}{(z^2-z)^2}$$

is a function of z which is unaltered by any of the transformations of its variable given by the six substitutions of the group. The function is well known, being connected with the six anharmonic ratios of four points in a line which can all be expressed in terms of any one of them by means of the substitutions.

Another illustration of a finite discrete group has already been furnished in the periodic elliptic transformation of § 258, whereby a crescent of the plane with its angle a submultiple of 2π was successively transformed, ultimately returning to itself: so that the whole plane is divided into portions equal in number to the periodic order of the substitution.

If a stereographic projection of the plane be made with regard to any external point, we shall have the whole sphere divided into a number of triangles, each bounded by two small circles and cutting at the same angle. By choice of centre of projection, the common corners of the crescents can be projected into the extremities of a diameter of the sphere: and then each of the crescents is projected into a lune. The effect of a substitution on the crescent is changed into a rotation round the diameter joining the vertices of a lune through an angle equal to the angle of the lune.

299. This is merely one particular illustration of a general correspondence between spherical rotations and plane homographies, as we now proceed to shew. The general correspondence is based upon the following proposition due to Cayley:—

* *Crelle*, t. cv, (1889), pp. 181—232.

† *Math. Ann.*, t. xli, (1892), pp. 1—82.

‡ *Lond. Math. Soc. Proc.*, vol. xxiii, (1892), pp. 48—88, ib., pp. 281—295.

When a sphere is displaced by a rotation round a diameter, the variables of the stereographic projections of any point in its original position and in its displaced position are connected by the relation

$$z' = \frac{(d+ic)z-(b-ia)}{(b+ia)z+(d-ic)},$$

where a, b, c, d are real quantities.

Rotation about a given diameter through an assigned angle gives a unique position for the displaced point: and stereographic projection, which is a conformal operation in that it preserves angles, also gives a unique point as the projection of a given point. Hence taking the stereographic projection on a plane of the original position and the displaced position of a point on the sphere, they will be uniquely related: that is, their complex variables are connected by a lineo-linear relation, which thus leads to a linear substitution for the plane-transformation corresponding to the spherical rotation.

Now the extremities of the axis are unaltered by the rotation; hence the projections of these points are the fixed points of the substitution. If the points be ξ, η, ζ and $-\xi$, $-\eta$, $-\zeta$, on a sphere of radius unity, and if the origin of projection be the north pole of the sphere, the fixed points of the substitution are

$$\frac{\xi+i\eta}{1-\zeta} \text{ and } -\frac{\xi+i\eta}{1+\zeta};$$

so that the substitution is of the form

$$\frac{z'+\dfrac{\xi+i\eta}{1+\zeta}}{z'-\dfrac{\xi+i\eta}{1-\zeta}} = K\frac{z+\dfrac{\xi+i\eta}{1+\zeta}}{z-\dfrac{\xi+i\eta}{1-\zeta}}.$$

To determine the multiplier K, we take a point P very near C, one extremity of the axis: let P' be the position after the rotation, so that $CP'=CP$. Then, in the stereographic projection, the small arcs which correspond to CP and CP' are equal in length, and they are inclined at an angle α. Hence the multiplier K is $e^{i\alpha}$: for when z, and therefore z', is nearly equal to $-\dfrac{\xi+i\eta}{1+\zeta}$, a fixed point of the substitution, the magnification is $|K|$ and the angular displacement is the argument of K, which is α.

Inserting the value of K, solving for z' and using the condition $\xi^2+\eta^2+\zeta^2=1$, we have

$$z' = \frac{(d+ic)z-(b-ia)}{(b+ia)z+(d-ic)},$$

where $\quad a=\xi\sin\tfrac{1}{2}\alpha, \quad b=\eta\sin\tfrac{1}{2}\alpha, \quad c=\zeta\sin\tfrac{1}{2}\alpha, \quad d=\cos\tfrac{1}{2}\alpha,$

so that $\quad a^2+b^2+c^2+d^2=1,$

the equivalent of the usual condition to which the four coefficients in any linear substitution are subject: it is evident that the substitution is elliptic. The proposition* is thus proved.

When the axis of rotation is the diameter perpendicular to the plane, we have, by § 256,

$$z = ke^{-\vartheta + i\phi}, \qquad z' = ke^{-\vartheta + i(\phi+\alpha)},$$

so that

$$z' = ze^{i\alpha},$$

agreeing with the above result by taking $\xi = 0 = \eta$, $\zeta = 1$ so that $a = 0 = b$, $c = \sin\frac{1}{2}\alpha$, $d = \cos\frac{1}{2}\alpha$.

It should be noted that the formula gives two different sets of coefficients for a single rotation: for the effect of the rotation is unaltered when it is increased by 2π, a change in α which leads to the other signs for all the constants a, b, c, d.

It thus appears that the rotation of a sphere about a diameter interchanges pairs of points on the surface, the stereographic projections of which on the plane of the equator are connected by an elliptic linear substitution: hence, in the one case as in the other, the substitution is periodic when α, the argument of the multiplier and the angle of rotation, is a submultiple of 2π.

In the discussion of functions related in their arguments to these linear substitutions, it proves to be convenient to deal with homogeneous variables, so that the algebraical forms which arise can be connected with the theory of invariants. We take $zz_2 = z_1$: the formulæ of transformation may then be represented by the equations

$$z_1' = \kappa(\alpha z_1 + \beta z_2), \quad z_2' = \kappa(\gamma z_1 + \delta z_2)$$

for the substitution $z' = (\alpha z + \beta)/(\gamma z + \delta)$. As we are about to deal with invariantive functions of position dependent upon rotations, it is important to have the determinant of homogeneous transformation equal to unity. This can be secured only if $\kappa = +1$ or if $\kappa = -1$: the two values correspond to the two sets of coefficients obtained in connection with the rotation. Hence, in the present case, the formulæ of homogeneous transformation are

$$z_1' = (d + ic)z_1 - (b - ia)z_2, \quad z_2' = (b + ia)z_1 + (d - ic)z_2,$$

where $a^2 + b^2 + c^2 + d^2$, being the determinant of the substitution, $= 1$; every rotation leads to two pairs of these homogeneous equations†. Each pair of equations will be regarded as giving a *homogeneous substitution*.

Moreover, rotations can be compounded: and this composition is, in the analytical expression of stereographically projected points, subject to the same algebraical laws as is the composition of linear substitutions. If, then, there

* Cayley, *Math. Ann.*, t. xv, (1879), pp. 238—240; Klein's *Vorlesungen über das Ikosaeder*, pp. 32—34.

† The succeeding account of the polyhedral functions are based on Klein's investigations, which are collected in the first section of his *Vorlesungen über das Ikosaeder* (Leipzig, Teubner, 1884): see also Cayley, *Camb. Phil. Trans.*, vol. xiii, pp. 4—68.

It will be seen that the results are intimately related to the results obtained in §§ 271—279, relative to the conformal representation of figures, bounded by circular arcs, on a half-plane.

be a complete group of rotations, that is, a group such that the composition of any two rotations (including repetitions) leads to a rotation included in the group, then there will be associated with it a complete group of linear homogeneous substitutions. The groups are finite together, the number of members in the group of homogeneous substitutions being double of the number in the group of rotations: and the substitutions can be arranged in pairs so that each pair is associated with one rotation.

300. Such groups of rotations arise in connection with the regular solids. Let the sphere, which circumscribes such a solid, be of radius unity: and let the edges of the solid be projected from the centre of the sphere into arcs of great circles on the surface. Then the faces of the polyhedron will be represented on the surface of the sphere by closed curvilinear figures, the angular points of which are summits of the polyhedron. There are rotations, of proper magnitude, about diameters properly chosen, which displace the polyhedron into coincidence (but not identity) with itself, and so reproduce the above-mentioned division of the surface of the sphere: when all such rotations have been determined, they form a group which may be called the group of the solid. Each such rotation gives rise to two homogeneous substitutions, so that there will thence be derived a finite group of discrete substitutions: and as these are connected with the stereographic projection of the sphere, they are evidently the group of substitutions which transform into one another the divisions of the plane obtained by taking the stereographic projection of the corresponding division of the surface of the sphere. For the construction of such groups of substitutions, it will therefore be sufficient to obtain the groups of rotations, considered in reference to the surface of the sphere.

I. The *Dihedral Group*. The simplest case is that in which the solid, hardly a proper solid, is composed of a couple of coincident regular polygons of n sides*: a reference has already been made to this case. We suppose the polygons to lie in the equator, so that their corners divide the equator into n equal parts: one polygon becomes the upper half of the spherical surface, the other the lower half. The two poles of the equator, and the middle points of the n arcs of the equator, are the corners of the corresponding solid.

Then the axes, rotations about which can bring the surface into such coincidence with itself that its partition of the spherical surface is topographically the same in the new position as in the old, are

(i) the polar axis,

(ii) a diameter through each summit on the equator,

(iii) a diameter through each middle point of an edge:

the last two are the same or are different according as n is odd or is even.

* The solid may also be regarded as a double pyramid.

For the polar axis, the necessary angle of rotation is an integral multiple of $\frac{2\pi}{n}$. Thus we have $\xi = 0 = \eta$, $\zeta = 1$ and therefore

$$a = 0 = b, \quad c = \sin\frac{\pi}{n}, \quad d = \cos\frac{\pi}{n};$$

the substitutions are

$$z_1' = e^{\frac{i\pi r}{n}} z_1, \quad z_2' = e^{-\frac{i\pi r}{n}} z_2,$$

for $r = 0, 1, \ldots, n-1$, and

$$z_1' = -e^{\frac{i\pi r}{n}} z_1, \quad z_2' = -e^{-\frac{i\pi r}{n}} z_2,$$

for the same values of r. These are included in the set

$$z_1' = e^{\frac{i\pi r}{n}} z_1, \quad z_2' = e^{-\frac{i\pi r}{n}} z_2,$$

for $r = 0, 1, 2, \ldots, 2n-1$, being $2n$ in number: the identical substitution is included for the same reason as before, when we associated a region of reference in the z-plane with the identical substitution.

For each of the axes lying in the equator, the angle of rotation is evidently π. Let an angular point of the polygon lie on the axis of ξ, say at $\xi = 1$, $\eta = 0$, $\zeta = 0$. Then so far as concerns (ii) in the above set, if we take the axis through the $(r+1)$th angular point, we have $\xi = \cos\frac{2r\pi}{n}$, $\eta = \sin\frac{2r\pi}{n}$, $\zeta = 0$; hence, as α is equal to π, we have, for the corresponding substitutions,

$$z_1' = ie^{+\frac{2r\pi i}{n}} z_2, \quad z_2' = ie^{-\frac{2r\pi i}{n}} z_1,$$

for $r = 0, 1, \ldots, n-1$, and

$$z_1' = -ie^{+\frac{2r\pi i}{n}} z_2, \quad z_2' = -ie^{-\frac{2r\pi i}{n}} z_1,$$

for the same values of r.

And so far as concerns (iii) in the above set, if we take an axis through the middle point of the rth side, that is, the side which joins the rth and the $(r+1)$th points, then $\xi = \cos\frac{(2r-1)\pi}{n}$, $\eta = \sin\frac{(2r-1)\pi}{n}$, $\zeta = 0$: hence as α is equal to π, we have, for the corresponding substitutions,

$$z_1' = ie^{+\frac{(2r-1)\pi i}{n}} z_2, \quad z_2' = ie^{-\frac{(2r-1)\pi i}{n}} z_1,$$

for $r = 0, 1, \ldots, n-1$, and

$$z_1' = -ie^{+\frac{(2r-1)\pi i}{n}} z_2, \quad z_2' = -ie^{-\frac{(2r-1)\pi i}{n}} z_1,$$

for the same values of r.

If n be even, the set of substitutions associated with (ii) are the same in pairs, and likewise the set associated with (iii); if n be odd, the set associated with (ii) is the same as the set associated with (iii). Thus in either case there are $2n$ substitutions: and they are all included in the form

$$z_1' = ie^{\frac{i\pi r}{n}} z_2, \quad z_2' = ie^{-\frac{i\pi r}{n}} z_1,$$

for $r = 0, 1, \ldots, 2n-1$.

Thus the *whole group of* $4n$ *substitutions,* in their homogeneous form, is

$$\left.\begin{aligned} z_1' &= e^{\frac{i\pi r}{n}} z_1 \\ z_2' &= e^{-\frac{i\pi r}{n}} z_2 \end{aligned}\right\} \qquad \left.\begin{aligned} z_1' &= ie^{\frac{i\pi r}{n}} z_2 \\ z_2' &= ie^{-\frac{i\pi r}{n}} z_1 \end{aligned}\right\},$$

for $r = 0, 1, \ldots, 2n-1$: and in the non-homogeneous form, the group is

$$z' = e^{\frac{2i\pi r}{n}} z, \quad z' = \frac{e^{\frac{2i\pi r}{n}}}{z},$$

where $r = 0, 1, \ldots, n-1$ for each of them. The non-homogeneous expressions are not in their normal form in which the determinant of the coefficients in the numerator and denominator is unity. Each expression gives two homogeneous substitutions.

It is easy geometrically to see that all the axes have been retained: and that they form a group, that is, composition of rotations about any two of the axes is a rotation about one of the axes. The period for each of the equatorial axes is 2; the period for a rotation $\frac{2\pi r}{n}$ about the polar axis depends on the reducibility of $\frac{r}{n}$.

Before passing to the construction of the functions which are unaltered for the dihedral group of substitutions, we shall obtain the tetrahedral group and construct the tetrahedral functions, for the explanations in regard to the dihedral functions arise more naturally in the less simple case.

II. The *Tetrahedral Group.* We take a regular cube as in the figure: then $ABCD$ is a tetrahedron, $A'B'C'D'$ is the polar tetrahedron.

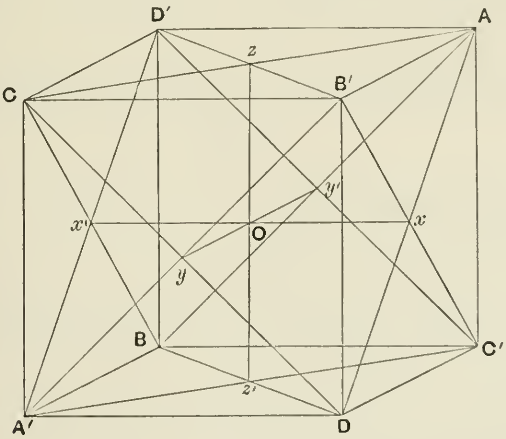

Fig. 119.

It is easy to see that the axes of rotation for the tetrahedron are

(i) the four diagonals of the cube AA', BB', CC', DD';

(ii) the three lines joining the middle points of the opposite edges of the tetrahedron.

The latter pass through the centre of the cube and are perpendicular to pairs of opposite faces. When the sphere circumscribing the cube is drawn, the three axes in (ii) intersect the sphere in six points which are the angles of a regular octahedron. Thus, though the axes of rotation for the three solids are not the same, the tetrahedron, the cube, and the octahedron may be considered together: in fact, in the present arrangement whereby the surface of the sphere is considered, the cube is merely the combination of the tetrahedron and its polar.

For each of the diagonals of the cube, the necessary angle of rotation for the tetrahedron is 0 or $\frac{2}{3}\pi$ or $\frac{4}{3}\pi$: the first of these gives identity, and the others give two rotations for each of the four diagonals of the cube, so that there are eight in all.

For each of the diagonals of the octahedron, the angle of rotation for the tetrahedron is π: there are thus three rotations.

With these we associate identity. Hence the number of rotations for the tetrahedron is $(8 + 3 + 1 =)$ 12 in all.

There are two sets of expressions for the tetrahedron according to the position of the coordinate axes of the sphere. One set arises when these are taken along Ox, Oy, Oz, the diagonals of the octahedron; the other arises, when a coordinate plane is made to coincide with a plane of symmetry of the tetrahedron such as $B'DBD'$.

Let the axes be the diagonals of the octahedron. The results are obtainable just as before, and so may now merely be stated:

For OB', $\xi = \eta = \zeta = \frac{1}{\sqrt{3}}$; when $\alpha = \frac{2}{3}\pi$, the substitution is

$$z' = \frac{z+i}{z-i},$$

and when $\alpha = \frac{4}{3}\pi$, the substitution is

$$z' = i\frac{z+1}{z-1}.$$

For OA, $\xi = -\eta = \zeta = \frac{1}{\sqrt{3}}$; when $\alpha = \frac{2}{3}\pi$, the substitution is

$$z' = -i\frac{z+1}{z-1},$$

and when $\alpha = \frac{4}{3}\pi$, the substitution is

$$z' = \frac{z-i}{z+i}.$$

For OC, $-\xi=\eta=\zeta=\dfrac{1}{\sqrt{3}}$; when $\alpha=\frac{2}{3}\pi$, the substitution is

$$z'=i\frac{z-1}{z+1},$$

and when $\alpha=\frac{4}{3}\pi$, the substitution is

$$z'=-\frac{z+i}{z-i}.$$

For OD', $-\xi=-\eta=\zeta=\dfrac{1}{\sqrt{3}}$; when $\alpha=\frac{2}{3}\pi$, the substitution is

$$z'=-\frac{z-i}{z+i},$$

and when $\alpha=\frac{4}{3}\pi$, the substitution is

$$z'=-i\frac{z-1}{z+1}.$$

For Ox, $\xi=1$, $\eta=0$, $\zeta=0$ and $\alpha=\pi$: the substitution is

$$z'=\frac{1}{z}.$$

For Oy, $\xi=0$, $\eta=1$, $\zeta=0$, and $\alpha=\pi$: the substitution is

$$z'=-\frac{1}{z}.$$

For Oz, $\xi=0$, $\eta=0$, $\zeta=1$ and $\alpha=\pi$: the substitution is

$$z'=-z.$$

And identity is $z'=z$.

Hence *the group of tetrahedral non-homogeneous substitutions is*

$$z'=\pm z,\quad \pm\frac{1}{z},\quad \pm i\frac{z-1}{z+1},\quad \pm i\frac{z+1}{z-1},\quad \pm\frac{z-i}{z+i},\quad \pm\frac{z+i}{z-i},$$

when the axes of reference in the sphere are the diameters bisecting opposite edges of the tetrahedron. Each of these substitutions gives rise to two homogeneous substitutions, making 24 *in all.*

To obtain the transformations in the case when the plane of xz is a plane of symmetry of the tetrahedron passing through one edge and bisecting the opposite edge, such as $B'DBD'$ in the figure, it is sufficient to rotate the preceding configuration through an angle $\frac{1}{4}\pi$ about the preceding Oz-axis, and then to construct the corresponding changes in the preceding formulæ.

For this rotation we have, with the preceding notation of § 299, $\xi=0=\eta$, $\zeta=1$, $\alpha=\frac{1}{4}\pi$: then $a=0=b$, $c=\sin\frac{1}{8}\pi$, $d=\cos\frac{1}{8}\pi$, so that $d\pm ic=e^{\pm\frac{1}{8}\pi i}$: and therefore the ζ' of the displaced point in the stereographic projection is connected with the ζ of the undisplaced point in the stereographic projection by the equation

$$\zeta'=\frac{d+ic}{d-ic}\zeta=\zeta e^{\frac{1}{4}\pi i}=\frac{1+i}{\sqrt{2}}\zeta.$$

If then Z be the variable of the projection of the undisplaced point and Z' that of the projection of displaced point with the present axes, and z and z' be the corresponding variables for the older axes, we have

$$Z = \frac{1+i}{\sqrt{2}} z, \qquad Z' = \frac{1+i}{\sqrt{2}} z',$$

that is,

$$z' = \frac{1-i}{\sqrt{2}} Z', \qquad z = \frac{1-i}{\sqrt{2}} Z.$$

Taking now the twelve substitutions in the form of the last set and substituting, we have *a group of tetrahedral non-homogeneous substitutions in the form*

$$Z' = \pm Z, \quad \pm\frac{i}{Z}, \quad \pm i\frac{Z\sqrt{2}-(1+i)}{(1-i)Z+\sqrt{2}}, \quad \pm i\frac{Z\sqrt{2}+(1+i)}{(1-i)Z-\sqrt{2}},$$

$$\pm i\frac{Z\sqrt{2}+(1-i)}{(1+i)Z-\sqrt{2}}, \quad \pm i\frac{Z\sqrt{2}-(1-i)}{(1+i)Z+\sqrt{2}},$$

when one of the coordinate planes is a plane through one edge of the tetrahedron bisecting the opposite edge: each of these gives rise to two homogeneous substitutions, making 24 *in all.*

301. The explanations, connected with these groups of substitutions, implied that certain aggregates of points remain unchanged by the operations corresponding to the substitutions. These aggregates are (i) the summits of the tetrahedron, (ii) the summits of the polar tetrahedron—these two sets together make up the summits of the cube: and (iii) the middle points of the edges, being also the middle points of the edges of the polar tetrahedron—this set forms the summits of an octahedron.

When these points are stereographically projected, we obtain aggregates of points which are unchanged by the substitutions. We therefore project stereographically with the extremity z of the axis Oz for origin of projection: and then the projections of x, x', y, y', z, z' are $1, -1, i, -i, \infty, 0$, which are the variables of these points.

Instead of taking factors $z-1$, $z+1$, ..., we shall take homogeneous forms $z_1 - z_2$, $z_1 + z_2$, $z_1 - iz_2$, $z_1 + iz_2$, z_2, z_1; the product of all these factors equated to zero gives the six points. This product is

$$t = z_1 z_2 (z_1^4 - z_2^4).$$

For the tetrahedron $ABCD$, the summits A, B, C, D are $\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$; $-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}$; $-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$; $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}$; respectively: and therefore the variables of the points in the stereographic projection are

$$\text{of } A,\ \frac{1-i}{\sqrt{3}-1};\ \text{of } B,\ \frac{-1-i}{\sqrt{3}+1};\ \text{of } C,\ \frac{-1+i}{\sqrt{3}-1};\ \text{of } D,\ \frac{1+i}{\sqrt{3}+1}.$$

Forming homogeneous factors as before, the product of the four equated to zero gives the stereographic projections of the four summits of the tetrahedron *ABCD*. This product is

$$\Psi = z_1^4 - 2\sqrt{-3}\, z_1^2 z_2^2 + z_2^4.$$

Similarly for the tetrahedron *A′B′C′D′*; the product of the factors corresponding to the stereographic projections of its four summits is

$$\Phi = z_1^4 + 2\sqrt{-3}\, z_1^2 z_2^2 + z_2^4.$$

And the product of the eight points for the cube is $\Phi\Psi$, that is,

$$W = z_1^8 + 14 z_1^4 z_2^4 + z_2^8.$$

All these forms t, Φ, Ψ are, by their mode of construction, unchanged (except as to a constant factor, which is unity in the present case) by the homogeneous substitutions: and therefore they are invariantive for the group of 24 linear homogeneous substitutions, derived from the group of 12 non-homogeneous tetrahedral substitutions. If Ψ be taken as a binary quartic, then Φ is its Hessian and t is its cubicovariant: the invariants are numerical and not algebraical: and the syzygy which subsists among the system of concomitants is

$$\Phi^3 - \Psi^3 = 12\sqrt{-3}\, t^2,$$

a relation easily obtained by reference merely to the expressions for the forms Φ, Ψ, t.

The object of this investigation is to form Z, the simplest rational function of z which is unaltered by the group of substitutions: for this purpose, it will evidently be necessary to form proper quotients of the foregoing homogeneous forms, of zero dimensions in z_1 and z_2. Let R be any rational function of z, which is unaltered by the tetrahedral substitutions. These substitutions give a series of values of z, for which Z has only one value: hence R and Z, being both functions of z and therefore of one another, are such that to a value of Z there is only one value of R, so that R is a rational function of Z.

In particular, the relation between R and Z may be lineo-linear: thus Z is determinate except as to linear transformations. This unessential indeterminateness can be removed, by assigning three particular conditions to determine the three constants of the linear transformation.

The number of substitutions in the z-group is 12: hence as there will thus be a group of 12 z-points interchanged by the substitutions, the simplest rational function of Z will be of the 12th degree in z, and therefore the numerator and the denominator of the fraction for Z, in their homogeneous forms, are of the 12th degree. The conditions assigned will be

(i) Z must vanish at the summits of the given tetrahedron:

(ii) Z must be infinite at the summits of the polar tetrahedron:

(iii) Z must be unity at the middle points of the sides.

Then Z, being a fractional function with its numerator and its denominator each of the 12th degree and composed of the functions Φ, Ψ, t, must, with the foregoing conditions, be given by

$$Z = \frac{\Psi^3}{\Phi^3}:$$

by means of the syzygy, we have

$$Z : Z - 1 : 1 = \Psi^3 : -12\sqrt{-3}t^2 : \Phi^3,$$

which is Klein's result. Removing the homogeneous variables, we have

$$Z : Z - 1 : 1 = (z^4 - 2\sqrt{-3}z^2 + 1)^3 : -12\sqrt{-3}z^2(z^4 - 1)^2 : (z^4 + 2\sqrt{-3}z^2 + 1)^3;$$

and then Z is a function of z which is unaltered by the group of 12 tetrahedral substitutions of p. 627. And every such function is a rational function of Z.

This is one form of the result, depending upon the first position of the axes: for the alternate form it is necessary merely to turn the axes through an angle of $\frac{1}{4}\pi$ round the z-axis, as was done in § 300 to obtain the new groups. The result is that a function Z, unaltered by the group of 12 substitutions of p. 628, is given by

$$Z : Z - 1 : 1 = (z^4 - 2\sqrt{3}z^2 - 1)^3 : -12\sqrt{3}z^2(z^4 + 1)^2 : (z^4 + 2\sqrt{3}z^2 - 1)^3.$$

It still is of importance to mark out the partition of the plane corresponding to the groups, in the same manner as was done in the case of the infinite groups in the preceding chapter. This partition of the plane is the stereographic projection of the partition of the sphere, a partition effected by the planes of symmetry of the tetrahedron. Some idea of the division may be gathered from the accompanying figure, which is merely a projection on the circumscribing sphere from the centre of the cube. The great circles

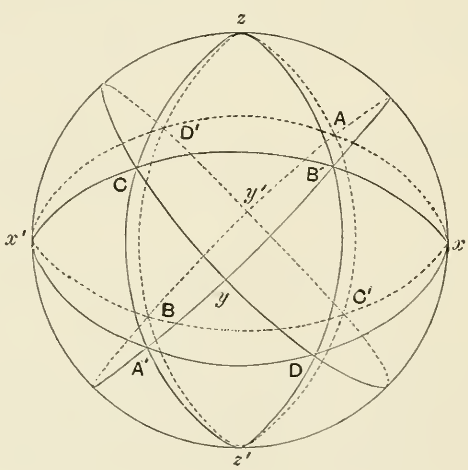

Fig. 120.

meet by threes in the summits of the tetrahedron and its polar, being the sections by the three planes of symmetry, which pass through every such summit, and the circles are equally inclined to one another there: they meet by twos in the middle points of the edges and they are equally inclined to one another there. They divide the sphere into 24 triangles, each of which has for angles $\frac{1}{2}\pi$, $\frac{1}{3}\pi$, $\frac{1}{3}\pi$. (See Case II., § 278.)

The corresponding division of the plane is the stereographic projection of this divided surface. Taking A as the pole of projection, which is projected

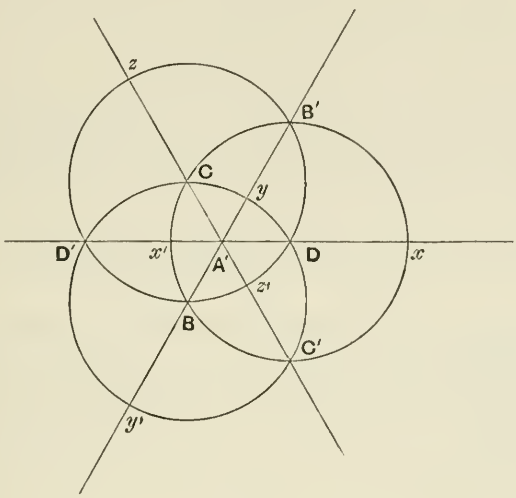

Fig. 121.

to infinity, then A' is the origin: the three great circles through A' become three straight lines equally inclined to one another; the other three great circles become three circles with their centres on the three lines concurrent in the origin. The accompanying figure shews the projection: the points in the plane have the same letters as the points on the sphere of which they are the projections: and the plane is thus divided into 24 parts. There are, in explicit form, only 12 non-homogeneous substitutions: but each of these has been proved to imply two homogeneous substitutions, so that we have the division of the plane corresponding to the 24 substitutions in the group. The fundamental polygon of reference is a triangle such as $CA'x'$.

302. It now remains to construct the function for the dihedral group. The sets of points to be considered are:—

(i) the angular points of the polygon: in the stereographic projection, these are

$$e^{\frac{2\pi si}{n}}, \text{ for } s=0, 1, \ldots, n-1;$$

(ii) the middle points of the sides: in the stereographic projection, these are

$$e^{\frac{\pi i(2s+1)}{n}}, \text{ for } s = 0, 1, \ldots, n-1; \text{ and}$$

(iii) the poles of the equator which are unaltered by each of the rotations: in the stereographic projection, these are 0 and ∞.

Forming the homogeneous products, as for the tetrahedron, we have, for (i),

$$U = z_1^n - z_2^n;$$

for (ii),

$$V = z_1^n + z_2^n;$$

and, for (iii),

$$W = z_1 z_2;$$

these functions being connected by a relation

$$-U^2 + V^2 = 4W^n.$$

Because the dihedral group contains $2n$ non-homogeneous substitutions, the rational function of z, say Z, must, in its initial fractional form, be of degree $2n$ in both numerator and denominator; and it must be constructed from U, V, W.

The function Z becomes fully determinate, if we assign to it the following conditions:

(i) Z must vanish at points corresponding to the summits of the polygon,

(ii) Z must be infinite at points corresponding to the poles of the equator,

(iii) Z must be unity at points corresponding to the middle points of the edges:

and then we find

$$Z : Z-1 : 1 = \{\tfrac{1}{2}(z^n - 1)\}^2 : \{\tfrac{1}{2}(z^n + 1)\}^2 : -z^n,$$

which gives the simplest rational function of z that is unaltered by the substitutions of the dihedral group.

The discussion of the polyhedral functions will not be carried further here: sufficient illustration has been provided as an introduction to the theory which, in its various bearings, is expounded in Klein's suggestive treatise already quoted.

Ex. 1. Shew that the anharmonic group of § 298 is substantially the dihedral group for $n=3$; and, by changing the axes, complete the identification. (Klein.)

Ex. 2. An octahedron is referred to its diagonals as axes of reference, and a partition of the surface of the sphere is made with reference to planes of symmetry and the axes of rotations whereby the figure is made to coincide with itself.

Shew that the number of these rotations is 24, that the sphere is divided into 48 triangles, that the non-homogeneous substitutions which transform into one another the partitions of the plane obtained from a stereographic projection are

$$z' = i^k z, \quad \frac{i^k}{z}, \quad i^k\frac{z+1}{z-1}, \quad i^k\frac{z-1}{z+1}, \quad i^k\frac{z-i}{z+i}, \quad i^k\frac{z+i}{z-i},$$

where $k=0, 1, 2, 3$; and that the corresponding octahedral function is

$$Z : Z-1 : 1 = (z^8 + 14z^4 + 1)^3 : (z^{12} - 33z^8 - 33z^4 + 1)^2 : 108z^4(z^4-1)^4. \quad \text{(Klein.)}$$

303. We now pass from groups that are finite in number to the consideration of functions connected with groups that are infinite in number. The best known illustration is that of the elliptic modular-functions; one example is the form of the modulus in an elliptic integral as a function of the ratio of the periods of the integral. The general definition of a *modular-function** is that it is a uniform function such that an algebraical equation subsists between $\psi\left(\frac{\alpha w+\beta}{\gamma w+\delta}\right)$ and $\psi(w)$, where α, β, γ, δ are integers subject to the relation $\alpha\delta-\beta\gamma=1$. The simplest case is that in which the two functions ψ are equal.

The elliptic quarter-periods K and iK' are defined by the integrals

$$2K=\int_0^1\{z(1-z)(1-k^2z)\}^{-\frac{1}{2}}dz=\int_0^1\{z(1-z)(1-cz)\}^{-\frac{1}{2}}dz,$$

$$2K'=\int_0^1\{z(1-z)(1-k'^2z)\}^{-\frac{1}{2}}dz=\int_0^1\{z(1-z)(1-c'z)\}^{-\frac{1}{2}}dz,$$

where $c+c'=1$. The ordinary theory of elliptic functions gives the equation

$$K\frac{dK'}{dc}-K'\frac{dK}{dc}=-\frac{\pi}{4cc'},$$

whatever be the value of c. To consider the nature of these quantities as functions of c, we note that $c=1$ is an infinity of K and an ordinary point of K', and that $c=0$ is a similar infinity of K' and an ordinary point of K: and these are all the singular points in the finite part of the plane. The value $c=\infty$ must also be considered. All other values of c are ordinary points for K and K'.

For values of c, such that $|c|<1$, we have

$$\frac{2K}{\pi}=1+\left(\frac{1}{2}\right)^2c+\left(\frac{1\,.\,3}{2\,.\,4}\right)^2c^2+\ldots,$$

so that, in the vicinity of the origin,

$$\frac{d}{dc}\left(\frac{K'}{K}\right)=-\frac{\pi}{4K^2cc'}$$

$$=-\frac{1}{\pi}\left\{\frac{1}{c}+\frac{1}{2}+\text{positive integral powers of } c\right\}.$$

Hence in the vicinity of the origin

$$\frac{K'}{K}=-\frac{1}{\pi}\log c+P(c),$$

where $P(c)$ is a uniform series converging for sufficiently small values of $|c|$: and therefore, still in the vicinity of the origin,

$$K'=-\frac{K}{\pi}\log c+KP(c).$$

* This is the definition of a modular-function which is adopted by Hermite, Dedekind, Klein, Weber and others.

Now let the modulus c describe a contour round the origin and return to its original value. Then K is unchanged, for the c-origin is not a singularity of K.

The new value of K' is evidently

$$-\frac{K}{\pi}(2\pi i + \log c) + KP(c),$$

that is, iK' changes into $2K + iK'$. Hence, *when c describes positively a small contour round the origin, the quarter-periods K and iK' become K and $2K + iK'$ respectively.*

In the same way from the equation

$$K'\frac{dK}{dc'} - K\frac{dK'}{dc'} = -\frac{\pi}{4cc'},$$

and from the expansion of K' in powers of c' when $|c'| < 1$, we infer that *when c' describes positively a small contour round its origin, that is, when c describes positively a small contour round the point $c = 1$, then iK' is unchanged and K changes to $K - 2iK'$.*

It thus appears that the quantities K and iK', regarded as functions of the elliptic modulus c, are subject to the linear transformations

$$\left.\begin{aligned} U(K) &= K \\ U(iK') &= 2K + iK' \end{aligned}\right\}, \qquad \left.\begin{aligned} V(K) &= K - 2iK' \\ V(iK') &= iK' \end{aligned}\right\},$$

without change of the quantity c; and the application of either substitution is equivalent to making c describe a closed circuit round one or other of the critical points in the finite part of the plane, the description being positive if the direct substitution be applied and negative if the inverse be applied.

When these substitutions are applied any number of times—the index being the same and composed in the same way for K as for iK'—then, denoting the composite substitution by P, we have results of the form

$$\left.\begin{aligned} PK &= \delta K + \gamma iK' \\ PiK' &= \beta K + \alpha iK' \end{aligned}\right\},$$

where β and γ are even integers, α and δ are odd integers of the forms $1 + 4p$, $1 + 4q$, say $\equiv 1$ (mod. 4), and, because the determinant of U and that of V are both unity, we have $\alpha\delta - \beta\gamma = 1$ by § 282. These equations give the partially indeterminate form of the values of the quarter-periods for an assigned value of the modulus c.

Conversely, we may regard c as a function of $w = \dfrac{iK'}{K}$, the quotient of the quarter-periods. The quotient is taken, for various reasons: thus it enables us to remove common factors, it is the natural form in the passage to q-series, and so on. The function is unaltered, when w is subjected to the infinite group of substitutions derived from the fundamental substitutions

$$Uw = w + 2, \qquad Vw = \frac{w}{1 - 2w}.$$

Denoting the function c by $\phi(w)$, we have

$$c = \phi(w) = \phi(w+2) = \phi\left(\frac{w}{1-2w}\right).$$

We have still to take account of the relation of iK'/K to c, when the latter has infinitely large values. For this purpose, we compare the differential expressions

$$k\{x(1-x)(1-k^2x)\}^{-\frac{1}{2}}\,dx, \quad \{y(1-y)(1-l^2y)\}^{-\frac{1}{2}}\,dy,$$

which are equal to one another if $k^2x = y$ and $kl = 1$. As x moves from 0 to 1, y moves from 0 to k^2, that is, from 0 to $1/l^2$; integrating between these limits, we have

$$kK = \Lambda + i\Lambda',$$

where Λ and Λ' are quarter-periods with modulus $l = 1/k$. As y moves from 0 to 1, x moves from 0 to $1/k^2$; integrating between these limits, we have

$$k(K + iK') = \Lambda,$$

so that

$$kiK' = -i\Lambda'.$$

In order to obtain the effect on K and iK' of an infinitely large circuit described positively by c, we make l describe a very small circuit round its origin negatively. By what has been proved, the effect of the latter is to change Λ and $i\Lambda'$ into Λ and $i\Lambda' - 2\Lambda$ respectively. Hence the new value of kiK' is

$$-i\Lambda' + 2\Lambda = k(3iK' + 2K);$$

and the new value of kK is

$$\Lambda + i\Lambda' - 2\Lambda = -k(2iK' + K).$$

Hence if w' denote the new value of w, consequent on the description of the infinitely large circuit by c, we have

$$w' = -\frac{3w+2}{2w+1} = U^{-1}V^{-1}w.$$

No new fundamental substitution is thus obtained; and therefore U, V are the only fundamental substitutions of the group for c, regarded as a modular-function.

Again, c' is a rational function of c and is therefore a modular-function: consequently also cc' is a modular-function. Being a rational function of c, it is subject to the two substitutions U and V, which are characteristically fundamental for $\phi(w)$. Now cc' is unchanged when we interchange c and c', that is, when we interchange K and K'; so that, if K_1 and iK_1' be new quarter-periods for a modulus cc', we have

$$K_1 = K', \qquad iK_1' = iK,$$

and therefore

$$w_1 = -\frac{1}{w}.$$

Thus cc' as a modular-function must be subject to the substitution

$$Tw = -\frac{1}{w}.$$

But $$TUTw = -\frac{1}{UTw} = -\frac{1}{2+Tw} = \frac{w}{1-2w} = Vw,$$

so that V is compounded of T and U. Hence the substitutions for cc', regarded as a modular-function, are the infinite group, derived from the fundamental substitutions

$$Uw = w + 2, \qquad Tw = -\frac{1}{w}.$$

Denoting the modular-function cc' by $\chi(w)$, we have

$$cc' = \chi(w) = \chi(w+2) = \chi\left(-\frac{1}{w}\right).$$

To obtain the change in w caused by changing c into c/c', we use the differential expression

$$\{y(1-y)(1-l^2y)\}^{-\frac{1}{2}}dy.$$

When the variable is transformed by the equation* $(1-y)(1-k^2x) = 1-x$, where $k'^2l^2 = -k^2$, the expression becomes

$$k'\{x(1-x)(1-k^2x)\}^{-\frac{1}{2}}dx.$$

When y describes the straight line from 0 to 1 continuously, x also describes the straight line from 0 to 1 continuously. Integrating between these limits, we have

$$\Lambda = k'K,$$

where Λ is a quarter-period. When y describes the straight line from 0 to $1/l$ continuously, x describes the straight line from 0 to ∞ continuously or, say, the line from 0 to $1/k^2$ and the line from $1/k^2$ to ∞ continuously. Integrating between these limits, we have

$$\begin{aligned}\Lambda + i\Lambda' &= k'(K+iK') + \tfrac{1}{2}k'\int_{\frac{1}{k^2}}^{\infty}\{x(1-x)(1-k^2x)\}^{-\frac{1}{2}}dx\\ &= k'(K+iK') + k'K,\end{aligned}$$

on using the transformation $k^2xu = 1$ and taking account of the path described by the variable u: and therefore

$$i\Lambda' = k'(K+iK').$$

Hence the change of modulus from k to ik/k', which changes c to $-c/c'$, gives the changes of quarter-periods in the form

$$\Lambda = k'K, \quad i\Lambda' = k'(K+iK');$$

and therefore the new value of w, say w_1, is

$$w_1 = w + 1 = Sw.$$

It therefore follows that, when $c - c/c'$ is regarded as a modular function of the quotient w of the quarter-periods K and iK', it must be subject to the substitutions

$$S(w) = w+1, \qquad U(w) = w+2, \qquad V(w) = \frac{w}{1-2w}.$$

* This is the equation expressing elliptic functions of $k'u$ in terms of elliptic functions of u.

Evidently $S^2 = U$, and U may therefore be omitted; V and S are the fundamental substitutions of the infinite group of transformations of w, the argument of the modular-function c^2/c'.

As a last example, we consider the function

$$J = \frac{(c^2 - c + 1)^3}{(c^2 - c)^2}.$$

It is a rational function of cc', and therefore is a modular-function having the substitutions Tw and Uw. By § 298, it is unaltered when we substitute $\frac{c}{c-1}$ for c. It has just been proved that this change causes a change of w into $w+1$, and therefore J, as a modular-function, must be subject to the substitution

$$Sw = w + 1.$$

Evidently $S^2 w = w + 2 = Uw$, so that U is no longer a fundamental substitution when S is retained. Hence we have the result that J is unaltered, when w is subjected to the infinite group of substitutions derived from the fundamental substitutions

$$Sw = w + 1, \qquad Tw = -\frac{1}{w},$$

so that we may write

$$J = \frac{(c^2 - c + 1)^3}{c^2 (c-1)^2} = J(w) = J(w+1) = J\left(-\frac{1}{w}\right).$$

This is the group of substitutions considered in § 284: they are of the form $\frac{\alpha w + \beta}{\gamma w + \delta}$, where $\alpha, \beta, \gamma, \delta$ are real integers subject to the single relation $\alpha\delta - \beta\gamma = 1$.

These illustrations, in connection with which the example in § 298 should be consulted, suffice to put in evidence the existence of modular-functions, that is, functions periodic for infinite groups of linear substitutions, the coefficients of which are real integers. The theory has been the subject of many investigations, both in connection with the modular equations in the transformation of elliptic functions and also as a definite set of functions. The investigations are due among others to Hermite, Fuchs, Dedekind, Hurwitz and especially to Klein*; and reference must be made to their memoirs, or to Klein-Fricke's treatise on elliptic modular-functions, or to Weber's *Elliptische Functionen*, for an exposition of the theory.

304. The method just adopted for infinite groups is very special, being suited only to particular classes of functions: in passing now to linear substitutions, no longer limited by the condition that their coefficients are real integers, we shall adopt more general considerations. The chief purpose of the investigation will be to obtain expressions of functions characterised by the property of reproduction when their argument is subjected to any one of the infinite group of substitutions.

* Some references are given in Enneper's *Elliptische Functionen*, (2[te] Aufl.), p. 482.

The infinite group is supposed of the nature of that in § 290: the members of it, being of the form

$$\left(z, \frac{\alpha_i z + \beta_i}{\gamma_i z + \delta_i}\right), \quad \text{or} \quad (z, f_i(z)),$$

are such that a circle, called the fundamental circle, is unaltered by any of the substitutions. This circle is supposed to have its centre at the origin and unity for its radius.

The interior of the circle is divided into an infinite number of curvilinear polygons, congruent by the substitutions of the group: each polygon contains one, and only one, of the points in the interior associated by the substitutions with a given point not on the boundary of the polygon. Hence corresponding to any point within the circle, there is one and only one point within the fundamental polygon, as there is only one such point in each of the polygons: of these homologous points the one, which lies in the fundamental polygon of reference, will be called the *irreducible* point. It is convenient to speak of the zero of a function, implying thereby the irreducible zero: and similarly for the singularities.

The part of the plane, exterior to the fundamental circle, is similarly divided: and the division can be obtained from that of the internal area by inversion with regard to the circumference and the centre of the fundamental circle. Hence there will be two polygons of reference, one in the part of the plane within the circle and the other in the part without the circle: and all terms used for the one can evidently be used for the other. Thus the irreducible homologue of a point without the circle is in the outer polygon of reference: for a substitution transforms a point within an internal polygon to a point within another internal polygon, and a point within an external polygon to a point within another external polygon.

Take a point z in the interior of the circle, and round it describe a small contour (say for convenience a circle) so as not to cross the boundary of the polygon within which z lies: and let z_i be the point given by the substitution $f_i(z)$. Then corresponding to this contour there is, in each of the internal polygons a contour which does not cross the boundary of its polygon: and as the first contour (say C_0) does not occupy the whole of its polygon and as the congruent contours do not intersect, the sum of the areas of all the contours C_i is less than the sum of the areas of all the polygons, that is, the sum is less than the area of the circle and so it is finite.

If μ_i be the linear magnification at z_i, we have

$$\mu_i = \frac{1}{|\gamma_i z + \delta_i|^2} = \left|\frac{dz_i}{dz}\right|,$$

and therefore, if m_i be the least value of the magnification for points lying within C_0, we have

$$C_i > m_i^2 C_0.$$

The point $-\dfrac{\delta_i}{\gamma_i}$ is the homologue of $z=\infty$ by the substitution $\left(z, \dfrac{\alpha_i z+\beta_i}{\gamma_i z+\delta_i}\right)$, and therefore $-\delta_i/\gamma_i$ lies without the circle: though, in the limit of i infinite, it may approach indefinitely near to the circumference*.

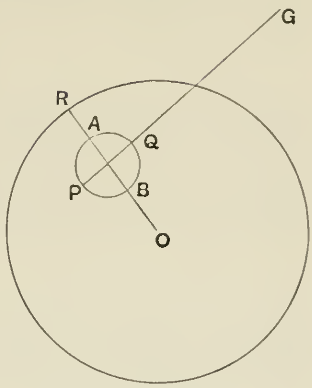

Fig. 122.

Let this point be G: and through G and O, the centre of the fundamental circle, draw straight lines passing through the centre of the circular contour. Then evidently

$$m_i = |\gamma_i|^{-2}\frac{1}{GP^2};$$

and, if M_i be the greatest magnification, then

$$M_i = |\gamma_i|^{-2}\frac{1}{GQ^2},$$

so that

$$\frac{M_i}{m_i} = \frac{GP^2}{GQ^2}.$$

Now G is certainly not inside the circle, so that GQ is not less than RA: thus

$$\frac{GP}{GQ} = 1+\frac{PQ}{GQ} = 1+\frac{AB}{GQ} < 1+\frac{AB}{RA} < \frac{RB}{RA},$$

which is independent of the point G, that is, of the particular substitution $f_i(z)$. Denoting $\left(\dfrac{RB}{RA}\right)^2$ by K, we have

$$\frac{M_i}{m_i} < K,$$

or

$$M_i < Km_i.$$

Evidently μ_i is finite.

Now

$$\frac{1}{|\gamma_i z+\delta_i|^2} = \mu_i < M_i < Km_i,$$

and therefore

$$\frac{1}{|\gamma_i z+\delta_i|^4} < K^2 m_i^2 < \frac{K^2}{C_0}C_i,$$

so that

$$\sum_{i=0}^{\infty} |\gamma_i z+\delta_i|^{-4} < \frac{K^2}{C_0}\sum_{i=0}^{\infty} C_i.$$

* For, in § 284, when the coefficients are real, a point associated with a given point may, for $i=\infty$, approach indefinitely near to a point on the axis of x: and then, by the transformation of § 290, we have the result in the text.

It has been seen that $\sum_{i=0}^{\infty} C_i$ is less than the area of the fundamental circle and is therefore finite: hence the quantity

$$\sum_{i=0}^{\infty} |\gamma_i z + \delta_i|^{-4}$$

is finite. It therefore follows that $\sum_{i=0}^{\infty} \mu_i^2$ is an absolutely converging series. Similarly, it follows that $\sum_{i=0}^{\infty} \mu_i^m$ is an absolutely converging series for all values of m that are greater than unity*. This series is evidently

$$\sum_{i=0}^{\infty} |\gamma_i z + \delta_i|^{-2m},$$

and the absolute convergence is established on the assumption that z lies within the fundamental circle.

Next, let z lie without the fundamental circle. If z coincide with some one of the points $-\delta_i/\gamma_i$, then the corresponding term of the series

$$\sum_{i=0}^{\infty} |\gamma_i z + \delta_i|^{-2m}$$

is infinite.

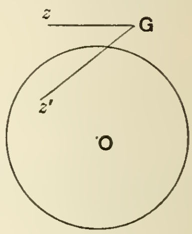

Fig. 123.

If it do not coincide with any one of the points $-\delta_i/\gamma_i$, let c be its distance from the nearest of them, so that

$$|\gamma_i z + \delta_i|^{-2m} < |\gamma_i|^{-2m} c^{-2m}.$$

Let z' be any point within the fundamental circle: then

$$|\gamma_i z' + \delta_i|^{-2m} = (Gz')^{-2m} |\gamma_i|^{-2m}.$$

Now $Gz' < 1 + OG < 1 + \left|\frac{\delta_i}{\gamma_i}\right|$, for any point within the circle, so that

$$|\gamma_i z' + \delta_i|^{-2m} > |\gamma_i|^{-2m} \left\{1 + \left|\frac{\delta_i}{\gamma_i}\right|\right\}^{-2m}.$$

Hence
$$\frac{|\gamma_i z + \delta_i|^{-2m}}{|\gamma_i z' + \delta_i|^{-2m}} < \left[\frac{1}{c}\left\{1 + \left|\frac{\delta_i}{\gamma_i}\right|\right\}\right]^{2m}.$$

Only a limited number of the points $-\delta_i/\gamma_i$ can be at infinity. Each of the corresponding substitutions gives the point at infinity as the homologue of $-\delta_i/\gamma_i$; and therefore, inverting with regard to the fundamental circle, we have a number of homologues of the origin coinciding with the origin, equal to the number of the points $-\delta_i/\gamma_i$ at infinity. The origin is not a singularity of the group, so that the number of homologues of the origin, coincident with it, must be limited.

* A completely general inference as to the convergence of the series, when $m=1$, cannot be made: the convergence depends upon the form of the division of the plane into polygons, and Burnside (l.c., p. 620) has proved that there is certainly one case in which $\sum_{i=0}^{\infty} \mu_i$ is an absolutely converging series.

Omitting the corresponding terms from the series, an omission which does not affect its convergence, we can assign a superior limit to $\left|\frac{\delta_i}{\gamma_i}\right|$: let it be $C-1$. Then

$$\frac{|\gamma_i z + \delta_i|^{-2m}}{|\gamma_i z' + \delta_i|^{-2m}} < \left(\frac{C}{c}\right)^{2m}.$$

Thus
$$\sum_{i=0}^{\infty} |\gamma_i z + \delta_i|^{-2m} < \left(\frac{C}{c}\right)^{2m} \sum_{i=0}^{\infty} |\gamma_i z' + \delta_i|^{-2m},$$

which is a finite quantity by the preceding investigation, for z' is a point within the circle.

Lastly, let z lie on the fundamental circle. If it coincide with one of the essential singularities of the group, then there is an infinite number of points $-\delta_i/\gamma_i$ which coincide with it: and so there will be an infinite number of terms in the series infinite in value. If it do not coincide with any of the essential singularities of the group, then there is a finite (it may be small, but it is not infinitesimal) limit to its distance from the nearest of the points $-\delta_i/\gamma_i$: the preceding analysis is applicable, and the series converges.

Hence, summing up our results, we have:—

The series
$$\sum_{i=0}^{\infty} |\gamma_i z + \delta_i|^{-2m}$$

is an absolutely converging series for any point in the plane, which is not coincident with any one of the points $-\delta_i/\gamma_i$ *(which all lie without the fundamental circle) or with any one of the essential singularities of the assigned group (which all lie on the circumference of the fundamental circle)**.

305. Let $H(z)$ denote a rational function of z, having a number of accidental singularities $a_1, \dots, a_p$, no one of which lies on the fundamental circle; and let it have no other singularities. Consider the series

$$\Theta(z) = \sum_{i=0}^{\infty} (\gamma_i z + \delta_i)^{-2m} H\left(\frac{\alpha_i z + \beta_i}{\gamma_i z + \delta_i}\right),$$

the group being the same as above. If z do not coincide with any of the points $a_1, \dots, a_p$, or with any of the points homologous with $a_1, \dots, a_p$ by the substitutions of the group, there is a maximum value, say M, for the modulus of H with any of the arguments $\dfrac{\alpha_i z + \beta_i}{\gamma_i z + \delta_i}$. Then

$$|\Theta(z)| < M \sum_{i=0}^{\infty} |\gamma_i z + \delta_i|^{-2m},$$

* The coefficients α, β, γ, δ of the substitutions of the group depend upon the coefficients of the fundamental substitutions, which may be regarded as parameters, arbitrary within limits. The series is proved by Poincaré to be a continuous function of these parameters, as well as of the variable z: this proposition, however, belongs to the development of the theory and can be omitted here as we do not propose to establish the general existence of all the functions.

and the right-hand side is finite, if in addition z do not coincide with any of the points $-\delta_i/\gamma_i$ or with any of the essential singularities of the group. Hence $\Theta(z)$ is an absolutely converging series for any value of z in the plane which does not coincide with (i) an accidental singularity of $H(z)$, or one of the points homologous with these singularities by the substitutions of the group, or with (ii) any of the points $-\delta_i/\gamma_i$, which are the various points homologous with $z=\infty$ by the substitutions of the group, or with (iii) any of the essential singularities of the group, which are points lying on the fundamental circle.

All these points are singularities of $\Theta(z)$.

If z coincide with $f_k(a)$ and if $f_i\{f_k(z)\}=z$, then the term $H\left(\frac{\alpha_i z+\beta_i}{\gamma_i z+\delta_i}\right)$ is infinite, the point being an accidental singularity of $H\left(\frac{\alpha_i z+\beta_i}{\gamma_i z+\delta_i}\right)$. The rest of the series is then of the same nature as $\Theta(z)$ in the more general case, and therefore converges. Hence the point is an accidental singularity of the function $\Theta(z)$ of the same order as for H, that is, the series of points, given by the accidental singularities of $H(z)$ and by the points homologous with them through the substitutions of the group, are accidental singularities of the function $\Theta(z)$.

In the same way it is easy to see that the points $-\delta_i/\gamma_i$ are either ordinary points or accidental singularities of $\Theta(z)$; and that the essential singularities of the group are essential singularities of $\Theta(z)$. Hence we have the result:—

The series $$\Theta(z)=\sum_{i=0}^{\infty}(\gamma_i z+\delta_i)^{-2m}H\left(\frac{\alpha_i z+\beta_i}{\gamma_i z+\delta_i}\right),$$

where the summation extends over the infinite number of members of an assigned discontinuous group, is a function of z, provided the integer m be >1 and $H(z)$ be a rational function of z. The singularities of Θ are:—

(i), *the accidental singularities of $H(z)$ and the points homologous with them by the substitutions of the group: all these points are accidental singularities of $\Theta(z)$;*

(ii), *the points $-\delta_i/\gamma_i$, which are the points homologous with $z=\infty$ by the substitutions of the group: all these points, if not ordinary points of $\Theta(z)$, are accidental singularities; and*

(iii), *the essential singularities of the group: these lie on the fundamental circle and they are essential singularities of $\Theta(z)$.*

If $H(z)$ had any essential singularity, then that point and all points homologous with it by substitutions of the group would be essential singularities of $\Theta(z)$. The function $\Theta(z)$, thus defined, is called* Thetafuchsian by Poincaré.

* *Acta Math.*, t. i, p. 210.

If the group belong to the first, the second or the sixth family, it is known that the circumference of the fundamental circle enters into the division of the interior of the circle (and also of the space exterior to the circle) only in so far as it contains the essential singularities of the group. But if the group belong to any one of the other four families, then parts of the circumference enter into the division of both spaces.

In the former case, when the group belongs to the set of families, made up of the first, the second, and the sixth, the circumference of the fundamental circle is a line over which the series cannot be continued: it is a *natural limit* (§ 81) both for a function existing in the interior of the circle and for a function existing in the exterior of the circle: but neither function exists for points on the circumference of the fundamental circle. The series represents one function within the circle and another function without the circle.

It has been proved that the area outside the fundamental circle can be derived from the area inside that circle, by inversion with regard to its circumference. Hence a function of z, existing only outside the fundamental circle, can be transformed into a function of $\frac{1}{z_0}$, and therefore also of $\frac{1}{z}$, existing for points only within the circle. *When, therefore, a group belongs to the first, the second or the sixth family, it is sufficient to consider only the function defined by the series for points within the fundamental circle:* it will be called the function $\Theta(z)$.

In the latter case, when the group belongs to the third, the fourth, the fifth or the seventh families, then parts of the circumference enter into the division of the plane both without and within the circle. Over these parts the function can be continued: and then *the series represents one (and only one) function in the two parts of the plane:* it will be called the function $\Theta(z)$.

306. The importance of the function $\Theta(z)$ lies in its pseudo-automorphic character for the substitutions of the group, as defined by the property now to be proved that, *if* $\frac{\alpha z+\beta}{\gamma z+\delta}$ *be any one of the substitutions of the group, then*

$$\Theta\left(\frac{\alpha z+\beta}{\gamma z+\delta}\right)=(\gamma z+\delta)^{2m}\,\Theta(z).$$

Let

$$\frac{\alpha_i\dfrac{\alpha z+\beta}{\gamma z+\delta}+\beta_i}{\gamma_i\dfrac{\alpha z+\beta}{\gamma z+\delta}+\delta_i}=\frac{\alpha_i' z+\beta_i'}{\gamma_i' z+\delta_i'},$$

which is, of course, another substitution of the infinite group: then

$$\gamma_i\frac{\alpha z+\beta}{\gamma z+\delta}+\delta_i=\frac{\gamma_i' z+\delta_i'}{\gamma z+\delta}.$$

Hence
$$\Theta\left(\frac{\alpha z+\beta}{\gamma z+\delta}\right)=\sum_{i=0}^{\infty}\left(\frac{\gamma_i' z+\delta_i'}{\gamma z+\delta}\right)^{-2m} H\left(\frac{\alpha_i' z+\beta_i'}{\gamma_i' z+\delta_i'}\right)$$
$$=(\gamma z+\delta)^{2m}\sum_{i=0}^{\infty}(\gamma_i' z+\delta_i')^{-2m} H\left(\frac{\alpha_i' z+\beta_i'}{\gamma_i' z+\delta_i'}\right)$$
$$=(\gamma z+\delta)^{2m}\,\Theta(z),$$
thus establishing the pseudo-automorphic character.

This function can evidently be made subsidiary to the construction of functions, which are automorphic for the group of substitutions, in the same manner as the σ-function in Weierstrass's theory of elliptic functions and the so-called Theta-functions in the theory of Jacobian and of Abelian transcendents. But before we consider these automorphic functions, it is important to consider the zeros and the accidental singularities of a pseudo-automorphic function such as $\Theta(z)$.

On the supposition that the function H, which enters as the additive element into the composition of Θ, has only accidental singularities, it has been proved that all the essential singularities of Θ lie on the circumference of the fundamental circle; and that the accidental singularities of Θ are, (i) the points homologous with the accidental singularities of H, and (ii) the points $-\delta_i/\gamma_i$, which all lie without the circle.

When the function $H(z)$ has one or more accidental singularities within the fundamental circle, then there is an irreducible point for each of them, which is an irreducible accidental singularity of $\Theta(z)$. Hence *in the case of a function which exists only within the circle, the number of irreducible accidental singularities is the same as the number of (non-homologous) accidental singularities of $H(z)$ lying within the fundamental circle.* If, then, all the infinities of the additive element $H(z)$ lie without the fundamental circle, and if the function $\Theta(z)$ exist only within the circle, then $\Theta(z)$ has no irreducible accidental singularities: but, in particular cases, it may happen that $\Theta(z)$ is then evanescent.

When the function $H(z)$ has one or more accidental singularities without the fundamental circle, then there is an irreducible point for each of them, this point lying in the fundamental polygon of reference in the space outside the circle: and this point is an irreducible accidental singularity of $\Theta(z)$, when $\Theta(z)$ exists both within and without the circle. Further, the point $-\delta_i/\gamma_i$ is an infinity of order $2m$: there is a homologous irreducible point within the polygon of reference without the circle, being, in fact, the irreducible point which is homologous with $z=\infty$. Hence taking the two fundamental polygons of reference—one within, for the internal division, and one without, for the external division,—it follows that *in the case of a function, which exists all over the plane, the number of irreducible accidental singularities* *is equal to the whole number of accidental singularities of the additive element* $H(z)$, *increased by* $2m$.

307. To obtain the number of irreducible zeros we use the result of § 43, Cor. IV., combined with the result just obtained as to the number of irreducible accidental singularities. A convention, similar to that adopted in the case of the doubly-periodic functions (§ 115), is now necessary: for if there be a zero on one side of the fundamental polygon, then the homologous point on the conjugate side of the polygon is also a zero and of the same degree: in that case, either we take both points as irreducible zeros and of half the degree, or we take one of them as the irreducible zero and retain its proper degree. Similarly, if a corner be a zero, every corner of the cycle is a zero: so that, if the cycle contain λ points and the sum of its angles be $\frac{2\pi}{\mu}$, then the corner is common to $\lambda\mu$ polygons; we may regard each of the corners of the fundamental polygon in that cycle as an irreducible zero, of degree equal to its proper degree divided by $\lambda\mu$, or we may take only one of them and count its degree as the proper degree divided by μ—the just distribution of zeros common to contiguous polygons being all that is necessary for the convention—so that the number of zeros to be associated with the area of each polygon is the same, while no zero is counted in more than its proper degree. A similar convention applies to the singularities.

With this convention, the excess of the number of irreducible zeros over the number of irreducible accidental singularities, each in its proper degree, is the value of

$$\frac{1}{2\pi i}\int\frac{\Theta'(z)}{\Theta(z)}dz,$$

taken positively round the fundamental polygon within the circle when the function $\Theta(z)$ exists only within the circle, and round the two fundamental polygons, within and without the circle respectively, when the function $\Theta(z)$ exists over the whole plane.

But should an infinity of $\frac{\Theta'(z)}{\Theta(z)}$ lie on the curve along which integration extends, (it will arise through either a zero or a pole of Θ), then, in order to avoid the difficulty in the integration and preserve the above convention, methods must be adopted depending upon the family of the group.

When all the cycles belong to the first sub-category (§ 292), we can proceed as follows: the general result can be proved to hold in every case. If an infinity occur on a side, another will occur on the conjugate side, the two being homologous by a fundamental substitution. A small semicircle is drawn with the point for centre and lying without the polygon, so that, when the element of the side is replaced by the semi-circumference, the point lies within the polygon: the homologous point on the conjugate side is excluded from the polygon when the element there is replaced by the homologous semi-circumference. The subject of integration is then finite along the modified sides.

A similar process is adopted when a corner is an infinity of $\dfrac{\Theta'(z)}{\Theta(z)}$. A small circular arc is drawn so as to have the point included in the polygon when the arc replaces the elements of the sides at the point: the homologous circular arcs at all the points in the cycle of the corner will exclude all those points, also poles, when they replace the elements of the sides at the point. The subject of integration is then finite everywhere along the modified path of integration.

First, let the function exist only within the circle. Let AB be any side of the polygon, $A'B'$ the conjugate side; and let

$$\zeta = \frac{\alpha z + \beta}{\gamma z + \delta}$$

be the corresponding fundamental substitution which transforms AB into $A'B'$, so that ζ may be regarded as the variable along $A'B'$.

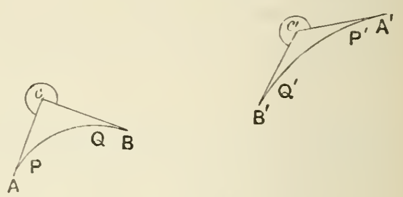

Fig. 124.

Then we have $$\Theta(\zeta) = (\gamma z + \delta)^{2m} \Theta(z),$$

and therefore $$\frac{\Theta'(\zeta)}{\Theta(\zeta)} d\zeta = \frac{2m\gamma}{\gamma z + \delta} dz + \frac{\Theta'(z)}{\Theta(z)} dz.$$

But as z moves from A to B, ζ moves from A' to B' (§ 287): and the latter is the negative direction of description. Hence, with the given notation, the sum of the parts of the integral, which arise through the two sides AB and $B'A'$, is

$$\int \frac{\Theta'(z)}{\Theta(z)} dz, \text{ for } AB + \int \left\{- \frac{\Theta'(\zeta)}{\Theta(\zeta)} d\zeta\right\}, \text{ for } B'A'$$

$$= -\int \frac{2m}{z + \frac{\delta}{\gamma}} dz, \text{ taken along } AB:$$

so that, if E denote the required excess, we have

$$E = -\frac{m}{\pi i} \int \frac{dz}{z + \frac{\delta}{\gamma}},$$

the new integral being taken along those sides of the polygon which are transformed into their conjugates by the fundamental substitutions of the group.

Consider the term which arises through the integration along AB: it is evidently

$$-\frac{m}{\pi i} \Big[\log(\gamma z + \delta) \Big]_A^B.$$

Now we have
$$\frac{d\zeta}{dz} = \frac{1}{(\gamma z + \delta)^2},$$
so that, if M be the magnification in transforming from A to A', and if ϕ_a be the angle through which a small arc is turned, we have at A
$$\frac{1}{(\gamma z + \delta)^2} = Me^{i\phi_a}.$$
Evidently ϕ_a is the excess of the inclination of $A'P'$, that is, of $A'C'$ to the line of real quantities over the inclination of AP, that is, of AC to that line: and therefore at A
$$\log(\gamma z + \delta) = -\tfrac{1}{2}\log M - \tfrac{1}{2}i\phi_a.$$
Since the whole integral must prove to be a real quantity, we omit the parts $-\frac{m}{2\pi i}\log M$ as in the aggregate constituting an evanescent (imaginary) quantity: hence we have
$$\frac{m}{2\pi}(-\phi_a + \phi_b)$$
as the part corresponding to the side AB. In this expression, ϕ_a is the angle required to turn AC into a direction parallel to $A'C'$, and ϕ_b is the angle required to turn QB, that is, CB into a direction parallel to $Q'B'$, that is, $C'B'$, both rotations being taken positively. Thus
$$\phi_a = \text{incl. } A'C' - \text{incl. } AC,$$
$$\phi_b = 2\pi - \text{incl. } BC + \text{incl. } B'C';$$
and therefore
$$\begin{aligned}\phi_a - \phi_b &= -2\pi + \text{incl. } A'C' - \text{incl. } B'C' + \text{incl. } BC - \text{incl. } AC\\ &= -2\pi + c_1' + c_1,\end{aligned}$$
where c_1 and c_1' are the angles ACB, $A'C'B'$ respectively. Hence, if we take c and c' to be the external angles ACB, $A'C'B'$ as in the figure, we have
$$c + c_1 = 2\pi = c' + c_1',$$
and therefore
$$\phi_b - \phi_a = c + c' - 2\pi.$$
The part corresponding to the arc AB in the above integral is therefore
$$\frac{m}{2\pi}(c + c' - 2\pi).$$
There are no sides of the second kind in the path of integration, because the function is supposed to exist only within the circle. Therefore the whole excess is given by
$$E = \frac{m}{2\pi}\Sigma(c + c' - 2\pi),$$
the summation extending over those sides of the polygon, being in number half of the sides of the first kind, which are transformed into their conjugates by the fundamental substitutions of the group.

Draw all the pairs of tangents at the extremities of the bounding arcs of the fundamental polygon of reference: then the angles, such as c and c' above, are internal angles of the rectilinear polygon formed by the straight lines. The remaining internal angles of this new polygon are the angles at which the arcs cut, which are the angles of the curvilinear polygon: and therefore their sum is the sum of the angles in the cycles, that is, the sum is equal to

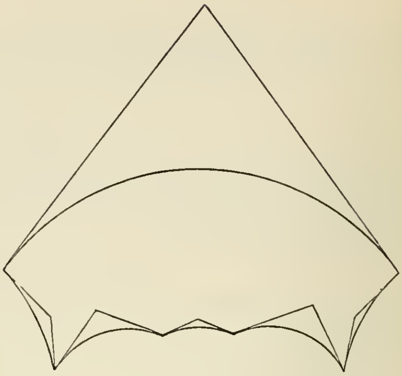

Fig. 125.

$$\Sigma \frac{2\pi}{\mu_i},$$

where $\frac{2\pi}{\mu_i}$ is the sum of the angles in one of the cycles. Now let $2n$ be the number of sides of the first kind in the curvilinear polygon, so that n is the number of fundamental substitutions in the group: hence the number of terms in the above summation for E is n, and therefore

$$E = -mn + \frac{m}{2\pi} \Sigma (c + c').$$

Moreover the rectilinear polygon has $4n$ sides: and therefore the sum of the internal angles is $(4n - 2)\pi$. But this sum is equal to $\Sigma (c + c') + \Sigma \frac{2\pi}{\mu_i}$, where the first summation extends to the different conjugate pairs and the second to the different cycles: thus

$$(4n - 2)\pi = \Sigma (c + c') + 2\pi \Sigma \frac{1}{\mu_i}.$$

Therefore
$$\begin{aligned} E &= -mn + m(2n - 1) - m\Sigma \frac{1}{\mu_i} \\ &= m\left(n - 1 - \Sigma \frac{1}{\mu_i}\right), \end{aligned}$$

where the summation extends over all the different cycles in the fundamental polygon. Hence *for a function, which is constructed from the additive element $H(z)$ and exists only within the fundamental circle of the group, the excess of the number of its irreducible zeros over the number of its irreducible accidental singularities is*

$$m\left(n - 1 - \Sigma \frac{1}{\mu_i}\right),$$

where m is the parametric integer of the function constructed in series, $2n$ is the number of sides of the first kind in the fundamental polygon, $\frac{2\pi}{\mu_i}$ is the sum *of the angles in a cycle of the first kind of corners and the summation extends to all these cycles.*

The number of irreducible accidental singularities has already been obtained; it is finite, and thus the number of irreducible zeros is finite.

Secondly, let the function exist all over the plane: then the irreducible points are (i) points lying within (or on) the boundary of the fundamental polygon of reference within the fundamental circle and (ii) points lying within (or on) the boundary of the fundamental polygon of reference without the fundamental circle, the outer polygon being the inverse of the inner polygon with regard to the centre. For such a function the excess of the number of irreducible zeros over the number of irreducible accidental singularities is the integral

$$\frac{1}{2\pi i}\int \frac{\Theta'(z)}{\Theta(z)}\,dz,$$

taken positively round the boundaries of both polygons. We shall assume that there are no zeros and no infinities on the path of integration; the result can, however, be shewn to be valid in the contrary case.

For the sides of the internal polygon that are of the first kind the value of the integral is, as before, equal to

$$m\left(n-1-\Sigma\frac{1}{\mu_i}\right):$$

and for the sides of the external polygon that are of the first kind, the value is also

$$m\left(n-1-\Sigma\frac{1}{\mu_i}\right).$$

Let the value of the integral along the sides of the second kind in the internal polygon be I. Those lines are also sides of the second kind in the external polygon; but they are described in the sense opposite to that for the internal polygon, the integral being always taken positively: hence the value of the integral along the sides of the second kind in the external polygon is $-I$.

Hence *the excess of the number of irreducible zeros over the number of irreducible accidental singularities of a function* $\Theta(z)$, *which is constructed from the additive element* $H(z)$ *and exists all over the plane, is*

$$2m\left(n-1-\Sigma\frac{1}{\mu_i}\right),$$

where the summation extends over all the cycles of the first category of either (but not both) of the fundamental polygons of reference.

As before, the number of irreducible zeros of such a function is finite, because the number of irreducible accidental singularities is finite.

In every case, this excess depends only upon

(i) the parametric integer m, used in the construction of the series:

(ii) the number of sides, $2n$, of the first kind in the polygon of reference:

(iii) the sum of the angles in the cycles of the first category.

Ex. Prove that a corner belonging to a cycle of the first category is in general a zero of order p, such that

$$p \equiv -m \pmod{\mu},$$

where $2\pi/\mu$ is the sum of the angles in the cycle: and discuss the nature of the corners which belong to cycles of the remaining categories. (Poincaré.)

308. We are now in a position to construct automorphic functions, using as subsidiary elements the pseudo-automorphic functions which have just been considered.

For, if we take a couple of these functions, Θ_1 and Θ_2, associated with a given infinite group, characterised by the same integer m, and arising through different additive elements $H(z)$, then we have

$$\Theta_1\left(\frac{\alpha z+\beta}{\gamma z+\delta}\right) = (\gamma z+\delta)^{2m}\,\Theta_1(z),$$

$$\Theta_2\left(\frac{\alpha z+\beta}{\gamma z+\delta}\right) = (\gamma z+\delta)^{2m}\,\Theta_2(z),$$

where $\dfrac{\alpha z+\beta}{\gamma z+\delta}$ is any one of the substitutions of the group; and therefore

$$\frac{\Theta_1\left(\dfrac{\alpha z+\beta}{\gamma z+\delta}\right)}{\Theta_2\left(\dfrac{\alpha z+\beta}{\gamma z+\delta}\right)} = \frac{\Theta_1(z)}{\Theta_2(z)}:$$

that is, the quotient of two such functions is automorphic. Denoting the quotient by $Pn(z)$*, we have

$$Pn\left(\frac{\alpha z+\beta}{\gamma z+\delta}\right) = Pn(z),$$

the automorphic property being possessed for each of the substitutions.

It thus appears that such functions exist: their essential property is that of being reproduced when the independent variable is subjected to any of the linear substitutions of the infinite group.

The foregoing is of course the simplest case, adduced at once to indicate the existence of the functions. The construction can evidently be generalised: for, if we have any number of functions $\Theta_1, \ldots, \Theta_r, \Phi_1, \ldots, \Phi_s$ with characteristic integers $m_1, \ldots, m_r, n_1, \ldots, n_s$ and all associated with one group

* Poincaré calls such functions *Fuchsian* functions: as already indicated (§ 297), I have preferred to associate the general name *automorphic* with them. But, because Poincaré himself has constructed one class of such functions by means of series as in the foregoing manner, his name, if any, should be associated with this class: the symbol $Pn(z)$ is therefore used.

while constructed from different additive elementary functions $H(z)$, then, denoting

$$\frac{\Theta_1(z)\ldots\ldots\Theta_r(z)}{\Phi_1(z)\ldots\ldots\Phi_s(z)}$$

by $Pn(z)$, we evidently have

$$Pn\left(\frac{\alpha z+\beta}{\gamma z+\delta}\right)=(\gamma z+\delta)^{2\sum\limits_{q=1}^{r} m_q-2\sum\limits_{q=1}^{s} n_q}Pn(z),$$

so that, provided only

$$\sum_{q=1}^{s} n_q=\sum_{q=1}^{r} m_q,$$

the function is automorphic. If we agree to call m, the integer characteristic of a pseudo-automorphic function, the *degree* of that function, then *the quotient of two products of pseudo-automorphic functions is automorphic, provided the products be of the same degree.*

There are evidently two classes of automorphic functions: those which exist all over the plane, and those which exist only within the fundamental circle. The classes are discriminated according to the composition of the functions from the subsidiary pseudo-automorphic functions.

When the pseudo-automorphic functions, which enter into the composition of the function, exist all over the plane, then the automorphic function exists all over the plane. But when the pseudo-automorphic functions, which enter into the composition of the function, exist only within the fundamental circle, then the automorphic function exists only within the circle.

309. It is evident that all the essential singularities of an automorphic function, thus constructed, lie on the fundamental circle. For whether the pseudo-automorphic functions exist only within that circle or over the whole plane, all their essential singularities lie on the circumference: so that, whatever be the constitution of the various subsidiary pseudo-automorphic functions, all the essential singularities of the automorphic function lie on the fundamental circle.

Next, the number of irreducible zeros of an automorphic function is equal to the number of its irreducible accidental singularities. For an irreducible zero of an automorphic function is either (i) an irreducible zero of a factor in the numerator or (ii) an irreducible accidental singularity of a factor in the denominator; and similarly with the irreducible accidental singularities of the function. The numerator and the denominator may have common zeros; this will not affect the result.

First, let the automorphic function exist only within the circle: then each of its factors exists only within the circle. The space without the circle is not significant for any of the factors of the function, because they do not there exist. Let $\epsilon_1, \ldots, \epsilon_r, \epsilon_1', \ldots, \epsilon_s'$ be the excesses of zeros over accidental singularities for the pseudo-automorphic functions within the fundamental circle: then

$$\epsilon_q = m_q \left(n - 1 - \Sigma \frac{1}{\mu_i} \right),$$

where n and $\Sigma \frac{1}{\mu_i}$ are the same for all these functions, and

$$\epsilon_q' = n_q \left(n - 1 - \Sigma \frac{1}{\mu_i} \right).$$

Now the excess of zeros over poles in the denominator becomes, after the above explanation, an excess of poles over zeros for the automorphic function: hence, for this automorphic function, the excess of zeros over accidental singularities is

$$\begin{aligned} &= \sum_{q=1}^{r} \epsilon_q - \sum_{q=1}^{s} \epsilon_q' \\ &= \left(n - 1 - \Sigma \frac{1}{\mu_i} \right) \left(\sum_{q=1}^{r} m_q - \sum_{q=1}^{s} n_q \right) \\ &= 0, \end{aligned}$$

by the condition $\sum_{q=1}^{r} m_q = \sum_{q=1}^{s} n_q$. Hence the number of irreducible zeros of the automorphic function is equal to the number of irreducible accidental singularities.

Secondly, let the automorphic function exist all over the plane; then all its factors exist all over the plane. For the present purpose, the sole analytical difference from the preceding case is that each of the quantities ϵ now has double its former value: and therefore the excess of the number of zeros over the number of poles is

$$2 \left(n - 1 - \Sigma \frac{1}{\mu_i} \right) \left(\sum_{q=1}^{r} m_q - \sum_{q=1}^{s} n_q \right),$$

which, as before, vanishes. Hence the number of irreducible zeros of the automorphic function is equal to the number of its irreducible accidental singularities.

It follows, as an immediate Corollary, that the *number of irreducible points for which an automorphic function assumes a given value is equal to the number of its irreducible accidental singularities.* For

$$Pn(z) - A,$$

where A is a constant, is an automorphic function: the number of its irreducible accidental singularities is equal to the number of its irreducible zeros, that is, it is equal to the number of irreducible points for which $Pn(z)$ assumes an assigned value.

Moreover, *each of these numbers is finite:* for the number of irreducible zeros and the number of irreducible accidental singularities of each of the component pseudo-automorphic factors is finite, and there is only a finite number of these factors in the automorphic function. The integer, which represents each number, will evidently be as characteristic of these functions as the corresponding integer was of functions with linear additive periodicity.

Note. The preceding method, due to Poincaré, of expressing the pseudo-automorphic functions as converging infinite series of functions of the variable, is not the only method of obtaining such functions. It was shewn that uniform analytical functions can be represented either as converging series of powers or as converging series of functions or as converging products of primary factors, not to mention the (less useful) forms intermediate between series and products. The representation of automorphic functions as infinite products of primary factors is considered in the memoirs of Von Mangoldt and Stahl, already referred to in § 297.

310. Let $Pn_1(z)$, $Pn_2(z)$, say P_1 and P_2, be two automorphic functions with the same group, constructed with the most general additive elements: and let the number of irreducible zeros of the former be κ_1, and of the latter be κ_2.

Then for an assigned value of P_1 there are κ_1 irreducible points: P_2 has a single value for each of these points, and therefore it has κ_1 values altogether for all the points, that is, it has κ_1 values for each value of P_1. Similarly, P_1 has κ_2 values for each value of P_2. Hence *there is an algebraical relation between P_1 and P_2 of degree κ_2 in P_1 and of degree κ_1 in P_2*, which may be expressed in the form

$$F_{12}(P_1, P_2) = 0.$$

Let $Pn(z)$, say P, be any other uniform automorphic function, having the same group as P_1 and P_2: and let κ be the number of its irreducible zeros. Then we have an algebraical equation

$$F_1(P, P_1) = 0,$$

which is of degree κ_1 in P and of degree κ in P_1; and another equation

$$F_2(P, P_2) = 0,$$

which is of degree κ_2 in P and of degree κ in P_2. The last two equations coexist, in virtue of the relation

$$F_{12}(P_1, P_2) = 0$$

satisfied by P_1 and P_2. Since $F_1 = 0 = F_2$ coexist, the ordinary theory of elimination leads to the result that the uniform function P can be expressed rationally in terms of P_1 and P_2, so that we have the theorem that *every automorphic function associated with a given group can be expressed rationally in terms of two general automorphic functions associated with that group: and between these two functions there exists an irreducible algebraical relation.*

The *class* (§ 178) of this algebraical relation can be obtained as follows. Let N denote the class of the group, determined as in § 293: then the fundamental polygon of reference, if functions exist only within the circle, or the two fundamental polygons of reference, if functions exist over the whole plane, can be transformed into a surface of multiple connectivity $2N+1$. The automorphic functions are functions of uniform position on this surface; and hence, as in Riemann's theory of functions, the *algebraical relation between two general uniform functions of position, that is, between two general automorphic functions is of class N, where N is the class of the group**.

It is now evident that the existence-theorem and the whole of Riemann's theory of functions can be applied to the present class of functions, whether actually automorphic or only pseudo-automorphic. There will be functions of the same kinds as on a Riemann's surface: the periods will be linear numerical multiples of constant quantities acquired by a function when its argument moves from any position to a homologous position or returns to its initial position. There will be functions everywhere finite on the surface, that is, finite for all values of the variable z except those which coincide with the essential singularities of the group. The number of such functions, linearly independent of one another, is N; and every such function, finite for all values of z except at the essential singularities, can be expressed as a linear function of these N functions with constant coefficients and (possibly) an additive constant. And so on, for other classes of functions†.

311. Because $Pn(z)$ is an automorphic function, we have

$$Pn\left(\frac{\alpha z+\beta}{\gamma z+\delta}\right)=Pn(z),$$

and therefore, as $\alpha\delta-\beta\gamma=1$,

$$Pn'\left(\frac{\alpha z+\beta}{\gamma z+\delta}\right)=(\gamma z+\delta)^2\,Pn'(z).$$

Hence, if $\Theta(z)$ be a pseudo-automorphic function with m for its characteristic integer, so that

$$\Theta\left(\frac{\alpha z+\beta}{\gamma z+\delta}\right)=(\gamma z+\delta)^{2m}\,\Theta(z),$$

we have

$$\frac{\Theta\left(\dfrac{\alpha z+\beta}{\gamma z+\delta}\right)}{\left\{Pn'\left(\dfrac{\alpha z+\beta}{\gamma z+\delta}\right)\right\}^m}=\frac{\Theta(z)}{\{Pn'(z)\}^m},$$

* It may happen that, just as in the general theory of algebraical functions, the class of the equation between two particular automorphic functions may be less than N: thus one might be expressed rationally in terms of the other. The theorems are true for functions constructed in the most general manner possible.

† The memoirs by Burnside, quoted in § 297, develop this theory in full detail for the group which has its (combined) polygons of reference bounded by $2n$ circles with their centres on the axis of real quantities, the group being such that the pseudo-automorphic functions exist over the whole plane.

that is, $\Theta(z)\{Pn'(z)\}^{-m}$ is an automorphic function. Such a function can be expressed rationally in terms of $Pn(z)$ and some other function, say of P and Q: hence the general type of a pseudo-automorphic function with a characteristic integer m is

$$\left(\frac{dP}{dz}\right)^m f(P, Q),$$

where f is a rational function.

COROLLARY. *Two automorphic functions P and Q, belonging to the same group, are connected by the equation*

$$\frac{dQ}{dz}=\frac{dP}{dz}f(P, Q).$$

For evidently unity is the characteristic integer of the first derivative of an automorphic function.

This equation can be changed to

$$\frac{dQ}{dP}=f(P, Q),$$

where f is a rational function: moreover P and Q are connected by an equation

$$F(P, Q)=0,$$

which is an algebraical rational equation, and can evidently be regarded as an integral of the above differential equation of the first order, all trace of the variable z having disappeared. Evidently the form of f is given by

$$\frac{\partial F}{\partial P}+f(P, Q)\frac{\partial F}{\partial Q}=0.$$

Again, denoting $\dfrac{\alpha z+\beta}{\gamma z+\delta}$ by ζ, and $Pn\left(\dfrac{\alpha z+\beta}{\gamma z+\delta}\right)$ by $\Pi(\zeta)$, we have

$$\Pi'(\zeta)=(\gamma z+\delta)^2 Pn'(z),$$

say

$$\Pi'=(\gamma z+\delta)^2 P'.$$

Then

$$\frac{\Pi''}{\Pi'}=(\gamma z+\delta)^2\left[\frac{2\gamma}{\gamma z+\delta}+\frac{P''}{P'}\right]$$

$$=2\gamma(\gamma z+\delta)+(\gamma z+\delta)^2\frac{P''}{P'},$$

so that

$$\frac{\Pi'''}{\Pi'}-\left\{\frac{\Pi''}{\Pi'}\right\}^2=(\gamma z+\delta)^2\left[2\gamma^2+2\gamma(\gamma z+\delta)\frac{P''}{P'}+(\gamma z+\delta)^2\left(\frac{P'''}{P'}-\frac{P''^2}{P'^2}\right)\right];$$

and therefore

$$\frac{\Pi'''}{\Pi'}-\tfrac{3}{2}\left\{\frac{\Pi''}{\Pi'}\right\}^2=(\gamma z+\delta)^4\left[\frac{P'''}{P'}-\tfrac{3}{2}\left\{\frac{P''}{P'}\right\}^2\right],$$

whence

$$\frac{\{\Pi, \zeta\}}{\Pi'^2}=\frac{\{P, z\}}{P'^2},$$

where $\{P, z\}$ is the Schwarzian derivative. It thus appears that, if P be an automorphic function, then $\{P, z\} P'^{-2}$ is a function automorphic for the same group.

But between two automorphic functions of the same group, there subsists an algebraical equation: hence there is an algebraical equation between P and $\{P, z\} P'^{-2}$, that is, $P(z)$, *an automorphic function of* z, *satisfies a differential equation of the third order, the degree of which is the integer representing the number of irreducible zeros of* P *and the coefficients of which, where they are not derivatives of* P, *are functions of* P *only and not of the independent variable.*

This equation can be differently regarded. Take

$$y_1 = P'^{\frac{1}{2}}, \qquad y_2 = zP'^{\frac{1}{2}};$$

then it is easy to prove that

$$\frac{1}{y_1}\frac{d^2y_1}{dP^2} = \frac{1}{y_2}\frac{d^2y_2}{dP^2} = \frac{1}{2}\frac{\{P, z\}}{P'^2}.$$

The last fraction has just been proved to be an automorphic function of z; and therefore it is rationally expressible in terms of P and any other general function, say Q, automorphic for the group. Then y_1 and y_2 are independent integrals of the equation

$$\frac{d^2y}{dP^2} = y\phi(P, Q),$$

where Q and P are connected by the algebraical equation

$$F(P, Q) = 0.$$

Conversely, the quotient of two independent integrals of the equation

$$\frac{d^2y}{dP^2} = y\phi(P, Q),$$

where Q and P are connected by the algebraical equation

$$F(P, Q) = 0,$$

can be taken as an argument of which P and Q are automorphic functions: the class of the equation $F = 0$ is the class of the infinite group of substitutions for which P and Q are automorphic*.

Ex. One of the simplest set of examples of automorphic functions is furnished by the class of homoperiodic functions (§ 116). Another set of such examples arises in the triangular functions, discussed in § 275; they are automorphic for an infinite group, and the triangles have a circle for their natural limit. A third set is furnished by the polyhedral functions (§§ 276—279).

As a last set of examples, we may consider the modular-functions which were obtained by a special method in § 303.

* Klein remarks (*Math. Ann.*, t. xix, p. 143, note 4) that the idea of uniform automorphic functions occurs in a posthumous fragment by Riemann (*Ges. Werke*, number xxv, pp. 413—416). It may also be pointed out that the association of such functions with the linear differential equation of the second order is indicated by Riemann.

First, we consider them in illustration of the algebraical relations between functions automorphic for the same group. It follows, from the construction of the group and the relation of c to w, that, in the division of the plane by the group with Uw and Vw for its fundamental substitutions, where

$$Uw = w + 2, \qquad Vw = \frac{w}{1-2w},$$

there is only a single point in each of the regions for which c has an assigned value; hence, regarding c as an automorphic function of w, the number κ (§ 310) is unity. If there be any other function C of w, automorphic for this group, then between C and c there is an algebraical relation of degree in C equal to the number κ for c, that is, of the first degree in C. Hence *every function automorphic for the group, whose fundamental substitutions are U and V, where*

$$Uw = w + 2, \qquad Vw = \frac{w}{1-2w},$$

is a rational algebraical function of c.

In the same way, it can be inferred that *every function automorphic for the group, whose fundamental substitutions are*

$$Uw = w + 2, \qquad Tw = -\frac{1}{w},$$

is a rational, algebraical, function of cc'; and that *every function automorphic for the group, whose fundamental substitutions are*

$$Sw = w + 1, \qquad Tw = -\frac{1}{w},$$

that is, automorphic for all substitutions of the form $\dfrac{aw+b}{cw+d}$, where a, b, c, d are real integers, such that $ad - bc = 1$, is a rational algebraical function of $J = \dfrac{(c^2-c+1)^3}{c^2(c-1)^2}$.

Secondly, in illustration of the general theorem relating to the differential equation of the third order which is characteristic of an automorphic function, we consider the quantity c as a function of the quotient of the quarter-periods. Let z denote $\dfrac{iK'}{K}$: then because every function automorphic for the same group of substitutions as c is a rational function of c, we have

$$\frac{\{c,\ z\}}{c'^2} = \text{rational function of } c\,;$$

and therefore, by a property of the Schwarzian derivative,

$$\{z,\ c\} = -\text{ same rational function of } c.$$

By known formulæ of elliptic functions, it is easy to shew that

$$\{z,\ c\} = \frac{1-c+c^2}{2c^2(1-c)^2},$$

thus verifying the general result.

Similarly, it follows that $\left\{\dfrac{iK'}{K},\ \theta\right\}$, where $\theta = cc'$, is a rational function of cc', the actual value being given by

$$\left\{\frac{iK'}{K},\ \theta\right\} = \frac{1-5\theta+16\theta^2}{2\theta^2(1-4\theta)^2};$$

and that $\left\{\dfrac{iK'}{K},\ J\right\}$ is a rational function of J, the actual value being given by

$$\left\{\frac{iK'}{K},\ J\right\} = \frac{16J^2-123J-330}{2J^2(4J-27)^2}.$$

In this connection a memoir by Hurwitz* may be consulted.

* *Math. Ann.*, t. xxiii, (1889), pp. 345—352.

The preceding application to differential equations is only one instance in the general theory which connects automorphic functions with linear differential equations having algebraical coefficients. This development belongs to the theory of differential equations rather than to the general theory of functions: its exposition must be reserved for another place.

Here my present task comes to an end. The range of the theory of functions is vast, its ramifications are many, its development seems illimitable: an idea of its freshness and its magnitude can be acquired by noting the results, and appreciating the suggestions, contained in the memoirs of the mathematicians who are quoted in the preceding pages.

GLOSSARY

OF TECHNICAL TERMS USED IN THE THEORY OF FUNCTIONS.

(*The numbers refer to the pages, where the term occurs for the first time in the book or is defined.*)

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